QCD Feynman Rules

The classical chromodynamics has a fairly simple Lagrangian

$$\mathcal{L} \;=\; \mathcal{L}_{\text{Yang-Mills}} \;+\mathcal{L}_{\text{quarks}} \;=\; -\tfrac{1}{4}\sum_{\mu,\nu,a}(F^a_{\mu\nu})^2 \;+\; \sum_{i,f}\overline{\Psi}_{if}(i\not{D}+m_f)\Psi^{if} \tag{1}$$

where i denotes the color of a quark and f its flavor. $D_\mu\Psi^i = \partial_\mu\Psi^i + igA^a_\mu(t^a)^i{}_j\Psi^j$ where t^a are matrices representing the gauge group generators in the quark representation; in QCD the quarks belong to the fundamental **3** representation of the $SU(3)_C$ so t^a is $\frac{1}{2}\times$ Gell-Mann matrix λ^a.

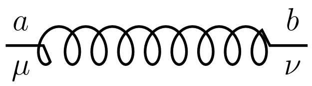
The Quantum ChromoDynamics is more complicated, even at the Lagrangian level: including the gauge-fixing and the ghost terms as well as the counterterms, we have

$$\begin{aligned}\mathcal{L} \;=\; & -\frac{1}{4}(F^a_{\mu\nu})^2 \;-\; \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \;+\; \partial_\mu\bar{c}^a D^\mu c^a \;+\; \sum_f \overline{\Psi}_{if}(i\not{D}+m_f)\Psi^{if} \\ & -\; \frac{\delta_3}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 \;+\; g\delta_1^{(3g)} f^{abc}A^b_\mu A^c_\nu \partial_\mu A^{a\nu} \;-\; \frac{g^2\delta_1^{(4g)}}{4}(f^{abc}A^b_\mu A^c_\nu)^2 \\ & +\; \delta_2^{(\text{gh})}\partial_\mu\bar{c}^a\partial^\mu c^a \;-\; g\delta_1^{(\text{gh})} f^{abc}\partial_\mu\bar{c}^a A^{b\mu}c^c \\ & +\; \sum_f \overline{\Psi}_{if}\left(i\delta_2^{(q_f)}\not{\partial} \;+\; \delta_m^{(q_f)} \;-\; g\delta_1^{(q_f)}\not{A}^a t^a\right)\Psi^{if}.\end{aligned} \tag{2}$$

In this formula, all sums over the colors (fundamental or adjoint) are implicit, as well as sums over the Lorentz or Dirac indices. But the sums over quark flavors are explicit because the quark masses depend on the flavor; the quark-related counterterms $\delta_2^{(q_f)}$, $\delta_1^{(q_f)}$, and $\delta_m^{(q_f)}$ could also be flavor-dependent.

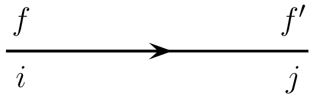
QCD Feynman rules follow from expanding the Lagrangian (2) into the free quadratic terms and the interaction terms (cubic, quartic, and all the counterterms). Thus we have:

— Gluon propagator



$$= \frac{-i\delta^{ab}}{k^2+i0}\left(g^{\mu\nu} \;+\; (\xi-1)\frac{k^\mu k^\nu}{k^2+i0}\right). \tag{3}$$

— Quark propagator

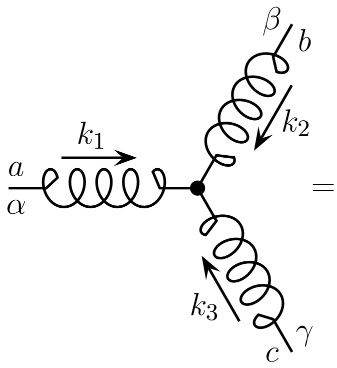


$$= \frac{i\delta^i_j\delta^f_{f'}}{\not{p}-m_f+i0}\,. \tag{4}$$

— Ghost propagator

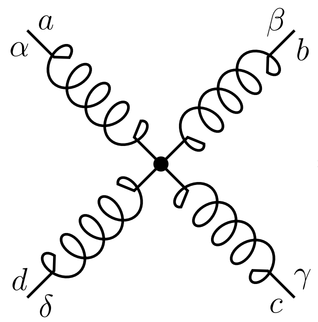
$$= \frac{i\delta^{ab}}{k^2 + i0}\,. \tag{5}$$

- Three-gluon vertex



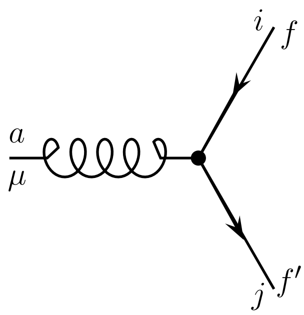
$$= -gf^{abc}\left[g^{\alpha\beta}(k_1 - k_2)^\gamma \;+\; g^{\beta\gamma}(k_2 - k_3)^\alpha \;+\; g^{\gamma\alpha}(k_3 - k_1)^\beta\right]. \tag{6}$$

- Four-gluon vertex



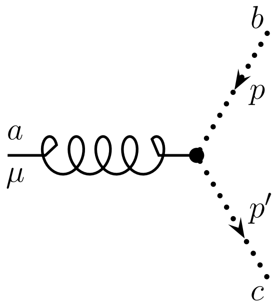
$$= -ig^2 \begin{bmatrix} f^{abe}f^{cde}(g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\delta}g^{\beta\gamma}) \\ + f^{ace}f^{bde}(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\delta}g^{\gamma\beta}) \\ + f^{ade}f^{bce}(g^{\alpha\beta}g^{\delta\gamma} - g^{\alpha\gamma}g^{\delta\beta}) \end{bmatrix}. \tag{7}$$

- Quark-gluon vertex



$$= -ig\gamma^\mu \times \delta_f^{f'} \times (t^a)^j{}_i\,. \tag{8}$$

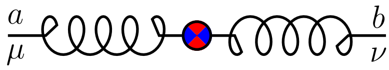
- Ghost-gluon vertex



$$= \; -gf^{abc}p'^{\mu}. \qquad (9)$$

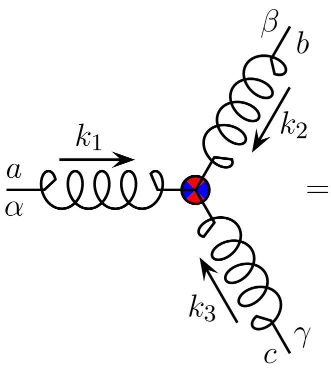
In addition, the renormalized theory has a whole bunch of the counterterm vertices:

∗ Two-gluon counterterm vertex



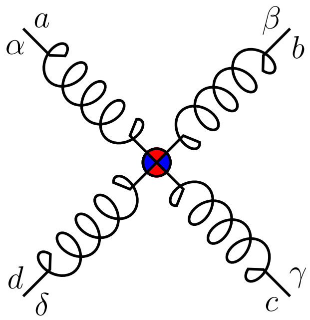
$$= \; -i\delta_3\delta^{ab}\left(k^2g^{\mu\nu} - k^{\mu}k^{\nu}\right). \qquad (10)$$

∗ Three-gluon counterterm vertex



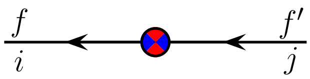
$$= \; -g\delta_1^{(3g)} \times f^{abc}\left[g^{\alpha\beta}(k_1 - k_2)^{\gamma} \;+\; g^{\beta\gamma}(k_2 - k_3)^{\alpha} \;+\; g^{\gamma\alpha}(k_3 - k_1)^{\beta}\right]. \qquad (11)$$

- Four-gluon counterterm vertex



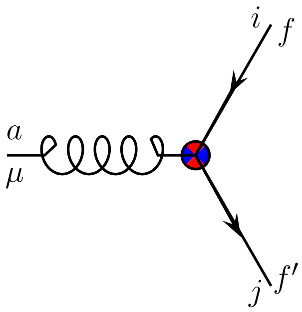
$$= \; -ig^2\delta_1^{(4g)} \times \begin{bmatrix} f^{abe}f^{cde}(g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\delta}g^{\beta\gamma}) \\ + f^{ace}f^{bde}(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\delta}g^{\gamma\beta}) \\ + f^{ade}f^{bce}(g^{\alpha\beta}g^{\delta\gamma} - g^{\alpha\gamma}g^{\delta\beta}) \end{bmatrix}. \qquad (12)$$

∗ Two-quark counterterm vertex



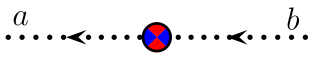
$$= \; \delta_f^{f'}\delta_i^j \times \left(i\delta_m^{(q_f)} \;-\; i\delta_2^{(q_f)} \times \not{p}\right). \qquad (13)$$

* Quark-gluon counterterm vertex



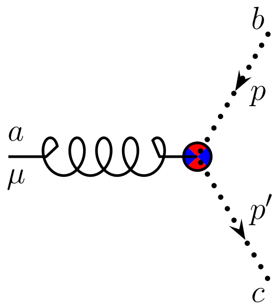
$$= -ig\delta_1^{(q_f)}\delta_f^{f'} \times \gamma^\mu \times (t^a)^j{}_i\,. \tag{14}$$

* Two ghost counterterm vertex



$$= \delta^{ab} \times i\delta_2^{(\mathrm{gh})} \times k^2. \tag{15}$$

* Ghost-gluon counterterm vertex

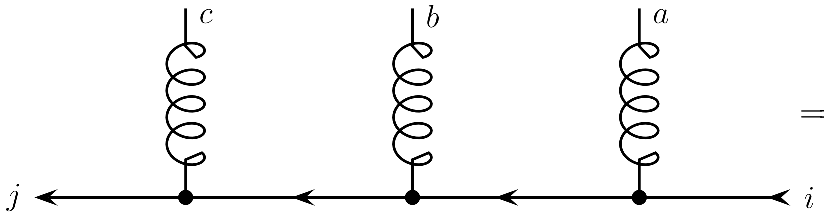


$$= -g\delta_1^{(\mathrm{gh})} \times f^{abc}p'^\mu. \tag{16}$$

⋆ Remember that the ghost fields are fermionic, so each closed loop of ghost propagators carries a minus sign.

⋆ The flavor f remains constant along any quark line, open or closed. For an open line, f matches both the incoming and the outgoing quarks (or antiquarks); for closed quark loops, we sum over all the flavors.

⋆ The color of a quark changes from propagator to propagator since the quark-quark-gluon vertices carry the $(t^a)^j{}_i$ factors. In matrix notations, the t^a generators should be multiplied right-to-left in the order of arrows on the quark line, for example



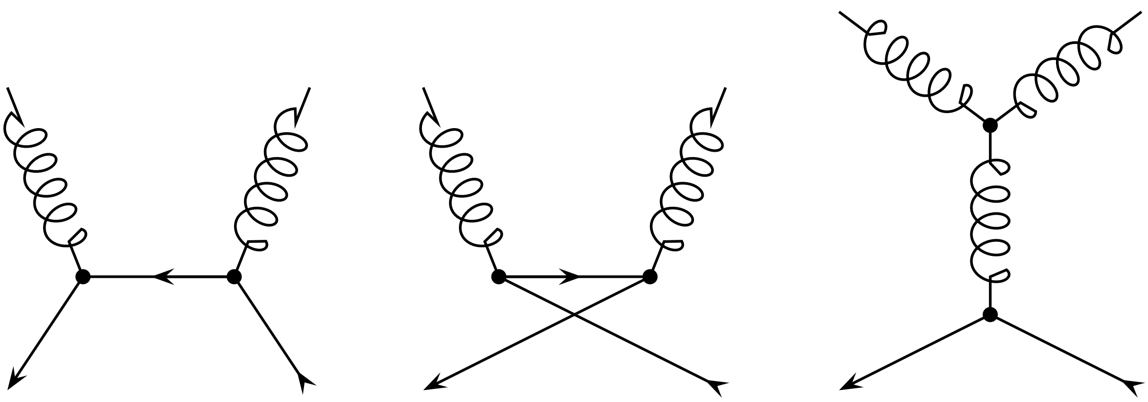
$$\Longrightarrow \quad \left(t^c t^b t^a\right)^j_{\ i} \times \text{other factors.}$$

For the closed quark lines, one starts at an arbitrary vertex, multiplies all the generators right-to-left in the order of the arrows, than takes the trace over the color indices, $\mathrm{tr}(\cdots t^c t^b t^a)$.

Ward Identities

QCD has weaker Ward identities than QED. In particular, consider the on-shell scattering amplitudes involving the longitudinally polarized gluons. *When one gluon is longitudinal and all other gluons are transverse, the amplitude vanishes. But when two or more gluons are longitudinal, the amplitude does not vanish; instead, it is related to the amplitudes involving the external ghosts instead of the longitudinal gluons.*

As an example, consider the tree level annihilation of a quark and an antiquark into a pair of gluons, $q\bar{q} \to gg$. In QED there are two tree diagrams for the $e^-e^+ \to \gamma\gamma$ annihilation, but in QCD there are three diagrams:



(17)

Let the incoming quark has momentum p_1 and color i, the incoming antiquark — momentum p_2 and color j, the first outgoing gluon — momentum k_1, polarization e_1^μ, and adjoint color a, and the second quark — k_2, e_2^ν, and b. Then the three diagrams evaluate to

$$\begin{aligned}
i\mathcal{M}_1 &= \bar{v}(p_2)\,(-ig\gamma^\nu e^*_{2\nu})\,\frac{i}{\not{q}_1 - m}\,(-ig\gamma^\mu e^*_{1\mu})\,u(p_1) \times \left(t^b t^a\right)^j_{\ i} \\
i\mathcal{M}_2 &= \bar{v}(p_2)\,(-ig\gamma^\mu e^*_{1\mu})\,\frac{i}{\not{q}_2 - m}\,(-ig\gamma^\nu e^*_{2\nu})\,u(p_1) \times \left(t^a t^b\right)^j_{\ i} \\
i\mathcal{M}_3 &= \bar{v}(p_2)\,(-ig\gamma^\kappa)\,u(p_1) \times (t^c)^j_{\ i} \times \frac{-ig_{\kappa\lambda}}{(k_1+k_2)^2} \times \\
&\quad \times (-g) f^{abc}\big[g^{\mu\nu}(-k_1+k_2)^\lambda \,+\, g^{\nu\lambda}(-k_2-(k_1+k_2))^\mu \,+\, g^{\lambda\mu}((k_1+k_2)+k_1)^\nu\big] \\
&\quad \times e^*_{1\mu} e^*_{2\nu}\,, \\
\mathcal{M}^{\rm net}_{\rm tree} &= \mathcal{M}_1 \,+\, \mathcal{M}_2 \,+\, \mathcal{M}_3\,.
\end{aligned} \tag{18}$$

Clearly, each term here is $O(g^2)$ and each term includes the polarization vectors for the two gluons, thus

$$\mathcal{M} \;=\; e^*_{1\mu} e^*_{2\nu} \times \mathcal{M}^{\mu\nu}\,. \tag{19}$$

So let us check the Ward identity $k_{1\mu} \times M^{\mu\nu} = 0$.

For the first diagram's amplitude we have

$$k_{1\mu} \times \mathcal{M}_1^{\mu\nu} \;=\; -g^2\left(t^b t^a\right)^j_{\ i} \times \bar{v}\gamma^\nu \frac{1}{\not{q}_1 - m}\,\not{k}_1 u. \tag{20}$$

In the second factor here, $q_1 = p_1 - k_1$, hence

$$\frac{1}{\not{q}_1 - m}\,\not{k}_1 \;=\; \frac{1}{\not{q}_1 - m}\,(\not{p}_1 - \not{q}_1) \;=\; -1 \;+\; \frac{1}{\not{q}_1 - m}\,(\not{p}_1 - m), \tag{21}$$

which for the on-shell quark gives

$$\frac{1}{\not{q}_1 - m}\,\not{k}_1\,u(p_1) \;=\; -u(p_1) \;+\; 0 \tag{22}$$

because $(\not{p}_1 - m)u(p_1) = 0$. Thus,

$$k_{1\mu} \times \mathcal{M}_1^{\mu\nu} \;=\; +g^2\left(t^b t^a\right)^j_{\ i} \times \bar{v}(p_2)\gamma^\nu u(p_1). \tag{23}$$

Likewise, for the second diagram

$$k_{1\mu} \times \mathcal{M}_2^{\mu\nu} \;=\; -g^2\big(t^b t^a\big)^j_{\;i} \times \bar{v}\, \not{k}_1 \frac{1}{\not{q}_2 - m}\, \gamma^\nu u. \tag{24}$$

where in the second factor

$$q_2 \,=\, k_1 \,-\, p_2 \quad\Longrightarrow\quad \not{k}_1 \frac{1}{\not{q}_2 - m} \,=\, 1 \,-\, (\not{p}_2 + m)\, \frac{1}{\not{q}_2 - m} \quad\Longrightarrow\quad v(p_2)\, \not{k}_1 \frac{1}{\not{q}_2 - m} \,=\, +v(p_2) \,-\, 0, \tag{25}$$

thus

$$k_{1\mu} \times \mathcal{M}_2^{\mu\nu} \;=\; -g^2\big(t^a t^b\big)^j_{\;i} \times \bar{v}(p_2)\gamma^\nu u(p_1). \tag{26}$$

In QED, $k_{1\mu} \times \mathcal{M}_1^{\mu\nu}$ and $k_{1\mu} \times \mathcal{M}_1^{\mu\nu}$ would have canceled each other, but in QCD eqs. (23) and (26) carry different color-dependent factors. So instead of cancellation, we have

$$k_{1\mu} \times \mathcal{M}_{1+2}^{\mu\nu} \;=\; g^2 \bar{v}\gamma^\nu u \times \big(t^b t^a \,-\, t^a t^b\big)^j_{\;i} \;=\; g^2 \bar{v}\gamma^\nu u \times -i f^{abc} \big(t^c\big)^j_{\;i}. \tag{27}$$

But the net color-dependent factor is similar to the third amplitude, so there is a hope that the Ward identity might work when all three diagrams are put together.

For the third diagram we have

$$\begin{aligned} k_{1\mu} \times \mathcal{M}_3^{\mu\nu} \;=\; & -i g^2 f^{abc} \big(t^c\big)^j_{\;i} \times \bar{v}\gamma^\lambda u \times \frac{1}{(k_1 + k_2)^2} \times \\ & \times k_{1\mu} \times \big[g^{\mu\nu}(k_2 - k_1)^\lambda \,+\, g^{\nu\lambda}(-k_1 - 2k_2))^\mu \,+\, g^{\lambda\mu}(2k_1 + k_2)^\nu\big]. \end{aligned} \tag{28}$$

On the second line here

$$\begin{aligned} k_{1\mu} \times [\cdots] \;&=\; k_1^\nu (k_2 - k_1)^\lambda \;+\; g^{\nu\lambda}(-k_1^2 - 2k_1 k_2) \;+\; k_1^\lambda (2k_1 + k_2)^\nu \\ &=\; -g^{\lambda\nu}\big((k_1 + k_2)^2 - k_2^2\big) \;+\; \big[k_1^\nu k_2^\nu + k_1^\lambda k_2^\nu + k_1^\nu k_2^\lambda\big] \\ \langle\!\langle\, \text{on shell} \,\rangle\!\rangle \\ &=\; -g^{\lambda\nu}(k_1 + k_2)^2 \;+\; (k_1 + k_2)^\lambda (k_1 + k_2)^\nu \;-\; k_2^\lambda k_2^\nu\,. \end{aligned} \tag{29}$$

Plugging the first term here into eq. (28), we obtain

$$k_{1\mu} \times \mathcal{M}_{3,a}^{\mu\nu} \;=\; +i g^2 f^{abc} \big(t^c\big)^j_{\;i} \times \bar{v}(p_2)\gamma^\nu u(p_1), \tag{30}$$

which precisely cancels the contributions of the first and the second diagrams, *cf.* eq. (27).

For the second term in eq. (29) we have

$$k_{1\mu}\times\mathcal{M}^{\mu\nu}_{3,b} \;=\; -ig^2 f^{abc}\big(t^c\big)^j_{\ i}\times\frac{(k_1+k_2)^\nu}{(k_1+k_2)^2}\times\bar v(\not k_1+\not k_2)u \tag{31}$$

where the last factor vanishes. Indeed, $k_1+k_2=p_1+p_2$, hence for the on-shell quark and antiquark

$$\bar v(p_2)(\not k_1+\not k_2)u(p_1) \;=\; \bar v(p_2)(\not p_2+m)u(p_1) \;+\; \bar v(p_2)(\not p_1-m)u(p_1) \;=\; 0\;+\;0. \tag{32}$$

But the third term's contribution does not vanish, which breaks the Ward identity for the net QCD amplitude:

$$k_{1\mu}\times\mathcal{M}^{\mu\nu}_{\rm net} \;=\; k_{1\mu}\times\mathcal{M}^{\mu\nu}_{3,c} \;=\; +ig^2 f^{abc}\big(t^c\big)^j_{\ i}\times v\not k_2 u\times\frac{1}{(k_1+k_2)^2}\times k_2^\nu \;\neq\; 0. \tag{33}$$

However, the net violation of the identity is proportional to the k_2^ν factor. Therefore, when we contract the amplitude $\mathcal{M}^{\mu\nu}_{\rm net}$ with the polarization vector of the second gluon, we obtain

$$k_{1\mu}\times\mathcal{M}^{\mu\nu}_{\rm net}e^*_{2\nu} \;=\; [\cdots]\times(k_2e^*_2), \tag{34}$$

which vanishes is the second gluon is transversely polarized! This agrees with the weakened Ward identity of QCD: Amplitudes involving one longitudinal gluon vanish if all the other gluons are transverse, but if two (or more) gluons are longitudinal, the amplitude does not have to vanish. Instead, such amplitudes are related to the amplitudes involving ghosts and antighosts.

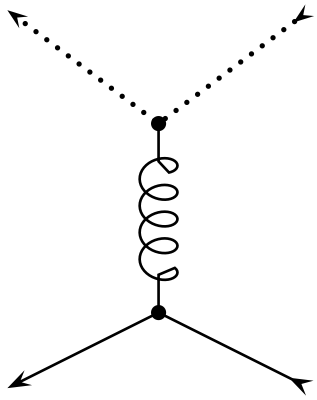
Indeed, consider the annihilation amplitude of two quarks into two longitudinal gluons, $\mathcal{M}(q\bar q\to g_Lg_L)$. In light of eq. (33),

$$\mathcal{M}(q\bar q\to g_Lg_L) \;=\; \frac{ig^2}{s}\,f^{abc}\big(t^c\big)^j_{\ i}\times v(p_2)\not k_2 u(p_1) \tag{35}$$

where $s=(k_1+k_2)^2=(p_1+p_2)^2$ is the center-of-mass energy.⋆ Let's compare this amplitude to the annihilation of the same quark and the same antiquark into a ghost and antighost. At the

⋆ To be precise, in Minkowski space, there two possible longitudinal polarizations for a gluon moving in the direction $\mathbf{n}$, namely $e^\mu_\pm=(1,\pm\mathbf{n})/\sqrt{2}$. The non-vanishing amplitude (35) is for one gluon having longitudinal polarization $L+$ (*i.e.*, parallel to the k^μ) while the other gluon has $L-$. For other combinations of longitudinal polarizations — both $L+$ or both $L-$ — the amplitude vanishes.

tree level, there is only one diagram for the later process,



thus

$$i\mathcal{M}_{\rm tree}(q+\bar{q}\to {\rm gh}+\overline{\rm gh}) \;=\; \bar{v}(p_2)(-ig\gamma^\lambda)u(p_1)\times (t^c)^j{}_i\times \frac{-ig_{\lambda\nu}}{s}\times -gf^{abc}k_2^\nu\,. \qquad (36)$$

By inspection,

$$\mathcal{M}(q+\bar{q}\to {\rm gh}+\overline{\rm gh}) \;=\; \mathcal{M}(q\bar{q}\to g_L+g_L). \qquad (37)$$

In the next set of notes we shall learn that such relations stems from the BRST symmetry, but right now we may use eq. (36) to understand how the physical cross-sections work in QCD.

The ghosts violate the spin-statistics theorem, so we must give up one one of its assumptions: relativity, positive particle energies, or the positive norm in the Hilbert space. The correct choice is to give up on the norm positivity in the extended Hilbert space including both physical and unphysical quanta — only the physical states must have positive norms, while the norm of the ghosts states comes out negative. Consequently, the cross-section for the annihilation-into-ghosts process comes out negative:

$$\frac{d\sigma}{d\Omega} \;=\; -\frac{|\mathcal{M}|^2}{64\pi^2 s}\,. \qquad (38)$$

By themselves, the negative cross-sections are impossible, but they make sense in the context of net unpolarized cross-section where the final states could be either gluons or ghosts,

$$\frac{d\sigma(q+\bar{q}\to\cdots)}{d\Omega} \;=\; \frac{d\sigma(q+\bar{q}\to g_T+g_T)}{d\Omega} + \frac{d\sigma(q+\bar{q}\to g_L+g_L)}{d\Omega} + \frac{d\sigma(q+\bar{q}\to {\rm gh}+\overline{\rm gh})}{d\Omega}\,. \qquad (39)$$

Thanks to eq. (37), the negative cross-section for the annihilation into ghosts precisely cancels the positive cross-section for the annihilation into longitudinal gluons. Thus, the un-physical processes cancel each other, and we are left with only the physical annihilation into the transverse gluons,

$$\frac{d\sigma(q+\bar{q}\to g+g\,\mathrm{or}\,\mathrm{gh}+\overline{\mathrm{gh}})}{d\Omega} = \frac{d\sigma(q+\bar{q}\to g_T+g_T\,\mathrm{only})}{d\Omega}\,. \tag{40}$$

Note: this relation is important for the unitarity of QCD.