QCD β Function

In these notes, I shall calculate to 1-loop order the δ_3 counterterm for the gluons and hence the β functions of a non-abelian gauge theory such as QCD. For simplicity, I am going to refer to the gauge fields as 'gluons' and to the fermions as 'quarks', but I am going to allow for any simple gauge group G and for the fermions in any complete multiplet of G. Once I finish the calculation, I'll spell out the group factors for $G = SU(N_c)$ and N_f fundamental multiplets of quarks, but until then I'll keep all the group factors generic.

As explained in class — see also my notes on minimal subtraction — the β function for the gauge coupling can be obtained from the infinite parts of the counterterms as

$$\frac{dg(\mu)}{d\log\mu} = \beta(g) = 2g \times N_{\text{loops}} \times \text{Residue of } \frac{1}{\epsilon} \text{ pole of} \big[\delta_1 - \delta_2 - \tfrac{1}{2}\delta_3\big]. \tag{1}$$

In class, I have also calculated the δ_2 and the δ_1 counterterms for the quarks to one-loop order: in the Feynman gauge ($\xi = 1$) for the gluons and the MS regularization scheme,

$$\delta_2(\text{quark}) = -\frac{g^2}{16\pi^2} \times \frac{1}{\epsilon} \times C\binom{\text{quark}}{\text{multiplet}}, \tag{2}$$

$$\delta_1(\text{quark}) = -\frac{g^2}{16\pi^2} \times \frac{1}{\epsilon} \times \left[C\binom{\text{quark}}{\text{multiplet}} + C(\text{adjoint})\right]. \tag{3}$$

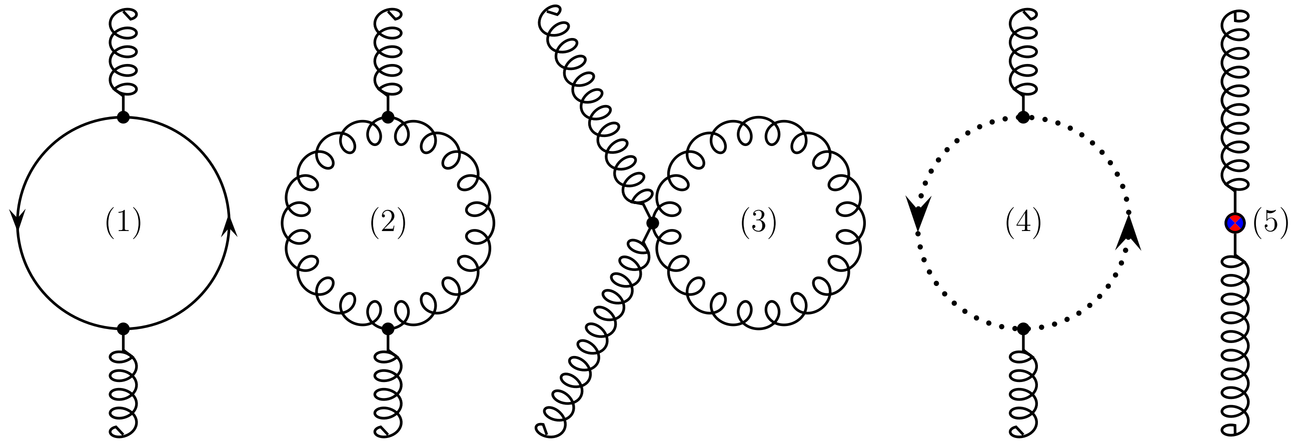
Note: in theories where the fermions form several different multiplets of the gauge group, each multiplet would have its own δ_2 and δ_1 counterterms, and they would differ from multiplet to multiplet. However, all the multiplets would have *the same difference*

$$\delta_1 - \delta_2 = -\frac{g^2}{16\pi^2} \times \frac{1}{\epsilon} \times C(\text{adjoint}). \tag{4}$$

In light of eq. (1), this means that all the fermions couple to the gauge fields with exactly the same renormalized coupling $g(\mu)$.

In these notes, I shall calculate the δ_3 counterterm to the one-loop order; once I have it, eqs. (1) and (4) would give me the (one-loop) $\beta(g)$.

At the one-loop order, the self-energy corrections to the gluons come from 5 diagrams:



where the fifth diagrams's contribution

$$\Sigma_5^{\mu\nu}(k) \;=\; -\delta_3 \times \left(k^2 g^{\mu\nu} \;-\; k^\mu k^\nu\right) \tag{5}$$

cancels the UV divergences of the first 4 diagrams. So let's calculate those divergences.

The first diagram — the quark loop — gives us

$$i\Sigma_1^{\mu\nu}(k) \;=\; -\int\frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\left((-ig\gamma^\mu)\frac{i}{\not{p}-m+i0}\,(-ig\gamma^\nu)\frac{i}{\not{p}+\not{k}-m_i0}\right)\times\mathrm{tr}\left(t^a_{(q)}\,t^b_{(q)}\right) \tag{6}$$

where the first trace is over the Dirac indices while the second trace is over the quarks' colors and flavors. For a single quark multiplet (m) of the gauge group G

$$\mathrm{tr}\left(t^a_{(m)}\,t^b_{(m)}\right) \;=\; \delta^{ab}\times R(m) \tag{7}$$

where $R(m)$ is the *index* of the multiplet (m). For several quark multiplets, their contributions add up, thus

$$\mathrm{tr}\left(t^a_{(q)}\,t^b_{(q)}\right) \;=\; \delta^{ab}\times R_{\mathrm{net}} \;=\; \delta^{ab}\times\sum_{\substack{\text{quark}\\\text{multiplets}}} R(\text{multiplet}). \tag{8}$$

In particular, in QCD the quarks comprise N_f copies of a fundamental $\mathbf{N}$ multiplet of the

$SU(N)$ gauge group — one fundamental multiplet for each flavor — hence

$$R_{\rm net} \;=\; N_f \times R(\text{fundamental}) \;=\; N_f \times \frac{1}{2}\,. \tag{9}$$

Apart from this group factor, the rest of the quark loop (6) looks exactly like the electron loop in QED. We have calculated that loop back in February — *cf.* my notes — so let me simply recycle the result in the present context:

$$\Sigma_1^{\mu\nu}(k) \;=\; \left(k^2 g^{\mu\nu} \;-\; k^\mu k^\nu\right) \times \frac{-g^2}{12\pi^2} \times R_{\rm net} \;\times \left(\frac{1}{\epsilon} + \text{finite}\right). \tag{10}$$

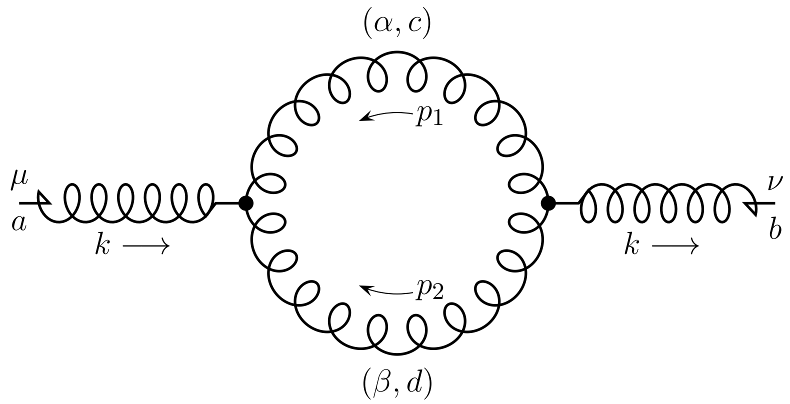
Consequently, the counterterm needed to cancel this divergence is

$$\delta_3(1^{\rm st}) \;=\; -\frac{g^2}{16\pi^2} \times \frac{1}{\epsilon} \times \frac{4}{3}\,R_{\rm net} \tag{11}$$

for a general gauge theory; for QCD

$$\delta_3(1^{\rm st}) \;=\; -\frac{g^2}{16\pi^2} \times \frac{1}{\epsilon} \times \frac{2}{3}\,N_f\,. \tag{12}$$

Now consider the second diagram — the gluon loop



(13)

Evaluating this diagram in the Feynman gauge, we get

$$i\Sigma_2^{\mu\nu}(k) \;=\; \frac{1}{2}\int\!\frac{d^4p_1}{(2\pi)^4}\,\frac{-i}{p_1^2+i0}\times\frac{-i}{p_2^2+i0}\times -gf^{acd}V^{\mu\alpha\beta}(k,p_1,p_2)\times -gf^{bcd}V^{\nu\alpha\beta}(-k,-p_1,-p_2) \tag{14}$$

where $\frac{1}{2}$ is the symmetry factor due to 2 similar gluon propagators, and the V's are the momentum- and Lorentz-index-dependent parts of the 3–gluon vertices,

$$\begin{aligned} V^{\mu\alpha\beta}(k,p_1,p_2) &= g^{\alpha\beta}(p_1-p_2)^\mu + g^{\beta\mu}(p_2-k)^\alpha + g^{\mu\alpha}(k-p_1)^\beta, \\ V^{\nu\alpha\beta}(-k,-p_1,-p_2) &= -V^{\nu\alpha\beta}(k,p_1,p_2). \end{aligned} \tag{15}$$

Let's start with the group factor in eq. (14). As explained in class,

$$\sum_{bc} f^{abc}\times f^{bcd} = \sum_{bc}\Bigl(-iT^a_{\rm adj}\Bigr)^{bc}\times\Bigl(+iT^a_{\rm adj}\Bigr)^{cb} = \operatorname{tr}\bigl(T^a_{\rm adj}T^d_{\rm adj}\bigr) = \delta^{ab}\times R(\text{adjoint}); \tag{16}$$

for an $SU(N)$ gauge group, $R(\text{adjoint})=N$. For a general gauge group,

$$R(\text{adjoint}) = C(\text{adjoint}),\quad \text{often denoted } C(G). \tag{17}$$

Plugging the group factor into eq. (14) and assembling all the constant factors, we obtain

$$\Sigma_2^{\mu\nu}(k) = -i\,\frac{g^2}{2}\,C(G)\times\int\!\frac{d^4p_1}{(2\pi)^4}\,\frac{\mathcal{N}_2^{\mu\nu}}{(p_1^2+i0)\times(p_2^2+i0)} \tag{18}$$

where the numerator is

$$\begin{aligned} \mathcal{N}_2^{\mu\nu} &= -V^{\mu\alpha\beta}(k,p_1,p_2)\times V^{\nu\alpha\beta}(-k,-p_1,-p_2) = +V^{\mu\alpha\beta}(k,p_1,p_2)\times V^{\nu\alpha\beta}(k,p_1,p_2) \\ &= D\times(p_1-p_2)^\mu(p_1-p_2)^\nu + g^{\mu\nu}\times(p_2-k)^2 + g^{\mu\nu}\times(k-p_1)^2 \\ &\quad + (p_1-p_2)^{(\mu}(p_2-k)^{\nu)} + (p_2-k)^{(\mu}(k-p_1)^{\nu)} + (k-p_1)^{(\mu}(p_1-p_2)^{\nu)}. \end{aligned} \tag{19}$$

As usual, the first step in evaluating the momentum integral like (18) is to simplify the denominator using the Feynman parameters. By momentum conservation $p_2\equiv -k-p_1$, hence

$$\frac{1}{(p_1^2+i0)(p_2^2+i0)} = \int_0^1\frac{dx}{\bigl[(1-x)p_1^2+x(p_1+k)^2+i0\bigr]^2} = \int_0^1\frac{dx}{[\ell^2-\Delta+i0]^2} \tag{20}$$

where

$$\ell = p_1 + xk \quad\text{and}\quad \Delta = -x(1-x)k^2. \tag{21}$$

Plugging this denominator int eq. (18) we get

$$\Sigma_2^{\mu\nu}(k) = -i\,\frac{g^2}{2}\,C(G)\times\int_0^1\!dx\int\!\frac{d^4\ell}{(2\pi)^4}\,\frac{\mathcal{N}_2^{\mu\nu}}{[\ell^2-\Delta+i0]^2}\,, \tag{22}$$

and now we need to re-express the numerator in terms of the shifted momentum ℓ. Using

$p_1 = +\ell - xk$ and $p_2 = -\ell - (1-x)k$, we obtain

$$p_1 - p_2 \;=\; 2\ell - (2x-1)k, \quad p_2 - k \;=\; -\ell + (x-2)k, \quad k - p_1 \;=\; -\ell + (x+1)k, \tag{23}$$

and hence

$$\begin{aligned}
\mathcal{N}_2^{\mu\nu} \;=\; & D \times \big(2\ell - (2x-1)k\big)^\mu \big(2\ell - (2x-1)k\big)^\nu \\
& +\; g^{\mu\nu} \times \big[(-\ell + (x-2)k)^2 \;+\; (-\ell + (x+1)k)^2\big] \\
& +\; (2\ell - (2x-1)k)^{(\mu}(-\ell + (x-2)k)^{\nu)} \;+\; (-\ell + (x-2)k)^{(\mu}(-\ell + (x+1)k)^{\nu)} \\
& +\; (-\ell + (x+1)k)^{(\mu}(2\ell - (2x-1)k)^{\nu)}
\end{aligned} \tag{24}$$

This whole big mess is a quadratic polynomial in ℓ and k, but the mixed terms like (ℓk) or $\ell^\mu k^\nu$ are odd with respect to $\ell \to -\ell$ and hence cancel out from the momentum integral (22). Thus, keeping only the terms carrying two or zero ℓ's, we arrive at

$$\begin{aligned}
\mathcal{N}_2^{\mu\nu} \;\cong\; & D \times \big[4\ell^\mu\ell^\nu \;+\; (2x-1)^2 k^\mu k^\nu\big] \;+\; g^{\mu\nu} \times \big[2\ell^2 \;+\; ((x-2)^2 + (x+1)^2)k^2\big] \\
& -\; 4\ell^\mu\ell^\nu \;-\; 2(2x-1)(x-2)k^\mu k^\nu \;+\; 2\ell^\mu\ell^\nu \;+\; 2(x-2)(x+1)k^\mu k^\nu \\
& -\; 4\ell^\mu\ell^\nu \;-\; 2(2x-1)(x+1)k^\mu k^\nu \\
=\; & g^{\mu\nu} \times \big[2\ell^2 \;+\; (2x^2 - 2x + 5)k^2\big] \;+\; (4D-6)\ell^\mu\ell^\nu \\
& +\; \big[(D-6) \;-\; (4D-6)x(1-x)\big] \times k^\mu k^\nu .
\end{aligned} \tag{25}$$

Moreover, in the context of the momentum integral (22)

$$\ell^\mu\ell^\nu \;\cong\; \frac{\ell^2}{D} \times g^{\mu\nu}, \tag{26}$$

hence

$$\begin{aligned}
\mathcal{N}_2^{\mu\nu} \;\cong\; & \big(k^2 g^{\mu\nu} \;-\; k^\mu k^\nu\big) \times \big[(6-D) \;+\; (4D-6)x(1-x)\big] \\
& +\; g^{\mu\nu} \times \left(\left(6 - \frac{6}{D}\right) \times \ell^2 \;+\; (D-1)(1 - 4x + 4x^2) \times k^2\right)
\end{aligned} \tag{27}$$

where I have re-arranged the k^2 terms so that the first line has the right tensor structure for the gluon's self-energy corrections. The second line has wrong tensor structure, and it does not integrate to zero, but we shall see that it cancels against similar bad terms from the two remaining diagrams.

To see how the cancellation works, let us postpone taking the momentum integral (22) until we have evaluated the sideways gluon loop and the ghost loop diagrams and brought them to a similar form

$$\Sigma_3^{\mu\nu}(k) \;=\; -i\,\frac{g^2}{2}\,C(G)\times\int_0^1\!dx\int\!\frac{d^4\ell}{(2\pi)^4}\,\frac{\mathcal{N}_3^{\mu\nu}}{[\ell^2-\Delta+i0]^2}\,, \tag{28}$$

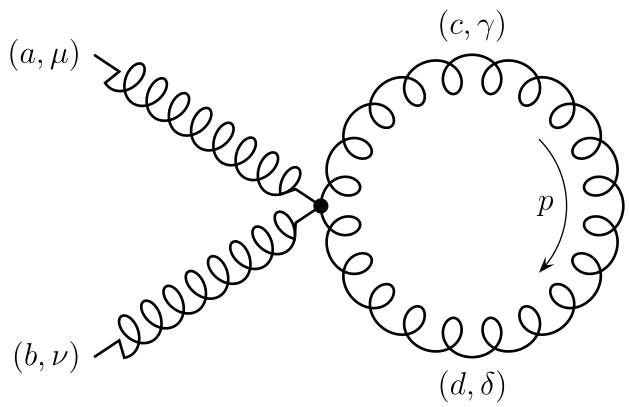
$$\Sigma_4^{\mu\nu}(k) \;=\; -i\,\frac{g^2}{2}\,C(G)\times\int_0^1\!dx\int\!\frac{d^4\ell}{(2\pi)^4}\,\frac{\mathcal{N}_4^{\mu\nu}}{[\ell^2-\Delta+i0]^2}\,, \tag{29}$$

for some numerators $\mathcal{N}_3^{\mu\nu}$ and $\mathcal{N}_4^{\mu\nu}$, then we are going to add up the numerators,

$$\mathcal{N}^{\mu\nu} \;=\; \mathcal{N}_2^{\mu\nu} \;+\; \mathcal{N}_3^{\mu\nu} \;+\; \mathcal{N}_4^{\mu\nu}, \tag{30}$$

and only then take the momentum integral.

For the sideways gluon loop



(31)

we have

$$i\Sigma_3^{\mu\nu}(k) \;=\; \frac{1}{2}\int\!\frac{d^4p}{(2\pi)^4}\,\frac{-ig_{\gamma\delta}\delta^{cd}}{p^2+i0}\times -ig^2\left[\begin{array}{r} f^{abe}f^{cde}(g^{\mu\gamma}g^{\nu\delta}-g^{\mu\delta}g^{\beta\gamma}) \\ +\,f^{ace}f^{bde}(g^{\mu\nu}g^{\gamma\delta}-g^{\mu\delta}g^{\gamma\nu}) \\ +\,f^{ade}f^{bce}(g^{\mu\nu}g^{\delta\gamma}-g^{\mu\gamma}g^{\delta\nu}) \end{array}\right] \tag{32}$$

where the overall factor $\frac{1}{2}$ comes from the symmetry of the propagator. The group factors in this amplitude evaluate to

$$\begin{aligned} \delta^{cd} \times f^{abe} f^{cde} &= 0, \\ \delta^{cd} \times f^{ace} f^{bde} &= f^{ace} f^{bce} = C(G) \times \delta^{ab}, \\ \delta^{cd} \times f^{ade} f^{bce} &= f^{ace} f^{bce} = C(G) \times \delta^{ab}, \end{aligned} \tag{33}$$

— *cf.* eq. (16) — hence

$$g_{\gamma\delta}\delta^{cd} \times [\cdots] = C(G)\delta^{ab} \times \Big(Dg^{\mu\nu} - g^{\mu\nu} + Dg^{\mu\nu} - g^{\mu\nu}\Big) = 2(D-1)C(G) \times \delta^{ab} g^{\mu\nu}. \tag{34}$$

Plugging this result into eq. (32), we obtain

$$\Sigma_3^{\mu\nu}(k) = +ig^2 C(G) \times (D-1) g^{\mu\nu} \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 + i0} \tag{35}$$

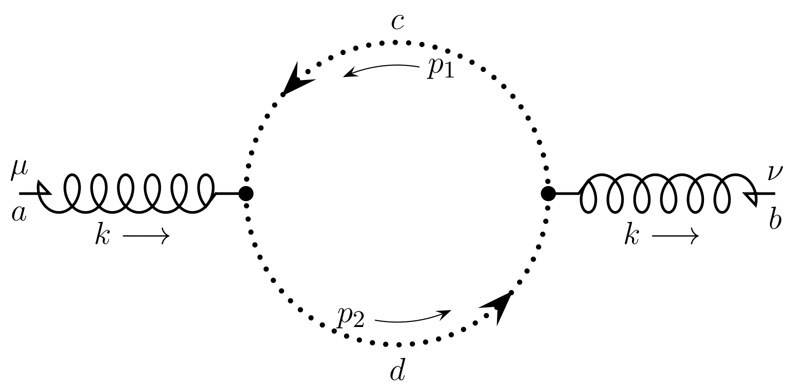
Instead of directly evaluate the momentum integral here, we are going to combine the integrand with the other one-loop diagrams. Since this diagram has only one propagator rather than two, we may identify the loop momentum p here as either p_1 or $p_2 = -p_1 - k$ — as long as our UV regulator allows constant shifts of the integration variable, both choices are equivalent. For symmetry's sake, let's take the average between the two choices and identify

$$\frac{1}{p^2 + i0} = \frac{1/2}{p_1^2 + i0} + \frac{1/2}{p_2^2 + i0} = \frac{p_1^2 + p_2^2}{2(p_1^2 + i0)(p_2^2 + i0)} = \int_0^1 dx \, \frac{(\ell - xk)^2 + (-\ell - (1-x)k)^2}{2[\ell^2 - \Delta + i0]^2} \tag{36}$$

Consequently, the amplitude (35) takes form (28) for the numerator

$$\mathcal{N}_3^{\mu\nu} = -(D-1) g^{\mu\nu} \times \big[(\ell - xk)^2 + (-\ell - (1-x)k)^2\big] = -(D-1) g^{\mu\nu} \times \big[2\ell^2 + (1 - 2x + 2x^2)k^2\big]. \tag{37}$$

Finally, there the ghost loop diagram



(38)

which evaluates to

$$i\Sigma_4^{\mu\nu}(k) \;=\; -\int\frac{d^4p_1}{(2\pi)^4}\,\frac{i}{p_1^2+i0}\times\frac{i}{p_2^2+i0}\times -gf^{acd}p_2^\mu\times -gf^{bdc}p_1^\nu. \tag{39}$$

Note: the ghost propagators are oriented and go in opposite directions, so this diagram does not have the symmetry factor $\frac{1}{2}$. Instead, it carries an overall minis sign for the fermionic loop. Another minus sign hides in the group factor:

$$f^{acd}f^{bdc} \;=\; -f^{acd}f^{bcd} \;=\; -C(G)\times\delta^{bc} \tag{40}$$

Consequently,

$$\Sigma_4^{\mu\nu}(k) \;=\; -i\frac{g^2}{2}C(G)\times\int\frac{d^4p_1}{(2\pi)^4}\,\frac{-2p_2^\mu p_1^\nu}{(p_1^2+i0)(p_2^2+i0)} \tag{41}$$

for $p_2 = +p_1 + k$. Combining the two denominator factors via the Feynman parameter integral, this amplitude takes for (29) for the numerator

$$\begin{aligned}\mathcal{N}_4^{\mu\nu} \;&=\; -2p_2^\mu p_1^\nu \;=\; -2(\ell-xk+k)^\mu(\ell-xk)^\nu \;\cong\; -2\ell^\mu\ell^\nu \;+\; 2x(1-x)k^\mu k^\nu\\ &\cong\; -\frac{2}{D}\,\ell^2\times g^{\mu\nu} \;+\; 2x(1-x)k^2g^{\mu\nu} \;-\; 2x(1-x)\big(k^2g^{\mu\nu}\;-\;k^\mu k^\nu\big).\end{aligned} \tag{42}$$

Now let's add up the numerators of the three diagrams:

$$\begin{aligned}\mathcal{N}^{\mu\nu} \;&=\; \mathcal{N}_2^{\mu\nu} \;+\; \mathcal{N}_3^{\mu\nu} \;+\; \mathcal{N}_4^{\mu\nu}\\ &=\; \big(k^2g^{\mu\nu}\;-\;k^\mu k^\nu\big)\times\big[(6-D)\;+\;(4D-6)x(1-x)\;+\;0\;-\;2x(1-x)\big]\\ &\quad+\; g^{\mu\nu}\ell^2\times\left[6\;-\;\frac{6}{D}\;-2(D-1)\;-\;\frac{2}{D}\right]\\ &\quad+\; g^{\mu\nu}k^2\times\big[(D-1)(1-4x+4x^2)\;-\;(D-1)(1-2x+2x^2)\;+\;2x(1-x)\big]\\ &=\; \big(k^2g^{\mu\nu}\;-\;k^\mu k^\nu\big)\times\big[(6-D)\;+\;4(D-2)x(1-x)\big]\\ &\quad+\; g^{\mu\nu}\ell^2\times\frac{-2(D-2)^2}{D}\;+\;g^{\mu\nu}k^2\times 2(2-D)x(1-x)\\ &\equiv\; \big(k^2g^{\mu\nu}\;-\;k^\mu k^\nu\big)\times\mathcal{N}_{\text{good}}\;+\;g^{\mu\nu}\times\mathcal{N}_{\text{bad}}\end{aligned} \tag{43}$$

where

$$\begin{aligned}\mathcal{N}_{\text{good}}(x) \;&=\; (6-D)\;+\;4(D-2)x(1-x),\\ \mathcal{N}_{\text{bad}}(x,\ell) \;&=\; -2(D-2)\times\left(\frac{D-2}{D}\,\ell^2\;-\;\Delta\right),\end{aligned} \tag{44}$$

for the same $\Delta = -x(1-x)k^2$ as in the denominator of the momentum integral.

The bad-tensor-structure term in the net numerator does not vanish, but it integrates to zero. Or rather, the dimensionally regularized integral of the bad term integrates to zero in any dimension D for which the integral converges (which takes $D < 2$). Indeed,

$$\begin{aligned}
\int\!\frac{d^D\ell}{(2\pi)^D}\,\frac{\mathcal{N}_{\rm bad}(\ell)}{[\ell^2-\Delta+i0]^2}\ &= \\
&=\ 2(D-2)i\int\!\frac{d^D\ell_E}{(2\pi)^D}\,\frac{\frac{D-2}{D}\,\ell_E^2\,+\,\Delta}{(\ell_E^2+\Delta)^2} \\
&=\ 2(D-2)i\int\!\frac{d^D\ell_E}{(2\pi)^D}\left(\frac{(D-2)/D}{\ell_e^2+\Delta}\,+\,\frac{(2/D)\Delta}{(\ell^2+\Delta)^2}\right) \\
&=\ \frac{2(D-2)i}{D}\int\!\frac{d^D\ell_E}{(2\pi)^D}\int\limits_0^\infty\!dt\,\bigl((D-2)\,+\,2\Delta\times t\bigr)\times\exp\bigl(-t(\ell_E^2+\Delta)\bigr) \\
&=\ \frac{2(D-2)i}{D}\int\limits_0^\infty\!dt\,\bigl((D-2)\,+\,2\Delta\times t\bigr)\times e^{-t\Delta}\times\left(\int\!\frac{d^D\ell_E}{(2\pi)^D}\,e^{-t\ell_E^2}\,=\,(4\pi t)^{-D/2}\right) \\
&=\ \frac{2(D-2)}{D(4\pi)^{D/2}}\times\left((D-2)\times\Gamma\left(1-\tfrac{D}{2}\right)\times\Delta^{\frac{D}{2}-1}\,+\,2\Delta\times\Gamma\left(1-\tfrac{D}{2}\right)\times\Delta^{\frac{D}{2}-2}\right) \\
&=\ \frac{4(D-2)}{D(4\pi)^{D/2}}\times\Delta^{\frac{D}{2}-2}\times\left(\left(\tfrac{D}{2}-1\right)\gamma\left(1-\tfrac{D}{2}\right)\,+\,\Gamma\left(2-\tfrac{D}{2}\right)\,=\,0\right).
\end{aligned} \tag{45}$$

Thus, the net vacuum polarization tensor for the gluons does have the right k dependence,

$$\Sigma^{\mu\nu}_{2+3+4}(k)\ =\ \bigl(k^2g^{\mu\nu}\,-\,k^\mu k^\nu\bigr)\times\Pi_{2+3+4}(k^2) \tag{46}$$

where

$$\Pi_{2+3+4}\ =\ -ig^2\,\frac{C(G)}{2}\times\int\limits_0^1\!dx\int\!\frac{d^4\ell}{(2\pi)^4}\,\frac{\mathcal{N}_{\rm good}}{(\ell^2-\Delta+i0)^2}\,. \tag{47}$$

Since the numerator $\mathcal{N}_{\rm good}$ does not depend on the loop momentum ℓ but only on the Feynman parameter x, the momentum integral here becomes the familiar

$$\int\!\frac{d^4\ell}{(2\pi)^4}\,\frac{1}{(\ell^2-\Delta+i0)^2}\ =\int\!\frac{d^4\ell_E}{(2\pi)^4}\,\frac{+i}{(\ell_E^2+\Delta)^2}\ \underset{\rm DR}{\longrightarrow}\ \frac{i}{16\pi^2}\left(\frac{1}{\epsilon}\,+\,{\rm finite}\right). \tag{48}$$

Consequently,

$$\Pi_{2+3+4}\ =\ +\frac{g^2C(G)}{32\pi^2}\left(\frac{1}{\epsilon}\,+\,{\rm finite}\right)\times\int\limits_0^1\!dx\,\mathcal{N}_{\rm good}(x), \tag{49}$$

and to get the divergent part of the amplitude we may evaluate the Feynman parameter integral for $D = 4$. Thus

$$\mathcal{N}_{\rm good}(x) \;\to\; 2 \;+\; 8x(1-x) \quad\Longrightarrow\quad \int\limits_0^1 dx\, \mathcal{N}_{\rm good}(x) \;\to\; \frac{10}{3}\,, \tag{50}$$

hence

$$\Pi_{2+3+4} \;=\; +\frac{g^2}{16\pi^2} \times \frac{5C(G)}{3} \times \left(\frac{1}{\epsilon} \,+\, \text{finite}\right), \tag{51}$$

and the δ_3 counterterm that cancels this divergence is

$$\delta_3(2^{\rm nd} + 3^{\rm rd} + 4^{\rm th}) \;=\; +\frac{g^2}{16\pi^2} \times \frac{5C(G)}{3} \times \frac{1}{\epsilon}\,. \tag{52}$$

Altogether — including the quark loops' contribution (11) — the one-loop δ_3 counterterm is

$$\delta_3^{\rm one\,loop} \;=\; \frac{g^2}{16\pi^2}\left(\frac{5}{3}\,C(G) \;-\; \frac{4}{3}\,R_{\rm net}({\rm quarks})\right) \times \frac{1}{\epsilon}\,. \tag{53}$$

Consequently, applying eqs. (1) and (4), we obtain

$$\left(\delta_1 \,-\, \delta_2 \,-\, \tfrac{1}{2}\delta_3\right)_{1\,\rm loop} \;=\; \frac{g^2}{16\pi^2}\left(-C(G) \;-\; \frac{5}{6}\,C(G) \;+\; \frac{2}{3}\,R_{\rm net}({\rm quarks})\right) \times \frac{1}{\epsilon} \tag{54}$$

and therefore

$$\beta(g)_{1\,\rm loop} \;=\; \frac{g^3}{16\pi^2}\left(-\frac{11}{3}\,C(G) \;+\; \frac{4}{3}\,R_{\rm net}({\rm quarks})\right). \tag{55}$$

For QCD and QCD-like theories with $SU(N_c)$ gauge group and N_f flavors of fundamental multiplets of quarks,

$$\beta(g)_{1\,\rm loop} \;=\; \frac{g^3}{16\pi^2}\left(-\frac{11}{3}\,N_c \;+\; \frac{2}{3}\,N_f\right). \tag{56}$$

For completeness sake, let me give you without proof the formula for the one-loop beta function for any gauge theory coupled to several kinds of 'matter' fields: Dirac fermions like the quarks, but also chiral Weyl fermions (left-handed or right-handed only), Majorana fermions, complex scalars, or real scalars. In general, the Dirac fermions, Weyl fermions, and complex scalars can be in any multiplets of the gauge group G, while the Majorana fermions and real scalars must be in real multiplets of G. Altogether,

$$\beta(g)_{1\,\text{loop}} = \frac{g^3}{16\pi^2} \times \sum_{\substack{\text{all physical}\\ \text{multiplets}}} R(\text{multiplet}) \times \begin{cases} -\frac{11}{3} & \text{for the gauge fields,} \\ +\frac{4}{3} & \text{for Dirac fermions,} \\ +\frac{2}{3} & \text{for Majorana fermions,} \\ +\frac{2}{3} & \text{for chiral Weyl fermions,} \\ +\frac{1}{3} & \text{for complex scalar fields,} \\ +\frac{1}{6} & \text{for real scalar fields.} \end{cases} \tag{57}$$

Note: the gauge fields' contribution here includes both the vector fields A^a_μ themselves and the ghosts c^a and $\bar{a}^a$, so do not count the ghosts as separate multiplets.

Also note that only the non-abelian gauge fields give negative contributions to the β function, all other fields' contributions are positive. Consequently, only the non-abelian gauge theories can be asymptotically free, and only when there are not too many fermionic or scalar fields coupled to the gauge fields. For example, QCD-like theory are asymptotically free only for $N_f < \frac{11}{2} N_c$.

Finally, a note on theories with product gauge groups $G = G_1 \otimes G_2 \otimes \cdots$. In such theories, each component group G_i — abelian or non-abelian — has its own gauge coupling g_i. At the one-loop level, the beta functions of each g_i are independent from each other (and also from the other couplings like Yukawa of $\lambda\phi^4$),

$$\forall i,\ \beta_i = \frac{g_i^3}{16\pi^2} \times \text{const} + \frac{g_i^3}{(4\pi)^4} \times O(g_i^2, \text{other}\, g_j^2, \text{yukawa}^2, \lambda) \tag{58}$$

Moreover, the constant factors in the one-loop terms are obtained exactly as in eq. (57) where we count multiplets of each G_i without paying attention to the other gauge groups G_j. For example, a $(\mathbf{m}, \mathbf{n})$ multiplet of $SU(m) \otimes SU(n)$ counts as m fundamental multiplets of $SU(n)$ when you calculate the β_n — or as n fundamental multiplets of $SU(m)$ when you calculate the β_m.