Problem **1**(a):

In $N \times N$ matrix form, the local $SU(N)$ symmetry acts on the adjoint matter field $\Phi(x)$ and the gauge field $\mathcal{A}_\mu(x)$ according to

$$\Phi'(x) \;=\; U(x)\Phi(x)U^\dagger(x), \qquad \mathcal{A}'_\mu(x) \;=\; U(x)\mathcal{A}_\mu(x)U^\dagger(x) \;+\; i\partial_\mu U(x)\,U^\dagger(x). \tag{S.1}$$

Consequently, the covariant derivatives (3.5) become

$$D_\mu\Phi(x) \;\to\; D'_\mu\Phi'(x) \;=\; \partial_\mu\Phi(x) \;+\; i\left[\mathcal{A}'_\mu(x),\Phi'(x)\right] \tag{S.2}$$

where the first term on the RHS expands to

$$\begin{aligned}
\partial_\mu\Phi' \;&=\; \partial_\mu\bigl(U\Phi U^\dagger\bigr)\\
&=\; (\partial_\mu U)\Phi U^\dagger \;+\; U(\partial_\mu\Phi)U^\dagger \;+\; U\Phi\bigl(\partial_\mu U^\dagger\bigr)\\
&=\; UU^\dagger(\partial_\mu U)\Phi U^\dagger \;+\; U(\partial_\mu\Phi)U^\dagger \;-\; U\Phi U^\dagger(\partial_\mu U)U^\dagger\\
&=\; U\Bigl((U^\dagger\partial_\mu U)\Phi \;+\; \partial_\mu\Phi \;-\; \Phi(U^\dagger\partial_\mu U)\Bigr)U^\dagger\\
&=\; U\Bigl(\partial_\mu\Phi \;+\; \bigl[\Phi,U^\dagger\partial_\mu U\bigr]\Bigr)U^\dagger
\end{aligned} \tag{S.3}$$

while the second term expands to

$$\begin{aligned}
\bigl[\mathcal{A}'_\mu,\Phi'\bigr] \;&=\; \bigl[U\mathcal{A}_\mu U^\dagger, U\Phi U^\dagger\bigr] \;+\; \bigl[i(\partial_\mu U)U^\dagger, U\Phi U^\dagger\bigr]\\
&=\; U\Bigl(\bigl[\mathcal{A}_\mu,\Phi\bigr] \;+\; \bigl[iU^\dagger\partial_\mu U,\Phi\bigr]\Bigr)U^\dagger
\end{aligned} \tag{S.4}$$

Combining the two expansions, we arrive at

$$D'_\mu\Phi' \;=\; U\Bigl(\partial_\mu\Phi + \cancel{\bigl[\Phi,U^\dagger\partial_\mu U\bigr]} \;+\; i\bigl[\mathcal{A}_\mu,\Phi\bigr] \;-\; \cancel{\bigl[iU^\dagger\partial_\mu U,\Phi\bigr]}\Bigr)U^\dagger \;=\; U\bigl(D_\mu\Phi\bigr)U^\dagger. \tag{S.5}$$

Thus, the $D_\mu\Phi(x)$ matrix transforms exactly like the $\Phi(x)$ matrix itself, which makes the D_μ derivative (3.4) covariant. $\mathcal{Q.E.D.}$

Problem $\mathbf{1}$(b):

Let's start with the second line of eq. (3.6). In the matrix form, the adjoint multiplet Φ is a matrix, the fundamental multiplet Ψ is a column vector, and their matrix product $\Phi\Psi$ is also a column vector. The covariant derivatives acts on these matrices and vectors as

$$D_\mu\Phi \;=\; \partial_\mu\Phi \;+\; i\big[\mathcal{A}_\mu,\Phi\big], \qquad D_\mu\Psi \;=\; \partial_\mu\Psi \;+\; i\mathcal{A}_\mu\Psi, \tag{S.6}$$

while

$$\begin{aligned} D_\mu(\Phi\Psi) &\stackrel{\text{def}}{=} \partial_\mu(\Phi\Psi) \;+\; i\mathcal{A}_\mu(\Phi\Psi) \\ &= (\partial_\mu\Phi)\Psi \;+\; \Phi(\partial_\mu\Psi) \;+\; i\big[\mathcal{A}_\mu,\Phi\big]\Psi \;+\; i\Phi\mathcal{A}_\mu\Psi \\ &= (D_\mu\Phi)\Psi \;+\; \Phi(D_\mu\Psi). \end{aligned} \tag{S.7}$$

Likewise, on the third line of eq. (3.6), Ξ is a matrix, $\Psi^\dagger$ is a row vector, and their matrix product $\Psi^\dagger\Xi$ is also a row vector. Therefore

$$D_\mu\Psi^\dagger \;=\; \partial_\mu\Psi^\dagger \;-\; i\Psi^\dagger\mathcal{A}_\mu\,, \qquad D_\mu\Xi \;=\; \partial_\mu\Xi \;+\; i\big[\mathcal{A}_\mu,\Xi\big], \tag{S.8}$$

while

$$\begin{aligned} D_\mu(\Psi^\dagger\Xi) &\stackrel{\text{def}}{=} \partial_\mu(\Psi^\dagger\Xi) \;-\; i(\Psi^\dagger\Xi)\mathcal{A}_\mu \\ &= (\partial_\mu\Psi^\dagger)\Phi \;+\; \Psi^\dagger(\partial_\mu\Xi) \;-\; i\Psi^\dagger\mathcal{A}_\mu\Xi \;+\; \Psi^\dagger\big[\Xi,\mathcal{A}_\mu\big] \\ &= (D_\mu\Psi^\dagger)\Xi \;+\; \Psi^\dagger(D_\mu\Xi). \end{aligned} \tag{S.9}$$

Finally, on the first line of eq. (3.6), Φ and Ξ are both $N\times N$ matrices and their product $\Phi\Xi$ is also a matrix. Consequently,

$$\begin{aligned} D_\mu(\Phi\Xi) &\stackrel{\text{def}}{=} \partial_\mu(\Phi\Xi) \;+\; i\big[\mathcal{A}_\mu,\Phi\Xi\big] \\ &= (\partial_\mu\Phi)\Xi \;+\; \Phi(\partial_\mu\Xi) \;+\; i\big[\mathcal{A}_\mu,\Phi\big]\Xi \;+\; i\Phi\big[\mathcal{A}_\mu,\Xi\big] \\ &= (D_\mu\Phi)\Xi \;+\; \Phi(D_\mu\Xi). \end{aligned} \tag{S.10}$$

$\mathcal{Q.E.D.}$

Problem **1**(c):

For the adjoint multiplet of fields $\Phi(x)$,

$$\begin{aligned} D_\mu D_\nu \Phi &= \partial_\mu(D_\nu \Phi) + i\big[\mathcal{A}_\mu, D_\nu \Phi\big] \\ &= \partial_\mu\Big(\partial_\nu \Phi + i\big[\mathcal{A}_\nu, \Phi\big]\Big) + i\Big[\mathcal{A}_\mu, \Big(\partial_\nu \Phi + i\big[\mathcal{A}_\nu, \Phi\big]\Big)\Big] \\ &= \partial_\mu \partial_\nu \Phi + i\big[\partial_\mu \mathcal{A}_\nu, \Phi\big] + i\big[\mathcal{A}_\nu, \partial_\mu \Phi\big] + i\big[\mathcal{A}_\mu, \partial_\nu \Phi\big] - \big[\mathcal{A}_\mu, \big[\mathcal{A}_\nu, \Phi\big]\big] \end{aligned} \tag{S.11}$$

where the two blue terms are *not* symmetric in indices $\mu \leftrightarrow \nu$. Consequently,

$$\begin{aligned} \big[D_\mu, D_\nu\big]\Phi &= i\big[\partial_\mu \mathcal{A}_\nu, \Phi\big] - \big[\mathcal{A}_\mu, \big[\mathcal{A}_\nu, \Phi\big]\big] - i\big[\partial_\nu \mathcal{A}_\mu, \Phi\big] + \big[\mathcal{A}_\nu, \big[\mathcal{A}_\mu, \Phi\big]\big] \\ &= i\big[\partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu, \Phi\big] - \big[\big[\mathcal{A}_\mu, \mathcal{A}_\nu\big], \Phi\big] \end{aligned} \tag{S.12}$$

where the second term follows from the Jacobi identity for the matrix commutator:

$$\begin{aligned} -\big[\mathcal{A}_\mu, \big[\mathcal{A}_\nu, \Phi\big]\big] + \big[\mathcal{A}_\nu, \big[\mathcal{A}_\mu, \Phi\big]\big] &= +\big[\mathcal{A}_\mu, \big[\Phi, \mathcal{A}_\nu\big]\big] + \big[\mathcal{A}_\nu, \big[\mathcal{A}_\mu, \Phi\big]\big] \\ &= -\big[\Phi, \big[\mathcal{A}_\nu, \mathcal{A}_\mu\big]\big] = -\big[\big[\mathcal{A}_\mu, \mathcal{A}_\nu\big], \Phi\big]. \end{aligned} \tag{S.13}$$

Altogether, we have

$$\big[D_\mu, D_\nu\big]\Phi = i\big[\big(\partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + i[\mathcal{A}_\mu, \mathcal{A}_\nu]\big), \Phi\big] \equiv i\big[\mathcal{F}_{\mu\nu}, \Phi\big] = ig\big[F_{\mu\nu}, \Phi\big] \tag{S.14}$$

where the second equality follows from the definition of the non-abelian $\mathcal{F}_{\mu\nu}$ and the third equality from $\mathcal{F}_{\mu\nu} = gF_{\mu\nu}$. Finally, in components

$$ig\big[F_{\mu\nu}, \Phi\big] = igF^b_{\mu\nu}\Phi^c \times \left[\frac{\lambda^b}{2}, \frac{\lambda^c}{2}\right] = igF^b_{\mu\nu}\Phi^c \times if^{bca}\,\frac{\lambda^c}{2} \tag{S.15}$$

and hence $[D_\mu, D_\nu]\Phi^a = -gf^{abc}F^b_{\mu\nu}\Phi^c$.

Problem **1**(d):

In matrix notations, the non-abelian gauge symmetries act on vector potentials $\mathcal{A}_\mu(x)$ according to

$$\mathcal{A}'_\mu(x) \;=\; U(x)\mathcal{A}_\mu(x)U^\dagger(x) \;+\; i\partial_\mu U(x)\,U^\dagger(x). \tag{S.16}$$

Taking

$$\mathcal{F}_{\mu\nu}(x) \;\stackrel{\rm def}{=}\; \partial_\mu\mathcal{A}_\nu(x) \;-\; \partial_\nu\mathcal{A}_\mu(x) \;+\; i\,[\mathcal{A}_\mu(x),\mathcal{A}_\nu(x)] \tag{S.17}$$

as the definition of the tension fields $\mathcal{F}_{\mu\nu}(x)$, we then have

$$\mathcal{F}'_{\mu\nu}(x) \;=\; \partial_\mu\mathcal{A}'_\nu(x) \;-\; \partial_\nu\mathcal{A}'_\mu(x) \;+\; i\left[\mathcal{A}'_\mu(x),\mathcal{A}'_\nu(x)\right], \tag{S.18}$$

whatever that evaluates to. Specifically, the first term here evaluates to

$$\begin{aligned}\partial_\mu\mathcal{A}'_\nu \;&=\; \partial_\mu\left(U\mathcal{A}_\nu U^\dagger \;+\; i(\partial_\nu U)U^\dagger\right)\\ &=\; U(\partial_\mu\mathcal{A}_\nu)U^\dagger \;+\; \left[(\partial_\mu U)U^\dagger, U\mathcal{A}_\nu U^\dagger\right] \;+\; i(\partial_\mu\partial_\nu U)U^\dagger \;-\; i(\partial_\nu U)U^\dagger\times(\partial_\mu U)U^\dagger\end{aligned} \tag{S.19}$$

where the second equality follows from

$$\partial_\mu\left(U\mathcal{A}_\nu U^\dagger\right) \;=\; U(\partial_\mu\mathcal{A}_\nu)U^\dagger \;+\; \left[(\partial_\mu U)U^\dagger, U\mathcal{A}_\nu U^\dagger\right] \tag{S.20}$$

— *cf.* similar formula (S.3) — and

$$\partial_\mu\left((\partial_\nu U)U^\dagger\right) \;=\; (\partial_\mu\partial_\nu U)U^\dagger \;+\; (\partial_\nu U)(\partial_\mu U^\dagger) \;=\; (\partial_\mu\partial_\nu U)U^\dagger \;-\; (\partial_\nu U)U^\dagger(\partial_\mu U)U^\dagger. \tag{S.21}$$

Likewise

$$\partial_\nu\mathcal{A}'_\mu \;=\; U(\partial_\nu\mathcal{A}_\mu)U^\dagger \;+\; \left[(\partial_\nu U)U^\dagger, U\mathcal{A}_\mu U^\dagger\right] \;+\; i(\partial_\nu\partial_\mu U)U^\dagger \;-\; i(\partial_\mu U)U^\dagger\times(\partial_\nu U)U^\dagger \tag{S.22}$$

and hence

$$\begin{aligned}\partial_\mu\mathcal{A}'_\nu \;-\; \partial_\nu\mathcal{A}'_\mu \;=\; U\big(\partial_\mu\mathcal{A}_\nu \;-\; \partial_\nu\mathcal{A}_\mu\big)U^\dagger \;+\; \left[(\partial_\mu U)U^\dagger, U\mathcal{A}_\nu U^\dagger\right] \;-\; \left[(\partial_\nu U)U^\dagger, U\mathcal{A}_\mu U^\dagger\right]\\ +\; 0 \;+\; i\left[(\partial_\mu U)U^\dagger, (\partial_\nu U)U^\dagger\right].\end{aligned} \tag{S.23}$$

At the same time, the commutator part of the tension field transforms into

$$\begin{aligned} i\left[\mathcal{A}'_\mu, \mathcal{A}'_\nu\right] &= i\left[\Big(U\mathcal{A}_\mu U^\dagger + i(\partial_\mu U)U^\dagger\Big), \Big(U\mathcal{A}_\nu U^\dagger + i(\partial_\nu U)U^\dagger\Big)\right] \\ &= i\left[U\mathcal{A}_\mu U^\dagger, U\mathcal{A}_\nu U^\dagger\right] - \left[(\partial_\mu U)U^\dagger, U\mathcal{A}_\nu U^\dagger\right] \\ &\quad - \left[U\mathcal{A}_\mu U^\dagger, (\partial_\nu U)U^\dagger\right] - i\left[(\partial_\mu U)U^\dagger, (\partial_\nu U)U^\dagger\right], \end{aligned} \tag{S.24}$$

Combining eqs. (S.23) and (S.24) leads to massive cancellation of 6 out of terms on the combined right hand side. Only the first terms on right hand sides of (S.23) and (S.24) survive the cancellation, thus

$$\begin{aligned} \partial_\mu\mathcal{A}'_\nu - \partial_\nu\mathcal{A}'_\mu + i\left[\mathcal{A}'_\mu, \mathcal{A}'_\nu\right] &= U\big(\partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu\big)U^\dagger + i\left[U\mathcal{A}_\mu U^\dagger, U\mathcal{A}_\nu U^\dagger\right] \\ &= U\Big(\partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu]\Big)U^\dagger, \end{aligned} \tag{S.25}$$

or in other words,

$$\mathcal{F}'_{\mu\nu}(x) = U(x)\mathcal{F}_{\mu\nu}(x)U^\dagger(x). \tag{S.26}$$

$\mathcal{Q}.\mathcal{E}.\mathcal{D}.$

Problem **1**(e):

There are two ways to prove the non-abelian Bianchi identity: using part (b) and (c) and the Jacobi identity for commutators, or the hard calculation based directly on eq. (S.17). Let me start with the easier proof.

In part (b) we have proved the Leibniz rule for covariant derivatives of a matrix product of two adjoint fields $\Phi(x)$ and $\Xi(x)$. Clearly, the same Leibniz rule also applies to the commutator $[\Phi, Xi]$:

$$D_\mu[\Phi, \Xi] = \big[D_\mu\Phi, \Xi\big] + \big[\Phi, D_\mu\Xi\big]. \tag{S.27}$$

In particular, for $\Phi = \mathcal{F}_{\mu\nu}$ and arbitrary Ξ, we have

$$D_\lambda([\mathcal{F}_{\mu\nu}, \Xi]) = [D_\lambda\mathcal{F}_{\mu\nu}, \Xi] + [\mathcal{F}_{\mu\nu}, D_\lambda\Xi]. \tag{S.28}$$

On the other hand, in part (c) we saw that for any adjoint field $\Xi(x)$, $[\mathcal{F}_{\mu\nu}, \Xi] = -i[D_\mu, D_\nu]\Xi$. Likewise, for the $D_\lambda\Xi(x)$ we also have $[\mathcal{F}_{\mu\nu}, D_\lambda\Xi] = -i[D_\mu, D_\nu]D_\lambda\Xi$. Consequently, eq. (S.28) becomes

$$-iD_\lambda[D_\mu,D_\nu]\Xi \;=\; [D_\lambda\mathcal{F}_{\mu\nu},\Xi] \;-\; i[D_\mu,D_\nu]D_\lambda\Xi \tag{S.29}$$

and hence

$$i[D_\lambda\mathcal{F}_{\mu\nu},\Xi] \;=\; [D_\lambda,[D_\mu,D_\nu]]\Xi. \tag{S.30}$$

Now, let's sum 3 such formulae, one for each cyclic permutations of the indices λ,μ,ν. On the left hand side, this gives us

$$i\left[\big(D_\lambda\mathcal{F}_{\mu\nu}+D_\mu\mathcal{F}_{\nu\lambda}+D_\nu\mathcal{F}_{\lambda\mu}\big),\Xi\right] \;=\; \cdots$$

while on the right hand side we obtain

$$\cdots \;=\; \Big([D_\lambda,[D_\mu,D_\nu]] \;+\; [D_\mu,[D_\nu,D_\lambda]] \;+\; [D_\nu,[D_\lambda,D_\mu]]\Big)\Xi \;=\; 0 \tag{S.31}$$

due to Jacobi identity for the double commutators of the three covariant derivatives D_λ, D_μ, and D_ν. Consequently

$$\left[\big(D_\lambda\mathcal{F}_{\mu\nu}+D_\mu\mathcal{F}_{\nu\lambda}+D_\nu\mathcal{F}_{\lambda\mu}\big),\Xi\right] \;=\; 0,$$

and this must be true for any adjoint field $\Xi(x)$. Moreover, for any x,λ,μ,ν, the $N\times N$ matrix

$$D_\lambda\mathcal{F}_{\mu\nu} \;+\; D_\mu\mathcal{F}_{\nu\lambda} \;+\; D_\nu\mathcal{F}_{\lambda\mu}$$

is traceless, and the only way it may commute with all traceless hermitian matrices $\Psi(x)$ is by being zero, thus

$$D_\lambda\mathcal{F}_{\mu\nu} \;+\; D_\mu\mathcal{F}_{\nu\lambda} \;+\; D_\nu\mathcal{F}_{\lambda\mu} \;=\; 0. \tag{S.32}$$

This is my first proof of the non-abelian Bianchi identity.

The second proof of the Bianchi identity follows directly from the definition (S.17) of the non-abelian tension fields and the covariant derivatives (4). Let's spell out $D_\lambda\mathcal{F}_{\mu\nu}$ in detail:

$$\begin{aligned} D_\lambda\mathcal{F}_{\mu\nu} &= \partial_\lambda\mathcal{F}_{\mu\nu} + i[\mathcal{A}_\lambda,\mathcal{F}_{\mu\nu}] \\ &= \partial_\lambda\big(\partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu + i[\mathcal{A}_\mu,\mathcal{A}_\nu]\big) + i\left[\mathcal{A}_\lambda,\big(\partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu + i[\mathcal{A}_\mu,\mathcal{A}_\nu]\big)\right] \\ &= \partial_\lambda\partial_\mu\mathcal{A}_\nu - \partial_\lambda\partial_\nu\mathcal{A}_\mu + i[\partial_\lambda\mathcal{A}_\mu,\mathcal{A}_\nu] + i[\mathcal{A}_\mu,\partial_\lambda\mathcal{A}_\nu] \\ &\quad + i[\mathcal{A}_\lambda,\partial_\mu\mathcal{A}_\nu] - i[\mathcal{A}_\lambda,\partial_\nu\mathcal{A}_\mu] - [\mathcal{A}_\lambda,[\mathcal{A}_\mu,\mathcal{A}_\nu]] \\ &= \Big(\partial_\lambda\partial_\mu\mathcal{A}_\nu - \partial_\lambda\partial_\nu\mathcal{A}_\mu\Big) + i\Big([\partial_\lambda\mathcal{A}_\mu,\mathcal{A}_\nu] - [\partial_\mu\mathcal{A}_\nu,\mathcal{A}_\lambda]\Big) \\ &\quad + i\Big([\mathcal{A}_\mu,\partial_\lambda\mathcal{A}_\nu] - [\mathcal{A}_\lambda,\partial_\nu\mathcal{A}_\mu]\Big) - \Big([\mathcal{A}_\lambda,[\mathcal{A}_\mu,\mathcal{A}_\nu]]\Big). \end{aligned} \tag{S.33}$$

On the bottom two lines here I have grouped terms in () so that after summing over cyclic permutations of the indices λ,μ,ν, we get a zero sum separately for each group. Indeed,

$$\begin{aligned} \Big(\partial_\lambda\partial_\mu\mathcal{A}_\nu - \partial_\lambda\partial_\nu\mathcal{A}_\mu\Big) + \text{cyclic} &= \Big(\partial_\lambda\partial_\mu\mathcal{A}_\nu - \partial_\nu\partial_\lambda\mathcal{A}_\mu\Big) + \text{cyclic} \\ &= 0 \quad \langle\!\langle \text{ by inspection} \rangle\!\rangle, \\ \Big([\partial_\lambda\mathcal{A}_\mu,\mathcal{A}_\nu] - [\partial_\mu\mathcal{A}_\nu,\mathcal{A}_\lambda]\Big) + \text{cyclic} &= 0 \quad \langle\!\langle \text{ by inspection} \rangle\!\rangle, \\ \Big([\mathcal{A}_\mu,\partial_\lambda\mathcal{A}_\nu] - [\mathcal{A}_\lambda,\partial_\nu\mathcal{A}_\mu]\Big) + \text{cyclic} &= 0 \quad \langle\!\langle \text{ by inspection} \rangle\!\rangle, \text{ and} \\ [\mathcal{A}_\lambda,[\mathcal{A}_\mu,\mathcal{A}_\nu]] + \text{cyclic} &= 0 \quad \langle\!\langle \text{ by Jacobi identity} \rangle\!\rangle. \end{aligned} \tag{S.34}$$

Therefore,

$$D_\lambda\mathcal{F}_{\mu\nu} + \text{cyclic} \equiv D_\lambda\mathcal{F}_{\mu\nu} + D_\mu\mathcal{F}_{\nu\lambda} + D_\nu\mathcal{F}_{\lambda\mu} = 0. \tag{S.35}$$

Problem **1**(f):

The Euler–Lagrange field equations follow from requiring zero first variation of the action $S = \int\mathcal{L}$ under infinitesimal variation of the independent fields $\mathcal{A}_\mu(x)$. Let's start by calculating the variation of the tension fields $\mathcal{F}_{\mu\nu}$:

$$\begin{aligned} \delta\mathcal{F}_{\mu\nu} &\equiv \delta\Big(\partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu + i[\mathcal{A}_\mu,\mathcal{A}_\nu]\Big) \\ &= \partial_\mu\delta\mathcal{A}_\nu - \partial_\nu\delta\mathcal{A}_\mu + i\big[\delta\mathcal{A}_\mu,\mathcal{A}_\nu\big] + i\big[\mathcal{A}_\mu,\delta\mathcal{A}_\nu\big] \\ &= \Big(\partial_\mu\delta\mathcal{A}_\nu + i\big[\mathcal{A}_\mu,\delta\mathcal{A}_\nu\big]\Big) - \Big(\partial_\nu\delta\mathcal{A}_\mu + i\big[\mathcal{A}_\nu,\delta\mathcal{A}_\mu\big]\Big) \\ &= D_\mu\delta\mathcal{A}_\nu - D_\nu\delta\mathcal{A}_\mu \end{aligned} \tag{S.36}$$

where we treat the matrix-valued variations $\delta\mathcal{A}_\nu(x)$ as adjoint fields so their covariant derivative work according to eq. (3.4), $D_\mu\Delta\mathcal{A}_\nu \equiv \partial_\mu\delta\mathcal{A}_\nu + i[\mathcal{A}_\mu,\delta\mathcal{A}_\nu]$ and likewise for the $D_\nu\delta\mathcal{A}_\mu$. In light of eq. (S.36), the trace in the Yang–Mills Lagrangian (3.9) varies by

$$\begin{aligned}\delta\,\mathrm{tr}\Big(\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}\Big) &= 2\,\mathrm{tr}\Big(\mathcal{F}^{\mu\nu}\delta\mathcal{F}_{\mu\nu}\Big) = 2\,\mathrm{tr}\Big(\mathcal{F}^{\mu\nu}\big(D_\mu\delta\mathcal{A}_\nu - D_\nu\delta\mathcal{A}_\mu\big)\Big)\\ &= 4\,\mathrm{tr}\Big(\mathcal{F}^{\mu\nu}D_\mu\delta\mathcal{A}_\nu\Big) \qquad \langle\!\langle\,\text{since } \mathcal{F}^{\mu\nu} = -\mathcal{F}^{\nu\mu}\,\rangle\!\rangle\\ &= -4\,\mathrm{tr}\Big(\big(D_\mu\mathcal{F}^{\mu\nu}\big)\delta\mathcal{A}_\nu\Big) + 4\partial_\mu\,\mathrm{tr}\Big(\mathcal{F}^{\mu\nu}\delta\mathcal{A}_\nu\Big)\end{aligned} \tag{S.37}$$

where the last equality follows from the Leibniz rule for the two adjoint fields $\Phi = \mathcal{F}^{\mu\nu}$ and $\Xi = \delta\mathcal{A}_\nu$:

$$\mathrm{tr}\big((D_\mu\Phi)\Xi\big) + \mathrm{tr}\big(\Phi(D_\mu\Xi)\big) = \mathrm{tr}\big(D_\mu(\Phi\Xi)\big) = \mathrm{tr}\big(\partial_\mu(\Phi\Xi)\big) + i\,\mathrm{tr}\big([\mathcal{A}_\mu,\Phi\Xi]\big) = \partial_\mu\,\mathrm{tr}\big(\Phi\xi\big) + 0 \tag{S.38}$$

(trace of a commutator is zero). Thus

$$\delta\mathcal{L}_{\mathrm{YM}} = \frac{2}{g^2}\,\mathrm{tr}\big((D_\mu\mathcal{F}^{\mu\nu})\delta\mathcal{A}_\nu\big) - \text{a total divergence} \tag{S.39}$$

so the net Yang-Mills action varies by

$$\delta S = \frac{2}{g^2}\int d^4x\;\mathrm{tr}\big(D_\mu\mathcal{F}^{\mu\nu}(x)\delta\mathcal{A}_\nu(x)\big) = \frac{1}{g^2}\int d^4x\sum_a D_\mu\mathcal{F}^{a\mu\nu}(x)\times\delta\mathcal{A}^a_\nu(x). \tag{S.40}$$

To make this variation vanish for any infinitesimal $\delta\mathcal{A}^a_\nu(x)$ we need $D_\mu\mathcal{F}^{a\mu\nu}(x)\equiv 0$.

Problem **2**(a):

In problem **1**(f) we saw that for infinitesimal variations of the gauge fields the YM Lagrangian varies by

$$\delta\mathcal{L}_{\mathrm{YM}} = \frac{1}{g^2}\sum_a D_\mu\mathcal{F}^{a\mu\nu}\times\delta\mathcal{A}^a_\nu + \partial_\mu(\cdots) = \sum_a D_\mu F^{a\mu\nu}\times\delta A^a_\nu + \partial_\mu(\cdots). \tag{S.41}$$

Now let's add the matter Lagrangian $\mathcal{L}_{\mathrm{mat}}(\phi, D\phi)$ for some matter fields in a non-trivial multiplet (or multiplets) of the gauge symmetry. When we vary the gauge fields $A^a_\nu(x)$ while keeping the matter fields $\phi(x)$ fixed, the covariant derivatives $D\phi$ vary due to $igA^a_\nu t^a\phi$ terms in $D_\nu\phi$, which leads to non-trivial variation

$$\delta\mathcal{L}_{\rm mat} \;=\; \sum_a \frac{\partial\mathcal{L}_{\rm mat}}{\partial A^a_\nu}\times\delta A^a_\nu \;\equiv\; -\sum_a J^{a\nu}\times\delta A^a_\nu\,. \tag{S.42}$$

Altogether, the net action of the theory varies by

$$\delta S \;=\; \int\! d^4x \sum_a \Big(D_\mu F^{a\mu\nu}(x)\;-\;J^{a\nu}(x)\Big)\times\delta A^a_\nu(x). \tag{S.43}$$

Requiring this variation to vanish for any $\delta A^a_\nu(x)$ leads to the field equations

$$D_\mu F^{a\mu\nu} \;=\; J^{a\nu}, \tag{S.44}$$

or in matrix notations $D_\mu F^{\mu\nu} = J^\nu$. This is the non-abelian version of the Maxwell equations $\partial_\mu F^{\mu\nu} = J^\nu$.

In the abelian EM theory, the equations $\partial_\mu F^{\mu\nu} = J^\nu$ require the electric current to be conserved, $\partial_\nu J^\nu = \partial_\nu\partial_\mu F^{\mu\nu} = 0$ since $F^{\mu\nu} = -F^{\nu\mu}$ and the derivatives commute with each other. The non-abelian tension fields $F^{\mu\nu}$ are also antisymmetric in $\mu\leftrightarrow\nu$, but the covariant derivatives do not commute, $D_\mu D_\nu \neq D_\nu D_\mu$. Therefore,

$$D_\nu J^\nu \;=\; D_\nu D_\mu F^{\mu\nu} \;=\; \tfrac{1}{2}[D_\mu,D_\nu]\mathcal{F}^{\mu nu} \;=\; \tfrac{i}{2}[\mathcal{F}_{\mu\nu},\mathcal{F}^{\mu\nu}] \tag{S.45}$$

where the last equality works exactly as in problem **1**(c) — the $\mathcal{F}^{a\mu\nu}$ fields form an adjoint multiplet of fields, and for any such multiplet packed into an hermitian $N\times N$ matrix Φ, $[D_\mu,D_\nu]\Phi \;=\; i[\mathcal{F}_{\mu\nu},\Phi]$. However, unlike a generic matrix Φ which may commute or not commute with the $\mathcal{F}_{\mu\nu}$, for any μ and ν the $\mathcal{F}^{\mu\nu}$ matrix always commutes with *itself*. Thus,

$$[\mathcal{F}_{\mu\nu},\mathcal{F}^{\mu\nu}] \;=\; 0 \quad \text{even before summing over } \mu \text{ and } \nu. \tag{S.46}$$

Of course, after the summing over μ and ν we still have a zero, thus $D_\nu D_\mu\mathcal{F}^{\mu\nu}(x)\equiv 0$.

Thus, consistency of the field equations (S.44) for the gauge fields requires the non-abelian currents $J^{a\mu}$ to be *covariantly conserved:*

$$D_\nu J^\nu \;=\; D_\mu D_\nu F^{\mu\nu} \;=\; 0, \tag{S.47}$$

or in components

$$\partial_\nu J^{a\nu} \;-\; f^{abc}\mathcal{A}^b_\nu J^{c\nu} \;=\; 0. \tag{S.48}$$

Note: because of the covariantizing term here, we do not have conserved net charges; alas,

$$\frac{d}{dt}\int d^3\mathbf{x}\, J^{a0}(\mathbf{x},t) \;\neq\; 0. \tag{S.49}$$

Problem **2**(b):

The currents J^a_μ come from the covariant derivatives in the Lagrangian for the scalar fields

$$\mathcal{L}_{\rm mat} \;=\; D_\mu\Psi^\dagger D^\mu\Psi \;-\; V(\Psi^d agger\Psi). \tag{S.50}$$

Expanding the covariant derivatives $D_\mu\Psi^\dagger$ and $D^\mu\Psi$ in components of Ψ_i, Ψ^{*i}, and A^a_μ, we obtain

$$D_\mu\Psi^{*i} \;=\; \partial_\mu\Psi^{*i} \;-\; \frac{ig}{2}\,A^a_\mu\Psi^{*j}\big(\lambda^a\big)_j^{\,i}, \qquad D^\mu\Psi_i \;=\; \partial^\mu\Psi_i \;+\; \frac{ig}{2}\,A^{a\mu}\big(\lambda^a\big)_i^{\,j}\Psi_j\,, \tag{S.51}$$

and hence

$$\begin{aligned}
J^a_\mu \;&=\; -\frac{\partial\mathcal{L}_{\rm mat}}{\partial A^a_\mu} \;=\; -D^\nu\Psi^{*i}\times\frac{\partial D_\nu\Psi_i}{\partial A^a_\mu} \;-\; \frac{\partial D_\nu\Psi^{*i}}{\partial A^a_\nu}\times D^\mu\Psi_i \\
&=\; \frac{-ig}{2}\Big(D^\nu\Psi^{*i}\times\delta^\mu_\nu\big(\lambda^a\big)_i^{\,j}\Psi_j \;-\; \delta^\mu_\nu\Psi^{*j}\big(\lambda^a\big)_j^{\,i}\times D^\nu\Psi_i\Big) \\
&=\; \frac{-ig}{2}\Big(D^\mu\Psi^\dagger\lambda^a\Psi \;-\; \Psi^\dagger\lambda^a D^\mu\Psi\Big) \\
&=\; -g\,{\rm Im}\Big(\Psi^\dagger\lambda^a D^\mu\Psi\Big).
\end{aligned} \tag{S.52}$$

To combine these N^2-1 currents into an hermitian traceless matrix $J^\mu = \frac{1}{2}\lambda^a J^{a\mu}$ we need a **lemma:** For any row vector $\Psi^\dagger$ and any column vector Ψ',

$$\sum_a \bigl(\Psi^\dagger\lambda^a\Psi'\bigr)\times\tfrac{1}{2}\lambda^a \;=\; \Psi'\otimes\Psi^\dagger \;-\; \frac{(\Psi^\dagger\Psi')}{N}\times\mathbf{1}_{N\times N} \tag{S.53}$$

where in the first term on the RHS $\Psi'\otimes\Psi^\dagger$ denotes an $N\times N$ matrix $\bigl(\Psi'\otimes\Psi^\dagger\bigr)_i^{\ j} = \Psi'_i\Psi^{*j\,\star}$ while the second term subtracts the trace part of the first term. Applying this lemma to column/row vectors $\Psi^\dagger$ and $D^\mu\Psi$ or $D^\mu\Psi^\dagger$ and Ψ, we obtain

$$J^\mu \;=\; \sum_a \frac{-ig}{2}\Bigl(D^\mu\Psi^\dagger\lambda^a\Psi \;-\; \Psi^\dagger\lambda^a D^\mu\Psi\Bigr)\times\tfrac{1}{2}\lambda^a \;=\; \tilde{J}^\mu \;-\; \frac{\mathrm{tr}(\tilde{J}^\mu)}{N}\times\mathbf{1} \tag{S.54}$$

where $\tilde{J}^\mu$ are hermitian but not traceless matrices

$$\tilde{J}^\mu(x) \;=\; \frac{-ig}{2}\Bigl(\Psi\otimes D^\mu\Psi^\dagger \;-\; D^\mu\Psi\otimes\Psi^\dagger\Bigr). \tag{S.55}$$

Under a local $SU(N)$ symmetry

$$\begin{aligned}\Psi(x) &\to U(x)\Psi(x), \quad D^\mu\Psi(x) \to U(x)D^\mu\Psi(x),\\ \Psi^\dagger(x) &\to \Psi^\dagger(x)U^\dagger(x), \quad D^\mu\Psi^\dagger(x) \to D^\mu\Psi^\dagger(x)U^\dagger(x).\end{aligned} \tag{S.56}$$

Consequently, the tensor product $D^\mu\Psi\otimes\Psi^\dagger$ transforms covariantly,

$$\Bigl(D^\mu\Psi(x)\otimes\Psi^\dagger(x)\Bigr) \;\to\; U(x)\Bigl(D^\mu\Psi(x)\otimes\Psi^\dagger(x)\Bigr)U^\dagger(x) \tag{S.57}$$

— indeed, in components

$$\begin{aligned}\Bigl(D^\mu\Psi(x)\otimes\Psi^\dagger(x)\Bigr)_i^{\ j} &\equiv D^\mu\Psi_i(x)\times\Psi^{*j}(x) \;\to U_i^{\ k}(x)D^\mu\Psi_k(x)\times\Psi^{*\ell}U_\ell^{\dagger\,j}\\ &\equiv \Bigl(U(x)\Bigl(D^\mu\Psi(x)\otimes\Psi^\dagger(x)\Bigr)U^\dagger(x)\Bigr)_i^{\ j}\end{aligned} \tag{S.58}$$

⋆ The tensor product $\Psi'\otimes\Psi^\dagger$ is the finite-matrix analogy of the Hilbert-space operator $|\Psi'\rangle\langle\Psi|$. In contrast, $\Psi^\dagger\Psi'$ is just a number, analogous to the Dirac product $\langle\Psi|\Psi'\rangle$.

— and likewise

$$\Big(\Psi(x)\otimes D^\mu\Psi^\dagger(x)\Big)\ \to\ U(x)\Big(\Psi(x)\otimes D^\mu\Psi^\dagger(x)\Big)U^\dagger(x). \tag{S.59}$$

Therefore, the $\tilde{J}^\mu(x)$ matrices transform in the similar manner,

$$\tilde{J}^\mu(x)\ \to\ U(x)\tilde{J}^\mu(x)U^\dagger(x), \tag{S.60}$$

which in turn makes the current matrices $J^\mu(x)$ transform covariantly according to eq. (3.3):

$$J^\mu(x)\ =\ \tilde{J}^\mu(x)\ -\ \frac{\operatorname{tr}(\tilde{J}^\mu)}{N}\times\mathbf{1}\ \to\ U(x)\tilde{J}^\mu(x)U^\dagger\ -\ \frac{\operatorname{tr}(\tilde{J}^\mu)}{N}\times\mathbf{1}\ =\ U(x)J^\mu(x)U^\dagger(x). \tag{S.61}$$

In other words, the currents $J^{a\mu}(x)$ transform into each other as members of an adjoint multiplet, $\mathcal{Q.E.D.}$

<u>Problem **2**(c)</u>:

First, let's derive the Leibniz rule for the adjoint multiplets of the form $Q^a=\Psi^\dagger\lambda^a\Psi'$:

$$D_\mu\big(\Psi^\dagger\lambda^a\Psi'\big)\ =\ D_\mu\Psi^\dagger\lambda^a\Psi'\ +\ \Psi^\dagger\lambda^a D_\mu\Psi'. \tag{S.62}$$

Proof:

$$\begin{aligned}
D_\mu\big(\Psi^\dagger\lambda^a\Psi'\big)\ &\equiv\ \partial_\mu\big(\Psi^\dagger\lambda^a\Psi'\big)\ -\ gf^{abc}A^b_\mu\big(\Psi^\dagger\lambda^c\Psi'\big)\\
&=\ (\partial_\mu\Psi^\dagger)\lambda^a\Psi'\ +\ \Psi^\dagger\lambda^a(\partial_\mu\Psi')\ -\ gA^b_\mu\,\Psi^\dagger\big(f^{abc}\lambda^c\ =\ -\tfrac{i}{2}[\lambda^a,\lambda^b]\big)\Psi'\\
&=\ \Big(\partial_\mu\Psi^\dagger\ -\ \frac{ig}{2}\,A^b_\mu\Psi^\dagger\lambda^b\Big)\lambda^a\Psi'\ +\ \Psi^\dagger\lambda^a\Big(\partial_\mu\Psi'\ +\ \frac{ig}{2}\,A^b_\mu\lambda^b\Psi'\Big)\\
&=\ D_\mu\Psi^\dagger\lambda^a\Psi'\ +\ \Psi^\dagger\lambda^a D_\mu\Psi'.
\end{aligned} \tag{S.63}$$

Thanks to this lemma, the non-abelian currents (S.52) satisfy

$$\begin{aligned}
D_\mu J^{a\mu}\ &=\ -\frac{ig}{2}D_\mu\Big(\Psi^\dagger\lambda^a D^\mu\Psi\ -\ D^\mu\Psi^\dagger\lambda^a\Psi\Big)\\
&=\ -\frac{ig}{2}\Big(\cancel{D_\mu\Psi^\dagger\lambda^a D^\mu\Psi}\ +\ \Psi^\dagger\lambda^a D_\mu D^\mu\Psi\ -\ D_\mu D^\mu\Psi^\dagger\lambda^a\Psi\ -\ \cancel{D^\mu\Psi^\dagger\lambda^a D_\mu\Psi}\Big)\\
&=\ g\operatorname{Im}\Big(\Psi^\dagger\lambda^a D_\mu D^\mu\Psi\Big).
\end{aligned} \tag{S.64}$$

Now let the scalar fields satisfy their covariant equations of motion

$$D_\mu \frac{\partial\mathcal{L}}{\partial(D_\mu\Psi_i)} \;=\; \frac{\partial\mathcal{L}}{\partial\Psi_i}\,, \qquad D_\mu \frac{\partial\mathcal{L}}{\partial(D_\mu\Psi^{*i})} \;=\; \frac{\partial\mathcal{L}}{\partial\Psi^{*i}}\,. \tag{S.65}$$

For the Lagrangian (3.13) these equations read

$$\begin{aligned} D_\mu D^\mu \Psi^{*i} \;&=\; -\frac{\partial V}{\partial\Psi_i} \;=\; -\Psi^{i*}\times\left(m^2+\frac{\lambda}{2}\Psi^\dagger\Psi\right), \\ D_\mu D^\mu \Psi_i \;&=\; -\frac{\partial V}{\partial\Psi^{*i}} \;=\; -\left(m^2+\frac{\lambda}{2}\Psi^\dagger\Psi\right)\times\Psi_i, \end{aligned} \tag{S.66}$$

so for the fields obeying these equations

$$\Psi^\dagger\lambda^a D_\mu D^\mu\Psi \;=\; -\left(m^2+\frac{\lambda}{2}\Psi^\dagger\Psi\right)\times\Psi^\dagger\lambda^a\Psi \;=\; (\text{real})\times(\text{real}) \tag{S.67}$$

for any hermitian matrix λ^a, and therefore

$$D_\mu J^{a\mu} \;=\; g\,\mathrm{Im}\Big(\Psi^\dagger\lambda^a D_\mu D^\mu\Psi\Big) \;=\; 0. \tag{S.68}$$

Problem **3**(●):

Regardless of the nature of the matter field multiplet (m) from the symmetry group's point of view — it could be a vector, a tensor, a spinor, whatever — let me treat the $\Psi_\alpha(x)$ as components of some big column vector $\Psi(x)$ of length size(m) so I can use matrix notations for the $\left(T^a_{(m)}\right)$ without bothering with indices $\alpha,\beta,\ldots$. Thus instead of $\left(T^a_{(m)}\right)_\alpha^{\ \beta}\Psi_\beta(x)$ I will write simply $T^a_{(m)}\Psi(x)$.

A local symmetry parametrized by infinitesimal $\Lambda^a(x)$ acts on the gauge and matter fields according to

$$\delta\mathcal{A}^a_\mu(x) \;=\; -\partial_\mu\Lambda^a(x) \;-\; f^{abc}\Lambda^b(x)\mathcal{A}^c_\mu(x), \qquad \delta\Psi(x) \;=\; i\Lambda^a(x)T^a_{(m)}\Psi(x). \tag{3.15+16}$$

Consequently, the covariant derivatives

$$D_\mu\Psi(x) \;=\; \partial_\mu\Psi(x) \;+\; i\mathcal{A}^a_\mu(x)T^a_{(m)}\Psi(x). \tag{3.17}$$

change by

$$
\begin{aligned}
\delta\big(D_\mu\Psi(x)\big) \;&=\; D_\mu\delta\Psi(x) \;+\; \big(\delta D_\mu \text{ due to } \delta\mathcal{A}_\mu\big)\Psi(x)\\
&=\; \partial_\mu\delta\Psi(x) \;+\; i\mathcal{A}^a_\mu(x)T^a_{(m)}\delta\Psi(x) \;+\; i\delta\mathcal{A}^a_\mu(x)T^a_{(m)}\Psi(x)\\
&=\; \cancel{i\partial_\mu\Lambda^a(x)T^a_{(m)}\Psi(x)} \;+\; i\Lambda^a(x)T^a_{(m)}\partial_\mu\Psi(x)\\
&\qquad -\; \mathcal{A}^b_\mu T^b_{(m)}\Lambda^a(x)T^a_{(m)}\Psi(x)\\
&\qquad -\; \cancel{i\partial_\mu\Lambda^a(x)T^a_{(m)}\Psi(x)} \;-\; if^{abc}\Lambda^b(x)\mathcal{A}^c_\mu(x)T^a_{(m)}\Psi(x)\\
&\qquad\qquad \text{relabel } a\to c\to b\to a \quad \longrightarrow \; -if^{cab}\Lambda^a(x)\mathcal{A}^b_\mu(x)T^c_{(m)}\Psi(x)\\
&=\; i\Lambda^a(x)T^a_{(m)}\partial_\mu\Psi(x) \;-\; \Lambda^a(x)\mathcal{A}^b_\mu(x)\Big(T^b_{(m)}T^a_{(m)} + if^{cab}T^c_{(m)}\Big)\Psi(x)
\end{aligned}
\tag{S.69}
$$

Now let's simplify the matrix inside the big () in the second term on the bottom line. The matrices $T^a_{(m)}$ represent the Lie algebra of the local symmetry, so they obey the same commutation relations as the generators $\hat{T}^a$,

$$
\big[T^a_{(m)}, T^b_{(m)}\big] \;=\; if^{abc}T^c_{(m)}. \tag{S.70}
$$

Also, f^{abc} is totally antisymmetric so $f^{cab} = +f^{abc}$. Consequently

$$
if^{cab}T^c_{(m)} \;=\; f^{abc}T^c_{(m)} \;=\; T^a_{(m)}T^b_{(m)}\backslash,-T^b_{(m)}T^a_{(m)} \quad\Longrightarrow\quad T^b_{(m)}T^a_{(m)} + if^{cba}T^c_{(m)} \;=\; T^a_{(m)}T^b_{(m)}, \tag{S.71}
$$

and therefore

$$
\begin{aligned}
\delta\big(D_\mu\Psi(x)\big) \;&=\; i\Lambda^a(x)T^a_{(m)}\partial_\mu\Psi(x) \;-\; \Lambda^a(x)\mathcal{A}^b_\mu(x)T^a_{(m)}T^b_{(m)}\Psi(x)\\
&=\; i\lambda^a(x)T^a_{(m)}\Big(\partial_\mu\Psi(x) \;+\; i\mathcal{A}^b_\mu(x)T^b_{(m)}\Psi(x)\Big)\\
&\equiv\; i\lambda^a(x)T^a_{(m)}D_\mu\Psi(x).
\end{aligned}
\tag{S.72}
$$

Thus the $D_\mu\Psi(x)$ transforms under the infinitesimal local symmetries exactly like the $\Psi(x)$ itself, which makes the derivative D_μ covariant. $\mathcal{Q}.\mathcal{E}.\mathcal{D}.$

Problem $\mathbf{3}(\star)$:

Under finite gauge symmetries $\mathcal{G}(x) \in G$, the field multiplet $\Psi(x)$ in representation (m) transform to

$$\Psi(x) \;\to\; R^{(m)}(\mathcal{G}(x))\,\Psi(x) \tag{S.73}$$

— or in components

$$\Psi_\alpha(x) \;\to\; \Big(R^{(m)}(\mathcal{G}(x))\Big)_\alpha^{\;\beta}\Psi_\beta(x) \tag{S.74}$$

— where $R^{(m)}(\mathcal{G})$ is the unitary matrix representing group element $\mathcal{G}$ in multiplet (m); for $\mathcal{G} = \exp(-i\Theta^a\hat{T}^a)$ for some finite real parameters Θ^a,

$$\Big(R^{(m)}(\mathcal{G}(x))\Big)_\alpha^{\;\beta} \;=\; \exp\big(-i\Theta^a T^a_{(m)}\big)_\alpha^{\;\beta}\,. \tag{S.75}$$

To simplify the form of covariant derivatives $D_\mu\Psi(x)$, let me combine the gauge fields $\mathcal{A}^a_\mu(x)$ with matrices $T^a_{(m)}$ representing the Lie algebra of G in multiplet (m) into a size$(m)\times$ size(m) matrix-valued connection

$$\mathcal{A}^{(m)}_\mu(x) \;\stackrel{\text{def}}{=}\; \sum_a \mathcal{A}^a_\mu(x)\times\Big(T^a_{(m)}\Big) \quad\Longrightarrow\quad D_\mu\Psi(x) \;=\; \partial_\mu\Phi(x) \;+\; i\mathcal{A}^{(m)}_\mu(x)\Psi(x). \tag{S.76}$$

To make sure these derivatives transform covariantly under finite local symmetries (S.73), the matrix-valued connection $\mathcal{A}^{(m)}_\mu(x)$ should transform as

$$\mathcal{A}^{(m)}_\mu(x) \;\to\; R^{(m)}(\mathcal{G}(x))\times\mathcal{A}^{(m)}_\mu(x)\times\Big(R^{(m)}(\mathcal{G}(x))\Big)^{-1} \;+\; i\partial_\mu R^{(m)}(\mathcal{G}(x))\times\Big(R^{(m)}(\mathcal{G}(x))\Big)^{-1}. \tag{S.77}$$

This works similarly to the $SU(N)$ symmetry I have discussed in class:

$$\begin{aligned} D_\mu\Psi \;\to\; D'_\mu\Psi' \;&=\; \partial_\mu\Big(R^{(m)}\Psi\Big) \;+\; i\bigg(R^{(m)}\mathcal{A}_\mu\Big(R^{(m)}\Big)^{-1} \;+\; i\partial_\mu R^{(m)}\times\Big(R^{(m)}\Big)^{-1}\bigg)\times R^{(m)}\Psi \\ &=\; \cancel{\partial_\mu R^{(m)}\times\Psi} \;+\; R^{(m)}\times\partial_\mu\Psi \;+\; iR^{(m)}\mathcal{A}_\mu\times\Psi \;-\; \cancel{\partial_\mu R^{(m)}\times\Psi} \\ &=\; R^{(m)}\times D_\mu\Psi\,. \end{aligned} \tag{S.78}$$

But the real problem here is to make sure that transforms (S.77) are consistent with having the same gauge fields $\mathcal{A}^a_\mu(x)$ for all multiplets of the gauge group.

Before I write down the transformation law for the $\mathcal{A}_\mu^a(x)$ fields in a multiplet-independent manner, let me note that the symmetries $\mathcal{G}(x)$ should be continuous functions of x. Consequently, for an infinitesimal displacement ϵ^μ, $\mathcal{G}(x+\epsilon)\times\mathcal{G}^{-1}(x)=1+O(\epsilon)$. But for any Lie group member infinitesimally close to unity, its displacement from unity is a linear combination of the Lie algebra generators $\hat{T}^a$, thus

$$\mathcal{G}(x+\epsilon)\times\mathcal{G}^{-1}(x)\;=\;1\;+\;i\epsilon^\mu C_\mu^a(x)\hat{T}^a\;+\;O(\epsilon^2) \tag{S.79}$$

for some real coefficients $C_\mu^a(x)$. In terms of derivatives of $\mathcal{G}(x)$,

$$\partial_\mu\mathcal{G}(x)\times\mathcal{G}^{-1}(x)\;=\;iC_\mu^a(x)\hat{T}^a. \tag{S.80}$$

Given the coefficients $C_\mu^a(x)$ in this equation, I can write an explicit formula for the finite gauge transform of the non-abelian gauge fields $\mathcal{A}_a^\mu(x)$:

$$\mathcal{A}_\mu^{\prime a}(x)\;=\;R_{\rm adj}^{ab}(\mathcal{G}(x))\times\mathcal{A}_\mu^b(x)\;-\;C_\mu^a(x) \tag{S.81}$$

where $R_{\rm adj}^{ab}(\mathcal{G})$ represents $\mathcal{G}$ in the *adjoint multiplet* of the Lie group G.

Now let me show that the gauge transform (S.81) leads to eqs. (S.77) for all multiplets (m) of the Lie group G. Any representation of G must respect the group product,

$$R^{(m)}(\mathcal{G}_2\times\mathcal{G}_1)\;=\;R^{(m)}(\mathcal{G}_2)\times R^{(m)}(\mathcal{G}_1). \tag{S.82}$$

Also, in the infinitesimal neighborhood of the unity,

$$R^{(m)}(1+i\epsilon^a\hat{T}^a)\;=\;1\;+\;i\epsilon^a T_{(m)}^a\,. \tag{S.83}$$

Consequently, for any multiplet (m),

$$\begin{aligned}R^{(m)}(\mathcal{G}(x+\epsilon))\times\Big(R^{(m)}(\mathcal{G}(x))\Big)^{-1}\;&=\;R^{(m)}\big(\mathcal{G}(x+\epsilon)\times\mathcal{G}^{-1}(x)\big)\\&=\;R^{(m)}\big(1+i\epsilon^\mu C_\mu^a\hat{T}^a\big)\\&=\;1\;+\;i\epsilon^\mu C_\mu^a T_{(m)}^a\end{aligned} \tag{S.84}$$

and hence

$$\partial_\mu R^{(m)}(\mathcal{G}(x))\times\Big(R^{(m)}(\mathcal{G}(x))\Big)^{-1}\;=\;iC_\mu^a T_{(m)}^a \tag{S.85}$$

with exactly the same coefficients $C_\mu^a(x)$ as in eq. (S.80). Therefore, the second term in eq. (S.77) for the transformation of the $\mathcal{A}_\mu^{(m)}(x)=\mathcal{A}_\mu^a(x)T_{(m)}^a$ agrees with the $-C_\mu^a(x)$ term in eq. (S.81) for the transformation of the component fields $\mathcal{A}_\mu^a(x)$.

As to the first term in eq. (S.77), it agrees with the first term in eq. (S.81) thanks to the Lemma (3.18): for any multiplet (m) and any group element $\mathcal{G}\in G$,

$$R^{(m)}(\mathcal{G})\times T_{(m)}^b\times\Big(R^{(m)}(\mathcal{G})\Big)^{-1}\;=\;\sum_a T_{(m)}^a R_{\rm adj}^{ab}(\mathcal{G}),\tag{S.86}$$

hence

$$R^{(m)}(\mathcal{G})\times\mathcal{A}_\mu^b T_{(m)}^b\times\Big(R^{(m)}(\mathcal{G})\Big)^{-1}\;=\;\mathcal{A}_\mu^b T_{(m)}^a R_{\rm adj}^{ab}(\mathcal{G})\;=\;T_{(m)}^a\Big(R_{\rm adj}^{ab}(\mathcal{G})\mathcal{A}_\mu^b\Big).\tag{S.87}$$

This completes my proof that he gauge fields $\mathcal{A}_\mu^a(x)$ transforming under finite local symmetries according to eq. (S.81) makes the derivatives D_μ covariant for all multiplets of the symmetry group G.

To complete these solutions, let me also prove the Lemma (S.86).
I assume $\mathcal{G}=\exp(-i\Theta^a\hat{T}^a)$ for some real parameters Θ^a and hence $R^{(m)}(\mathcal{G})=\exp(-i\Theta^aT_{(m)}^a)$. By the multiple-commutator formula

$$e^ABe^{-A}\;=\;B+[A,B]+\tfrac{1}{2}[[A,[A,B]]+\tfrac{1}{6}[A,[A,[A,B]]]+\cdots\;=\;\sum_{n=0}^\infty\frac{1}{n!}[A,[\cdots[A,B]\cdots]]^{n\,\rm times}\tag{S.88}$$

we have

$$\begin{aligned}R^{(m)}(\mathcal{G})T_{(m)}^a\Big(R^{(m)}(\mathcal{G})\Big)^{-1}\;&=\;\exp\Big(-i\Theta^bT_{(m)}^b\Big)\,T_{(m)}^a\exp\Big(+i\Theta^bT_{(m)}^b\Big)\\&=\;T_{(m)}^a\;-\;i\Big[\Theta^bT_{(m)}^b,T_{(m)}^a\Big]\;+\;\frac{(-i)^2}{2}\Big[\Theta^cT_{(m)}^c,\Big[\Theta^bT_{(m)}^b,T_{(m)}^a\Big]\Big]\\&\qquad+\;\frac{(-i)^3}{3!}\Big[\Theta^dT_{(m)}^d,\Big[\Theta^cT_{(m)}^c,\Big[\Theta^bT_{(m)}^b,T_{(m)}^a\Big]\Big]\Big]\;+\;\cdots\end{aligned}\tag{S.89}$$

where all the commutators follow from the Lie algebra:

$$\begin{aligned}
-i\left[\Theta^b T^b_{(m)}, T^a_{(m)}\right] &= \Theta^b f^{bac} T^c_{(m)}, \\
(-i)^2\left[\Theta^c T^c_{(m)}, \left[\Theta^b T^b_{(m)}, T^a_{(m)}\right]\right] &= \Theta^b f^{bad} \times (-i)\left[\Theta^c T^c_{(m)}, T^d_{(m)}\right] \\
&= \left(\Theta^b f^{bad}\right)\left(\Theta^c f^{cde}\right) T^e_{(m)}, \\
(-i)^3\left[\Theta^d T^d_{(m)}, \left[\Theta^c T^c_{(m)}, \left[\Theta^b T^b_{(m)}, T^a_{(m)}\right]\right]\right] &= \left(\Theta^b f^{bae}\right)\left(\Theta^c f^{cef}\right)\left(\Theta^d f^{dfg}\right) T^g_{(m)}, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned} \tag{S.90}$$

All the contractions of Θ's with structure constants on the right hand sides here may be interpreted in terms of the adjoint multiplet of the group G where $(T^a_{\text{adj}})^{bc} = -if^{abc}$:

$$\begin{aligned}
\Theta^b f^{bac} &= \left(i\Theta^b T^b_{\text{adj}}\right)^{ac}, \\
\left(\Theta^b f^{bad}\right)\left(\Theta^c f^{cde}\right) &= \left(i\Theta^b T^b_{\text{adj}}\right)^{ad}\left(i\Theta^c T^c_{\text{adj}}\right)^{de} = \left(\left(i\Theta^b T^b_{\text{adj}}\right)^2\right)^{ae}, \\
\left(\Theta^b f^{bae}\right)\left(\Theta^c f^{cef}\right)\left(\Theta^d f^{dfg}\right) &= \left(\left(i\Theta^b T^b_{\text{adj}}\right)^3\right)^{ag}, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned} \tag{S.91}$$

Combining eqs. (S.89) through (S.91), we obtain

$$\begin{aligned}
R^{(m)}(\mathcal{G}) T^a_{(m)}\left(R^{(m)}(\mathcal{G})\right)^{-1} &= T^a_{(m)} + \left(i\Theta^b T^b_{\text{adj}}\right)^{ac} T^c_{(m)} + \tfrac{1}{2}\left(\left(i\Theta^b T^b_{\text{adj}}\right)^2\right)^{ae} T^e_{(m)} \\
&\quad + \tfrac{1}{6}\left(\left(i\Theta^b T^b_{\text{adj}}\right)^3\right)^{ag} T^g_{(m)} + \cdots \\
&= \left(\exp\left(i\Theta^b T^b_{\text{adj}}\right)\right)^{ac} T^c_{(m)} \\
&\quad \langle\!\langle \text{ using antisymmetry of } (T^b_{\text{adj}})^{ac} = -(T^b_{\text{adj}})^{ca} \rangle\!\rangle \\
&= T^c_{(m)}\left(\exp\left(-i\Theta^b T^b_{\text{adj}}\right)\right)^{ca} \\
&= T^c_{(m)} \times R^{ca}_{\text{adj}}(\mathcal{G}),
\end{aligned} \tag{S.92}$$

which proves the Lemma (S.86). ‘