Problem **1**(a):

All the commutators in this question follow from the bosonic commutation relations (5.1) via the Leibniz rule:

$$[\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}^\dagger_\gamma] = [\hat{a}^\dagger_\alpha, \hat{a}^\dagger_\gamma]\hat{a}_\beta + \hat{a}^\dagger_\alpha[\hat{a}_\beta, \hat{a}^\dagger_\gamma] = 0 + \hat{a}^\dagger_\alpha \delta_{\beta,\gamma} = \delta_{\beta,\gamma}\hat{a}^\dagger_\alpha\,, \tag{S.1}$$

$$[\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}_\delta] = [\hat{a}^\dagger_\alpha, \hat{a}_\delta]\hat{a}_\beta + \hat{a}^\dagger_\alpha[\hat{a}_\beta, \hat{a}_\delta] = -\delta_{\alpha,\delta}\hat{a}_\beta + 0 = -\delta_{\alpha,\delta}\hat{a}_\beta\,, \tag{S.2}$$

$$[\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}^\dagger_\gamma \hat{a}_\delta] = [\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}^\dagger_\gamma]\hat{a}_\delta + \hat{a}^\dagger_\gamma[\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}_\gamma \hat{a}_\delta] = \delta_{\beta,\gamma}\hat{a}^\dagger_\alpha \hat{a}_\delta - \delta_{\alpha,\delta}\hat{a}^\dagger_\gamma \hat{a}_\beta\,, \tag{S.3}$$

$$\begin{aligned}
[\hat{a}^\dagger_\mu \hat{a}_\nu, \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\gamma \hat{a}_\delta] &= [\hat{a}^\dagger_\mu \hat{a}_\nu, \hat{a}^\dagger_\alpha]\hat{a}^\dagger_\beta \hat{a}_\gamma \hat{a}_\delta + \hat{a}^\dagger_\alpha[\hat{a}^\dagger_\mu \hat{a}_\nu, \hat{a}^\dagger_\beta]\hat{a}_\gamma \hat{a}_\delta \\
&\qquad + \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta[\hat{a}^\dagger_\mu \hat{a}_\nu, \hat{a}_\gamma]\hat{a}_\delta + \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\gamma[\hat{a}^\dagger_\mu \hat{a}_\nu, \hat{a}_\delta] \\
&= \delta_{\nu\alpha}\hat{a}^\dagger_\mu \hat{a}^\dagger_\beta \hat{a}_\gamma \hat{a}_\delta + \delta_{\nu\beta}\hat{a}^\dagger_\alpha \hat{a}^\dagger_\mu \hat{a}_\gamma \hat{a}_\delta - \delta_{\mu\gamma}\hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\nu \hat{a}_\delta - \delta_{\mu\delta}\hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\gamma \hat{a}_\nu\,.
\end{aligned} \tag{S.4}$$

Problem **1**(a):

First, let's prove by induction that for integer $n \geq 0$, $[\hat{a}, (\hat{a}^\dagger)^n] = n \times (\hat{a}^\dagger)^{n-1}$. The induction base is easy to check: For $n = 0$ we have $[\hat{a}, (\hat{a}^\dagger)^0] = [\hat{a}, 1] = 0 = 0 \times \text{whatever}$, while for $n = 1$ we have $[\hat{a}, (\hat{a}^\dagger)^n] = [a, \hat{a}^\dagger] = 1 = 1 \times (\hat{a}^\dagger)^0$. Now suppose $[\hat{a}, (\hat{a}^\dagger)^n] = n(\hat{a}^\dagger)^{n-1}$ for some n; then for $n + 1$ we have

$$\begin{aligned}
[\hat{a}, (\hat{a}^\dagger)^{n+1}] = [\hat{a}, (\hat{a}^\dagger)^n \times \hat{a}^\dagger] &= [a, (\hat{a}^\dagger)^n] \times \hat{a}^\dagger + (\hat{a}^\dagger)^n \times [a, \hat{a}^\dagger] \\
&= n(\hat{a}^\dagger)^{n-1} \times \hat{a}^\dagger + (\hat{a}^\dagger)^n \times 1 = (n+1) \times (\hat{a}^\dagger)^n.
\end{aligned} \tag{S.5}$$

Similarly, for any integer $n \geq 0$, $[\hat{a}^\dagger, (\hat{a})^n] = -n(\hat{a})^{n-1}$; again, the proof is by induction, which is so similar to the above that I don't need to spell it out.

Next, consider an analytic function f of the creation operator. Analytic functions can be expanded into power series, $f(x) = f_0 + f_1 x + f_2 x^2 + \cdots$; substituting $x \mapsto \hat{a}^\dagger$ into such series for f, we build the operator

$$f(\hat{a}^\dagger) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} f_n \times (\hat{a}^\dagger)^n = f_0 + f_1 \times \hat{a}^\dagger + f_2 \times (\hat{a}^\dagger)^2 + \cdots. \tag{S.6}$$

Likewise, for $f'(x) \stackrel{\text{def}}{=} df/dx = 0 + f_1 + 2f_2 x + 3f_3 x^2 + \cdots$ we have

$$f'(\hat{a}^\dagger) \;=\; \sum_{n=0}^{\infty} n f_n \times (\hat{a}^\dagger)^{n-1}. \tag{S.7}$$

Consequently,

$$[\hat{a}, f(\hat{a}^\dagger)] \;=\; \sum_{n=0}^{\infty} f_n \times [\hat{a}, (\hat{a}^\dagger)^n] \;=\; \sum_{n=0}^{\infty} f_n \times n \times (\hat{a}^\dagger)^{n-1} \;=\; f'(\hat{a}^\dagger). \tag{S.8}$$

Similarly, for an analytic function of the annihilation operator, $f(\hat{a}) = f_0 + f_1 \times \hat{a} + f_2 \times (\hat{a})^2 + \cdots$, we have

$$[\hat{a}^\dagger, f(\hat{a})] \;=\; \sum_{n=0}^{\infty} f_n \times [\hat{a}^\dagger, (\hat{a})^n] \;=\; \sum_{n=0}^{\infty} f_n \times (-n) \times (\hat{a})^{n-1} \;=\; -f'(\hat{a}). \tag{S.9}$$

$\mathcal{Q.E.D.}$

Problem **1**(c):

In light of part (b), $[\hat{a}, \exp(c\hat{a}^\dagger)] = \exp'(c\hat{a}^\dagger) = c\exp(c\hat{a}^\dagger)$ and $[\hat{a}^\dagger, \exp(c\hat{a})] = -\exp'(c\hat{a}) = -c\exp(c\hat{a})$. Consequently,

$$e^{c\hat{a}}\hat{a}^\dagger e^{-c\hat{a}} \;=\; \Big(\hat{a}^\dagger e^{c\hat{a}} \;-\; \big[\hat{a}^\dagger, e^{c\hat{a}}\big]\Big)e^{-c\hat{a}} \;=\; \Big(\hat{a}^\dagger e^{c\hat{a}} \;-\; (-c)e^{c\hat{a}}\Big)e^{-c\hat{a}} \;=\; \hat{a}^\dagger \;+\; c \tag{S.10}$$

and likewise

$$e^{c\hat{a}^\dagger}\hat{a}e^{-c\hat{a}^\dagger} \;=\; \Big(\hat{a}e^{c\hat{a}^\dagger} \;-\; \big[\hat{a}, e^{c\hat{a}^\dagger}\big]\Big)e^{-c\hat{a}^\dagger} \;=\; \Big(\hat{a}e^{c\hat{a}^\dagger} \;-\; (+c)e^{c\hat{a}^\dagger}\Big)e^{-c\hat{a}^\dagger} \;=\; \hat{a} \;-\; c. \tag{S.11}$$

Now, for any two operators $\hat{X}$ and $\hat{Y}$,

$$\Big(e^{\hat{X}}\hat{Y}e^{-\hat{X}}\Big)^n \;=\; e^{\hat{X}}\hat{Y}e^{-\hat{Y}} \times e^{\hat{X}}\hat{Y}e^{-\hat{Y}} \times \cdots \times e^{\hat{X}}\hat{Y}e^{-\hat{Y}} \;=\; e^{\hat{X}}\hat{Y} \times \hat{Y} \times \cdots \hat{Y}e^{-\hat{X}} \;=\; e^{\hat{X}}\hat{Y}^n e^{-\hat{X}}. \tag{S.12}$$

Consequently, for any analytic function $f(\hat{Y}) = f_0 + f_1\hat{Y} + f_2\hat{Y} + \cdots$,

$$\begin{aligned} f\left(e^{\hat{X}}\hat{Y}e^{-\hat{X}}\right) &= \sum_n f_n\left(e^{\hat{X}}\hat{Y}e^{-\hat{X}}\right)^n = \sum_n f_n \times e^{\hat{X}}\hat{Y}^n e^{-\hat{X}} \\ &= e^{\hat{X}}\left(\sum_n f_n\hat{Y}^n\right)e^{-\hat{X}} = e^{\hat{X}}f(\hat{Y})e^{-\hat{X}}. \end{aligned} \tag{S.13}$$

In particular, for $\hat{X} = c\hat{a}$ and $\hat{Y} = \hat{a}^\dagger$,

$$e^{c\hat{a}}f(\hat{a}^\dagger)e^{-c\hat{a}} = f\left(e^{c\hat{a}}\hat{a}^\dagger e^{-c\hat{a}}\right) = f(\hat{a}^\dagger + c), \tag{S.14}$$

and likewise, for $\hat{X} = c\hat{a}^\dagger$ and $\hat{Y} = \hat{a}$,

$$e^{c\hat{a}^\dagger}f(\hat{a})e^{-c\hat{a}^\dagger} = f\left(e^{c\hat{a}^\dagger}\hat{a}e^{-c\hat{a}^\dagger}\right) = f(\hat{a} - c). \tag{S.15}$$

$\mathcal{Q.E.D.}$

Problem **1**(d):

Since all the creation operators commute with each other, we may decompose any analytic function of multiple creation operators into a power series with respect to any particular $\hat{a}^\dagger_\alpha$ as

$$f(\text{multiple } \hat{a}^\dagger) = \sum_n F_n(\text{other } \hat{a}^\dagger_\beta) \times (\hat{a}^\dagger_\alpha)^n \tag{S.16}$$

where F_n are some analytic functions of the *other* creation operators $\hat{a}^\dagger_{\beta\neq\alpha}$. The same F_n appear in the partial derivative of $f(\hat{a}^\dagger)$ with respect to the $\hat{a}^\dagger_\alpha$,

$$\frac{\partial f(\text{multiple } \hat{a}^\dagger)}{\partial \hat{a}^\dagger_\alpha} = \sum_n n \times F_n(\text{other } \hat{a}^\dagger_\beta) \times (\hat{a}^\dagger_\alpha)^{n-1}. \tag{S.17}$$

Note that the creation operators $\hat{a}^\dagger_\beta$ with $\beta \neq \alpha$ commute with the $\hat{a}_\alpha$ annihilation operator, hence any function of such $\hat{a}^\dagger_{\beta\neq\alpha}$ also commutes with the $\hat{a}_\alpha$,

$$\Big[\hat{a}_\alpha, F_n(\text{other } \hat{a}^\dagger_\beta)\Big] \;=\; 0, \tag{S.18}$$

therefore

$$\begin{aligned}\Big[\hat{a}_\alpha, f(\text{multiple } \hat{a}^\dagger)\Big] \;&=\; \sum_n F_n(\text{other } \hat{a}^\dagger_\beta) \times \Big[\hat{a}_\alpha, (\hat{a}^\dagger_\alpha)^n\Big] \\ &=\; \sum_n F_n(\text{other } \hat{a}^\dagger_\beta) \times n(\hat{a}^\dagger_\alpha)^{n-1} \;=\; \frac{\partial f(\text{multiple } \hat{a}^\dagger)}{\partial \hat{a}^\dagger_\alpha}\,.\end{aligned} \tag{S.19}$$

This proves the first equation (4.4).

Similarly, any analytic function of multiple annihilation operators $\hat{a}_\beta$ — which also commute with each other — may be decomposed into a power series in any particular $\hat{a}_\alpha$ as

$$f(\text{multiple } \hat{a}) \;=\; \sum_n F_n(\text{other } \hat{a}_\beta) \times (\hat{a}_\alpha)^n \tag{S.20}$$

where the F_n are analytic functions of the remaining annihilation operators $\hat{a}_{\beta\neq\alpha}$ but not of the $\hat{a}_\alpha$ itself. Consequently, as operators all the $F_n(\text{other } \hat{a}_\beta)$ commute with the $\hat{a}^\dagger_\alpha$ and hence

$$\begin{aligned}\Big[\hat{a}^\dagger_\alpha, f(\text{multiple } \hat{a})\Big] \;&=\; \sum_n F_n(\text{other } \hat{a}_\beta) \times \Big[\hat{a}^\dagger_\alpha, (\hat{a}_\alpha)^n\Big] \\ &=\; \sum_n F_n(\text{other } \hat{a}_\beta) \times -n(\hat{a}_\alpha)^{n-1} \;=\; -\frac{\partial f(\text{multiple } \hat{a})}{\partial \hat{a}_\alpha}\,.\end{aligned} \tag{S.21}$$

This proves the second equation (4.4).

Now let's proceed similarly to part (c). Applying the first two eqs. (4.4) to $f(\text{multiple } x) = \exp\Big(\sum_\beta c_\beta x_\beta\Big)$, we have

$$\begin{aligned}\Big[\hat{a}_\alpha, \exp\Big(\sum\nolimits_\beta c_\beta \hat{a}^\dagger_\beta\Big)\Big] \;&=\; +\frac{\partial}{\partial \hat{a}^\dagger_\alpha}\exp\Big(\sum\nolimits_\beta c_\beta \hat{a}^\dagger_\beta\Big) \;=\; +c_\alpha \times \exp\Big(\sum\nolimits_\beta c_\beta \hat{a}^\dagger_\beta\Big)\,, \\ \Big[\hat{a}^\dagger_\alpha, \exp\Big(\sum\nolimits_\beta c_\beta \hat{a}_\beta\Big)\Big] \;&=\; -\frac{\partial}{\partial \hat{a}_\alpha}\exp\Big(\sum\nolimits_\beta c_\beta \hat{a}_\beta\Big) \;=\; -c_\alpha \times \exp\Big(\sum\nolimits_\beta c_\beta \hat{a}_\beta\Big)\,,\end{aligned} \tag{S.22}$$

and consequently

$$\begin{aligned}\exp\left(\sum\nolimits_\beta c_\beta \hat{a}_\beta\right) \times \hat{a}^\dagger_\alpha \times \exp\left(-\sum\nolimits_\beta c_\beta \hat{a}_\beta\right) \;&=\; \hat{a}^\dagger_\alpha \;+\; c_\alpha\,,\\ \exp\left(\sum\nolimits_\beta c_\beta \hat{a}^\dagger_\beta\right) \times \hat{a}_\alpha \times \exp\left(-\sum\nolimits_\beta c_\beta \hat{a}^\dagger_\beta\right) \;&=\; \hat{a}_\alpha \;-\; c_\alpha\,.\end{aligned} \tag{S.23}$$

Finally, applying eq. (S.13) to the these formulae, we obtain the last two eqs. (4.4) for any analytic function f. $\mathcal{Q.E.D.}$

Problem **2**(a):

First, let's verify eq. (5.7) for a state $|\gamma_1,\ldots,\gamma_N\rangle$, with wave-function

$$\psi(\mathbf{x}_1,\ldots,\mathbf{x}_N) \;=\; \frac{1}{T\sqrt{D}} \times \phi_{(\gamma_1}(\mathbf{x}_1)\cdots\phi_{\gamma_N)}(\mathbf{x}_N) \tag{S.24}$$

where () surrounding the indices $(\gamma_1\cdots\gamma_N)$ denote total symmetrization, *i.e.* summing over all $N!$ permutations, T is the number of trivial permutations (of indices which happen to coincide), and D is the number of distinct permutations (of indices which do not coincide). For this state,

$$\hat{a}_\alpha\,|\gamma_1,\ldots,\gamma_N\rangle \;=\; \sqrt{n_\alpha+1}\;|\gamma_1,\ldots,\gamma_N,\alpha\rangle\,, \tag{S.25}$$

which has wave-function

$$\psi'(\mathbf{x}_1,\ldots,\mathbf{x}_N,\mathbf{x}_{N+1}) \;=\; \frac{\sqrt{n_\alpha+1}}{T'\sqrt{D'}} \times \phi_{(\gamma_1}(\mathbf{x}_1)\cdots\phi_{\gamma_N}(\mathbf{x}_N)\phi_{\alpha)}(\mathbf{x}_{N+1}). \tag{S.26}$$

Here total symmetrization on the RHS means summing over all the $(N+1)!$ permutations of indices $(\gamma_1\cdots\gamma_N\alpha)$. Let's group these permutations in $N+1$ blocks of $N!$, namely first permute the γ's among themselves, and then put α anywhere in that list,

$$\phi_{(\gamma_1}(\mathbf{x}_1)\cdots\phi_{\gamma_N}(\mathbf{x}_N)\phi_{\alpha)}(\mathbf{x}_{N+1}) \;=\; \sum_{i=1}^{N+1}\phi_\alpha(\mathbf{x}_i)\times\phi_{(\gamma_1}(\mathbf{x}_1)\cdots\cancel{\phi(\mathbf{x}_i)}\cdots\phi_{\gamma_N)}(\mathbf{x}_{N+1}). \tag{S.27}$$

But the symmetrization over γ's here is exactly as in eq. (S.24), except for the relevant coordinates being $(\mathbf{x}_1,\ldots,\cancel{\mathbf{x}_i},\ldots,\mathbf{x}_{N+1})$ instead of $(\mathbf{x}_1,\ldots,\mathbf{x}_N)$. Therefore,

$$\psi'(\mathbf{x}_1,\ldots,\mathbf{x}_N,\mathbf{x}_{N+1}) \;=\; \frac{\sqrt{n_\alpha+1}}{T'\sqrt{D'}}\times T\sqrt{D}\times\sum_{i=1}^{N+1}\phi_\alpha(\mathbf{x}_i)\times\psi(\mathbf{x}_1,\ldots,\cancel{\mathbf{x}_i},\ldots,\mathbf{x}_{N+1}), \tag{S.28}$$

exactly as in eq. (5.7), except maybe the overall coefficient. To check this coefficient, we use eqs. (5.6). Given occupation numbers n_β of the original state $|\gamma_1,\ldots,\gamma_N\rangle$, the new state $|\gamma_1,\ldots,\gamma_N,\alpha\rangle$ has $n'_\beta = n_\beta + \delta_{\alpha\beta}$, hence

$$\begin{aligned}
\frac{T'}{T} &= \prod_\beta n'_\beta! \Big/ \prod_\beta n_\beta! = \frac{(n_\alpha+1)!}{n_\alpha!} = n_\alpha + 1,\\
\frac{T'\sqrt{D'}}{T\sqrt{D}} &= \sqrt{\frac{T'\times(N+1)!}{T\times N!}} = \sqrt{(n_\alpha+1)(N+1)}\,,\\
\frac{\sqrt{n_\alpha+1}}{T'\sqrt{D'}}\times T\sqrt{D} &= \frac{1}{\sqrt{N+1}}\,.
\end{aligned} \tag{S.29}$$

Thus, the coefficient in eq. (S.28) is also exactly as in eq. (5.7).

At this point, we have proved eq. (5.7) for states $|N,\Psi\rangle$ that happen to be $|\gamma_1,\ldots,\gamma_N\rangle$ for some $\gamma_1,\ldots,\gamma_N$. To prove it for all N-boson states $|N,\psi\rangle$ we now use *linearity:* the operator $\hat{a}^\dagger_\alpha$ is linear, and eq. (5.7) is manifestly linear with respect to ψ and ψ', so if it holds for any set of states, it also holds for all their linear combinations. But states $|\gamma_1,\ldots,\gamma_N\rangle$ make up a complete basis of the N-boson Hilbert space, so any $|N,\psi\rangle$ is a linear combination of such states. Therefore, eq. (5.7) must hold for any N-boson wave function $\psi(\mathbf{x}_1,\ldots,\mathbf{x}_N)$. $\mathcal{Q.E.D.}$

Problem **2**(b):

The operator $\hat{a}_\alpha$ is the hermitian conjugate of the operator $\hat{a}^\dagger_\alpha$, so for any two states $|N,\psi\rangle$ and $\langle N-1,\widetilde{\psi}|$,

$$\langle N-1,\widetilde{\psi}|\,\hat{a}_\alpha\,|N,\psi\rangle = \langle N,\psi|\,\hat{a}^\dagger_\alpha\,|N-1,\widetilde{\psi}\rangle^*\,. \tag{S.30}$$

In wave-function terms, this means

$$\begin{aligned}
&\int\!d^3\mathbf{x}_1\cdots\int\!d^3\mathbf{x}_{N-1}\,\widetilde{\psi}^*(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1})\times\psi''(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1}) = \\
&\qquad = \int\!d^3\mathbf{x}_1\cdots\int\!d^3\mathbf{x}_N\,\psi(\mathbf{x}_1,\ldots,\mathbf{x}_N)\times\left[\widetilde{\psi}'(\mathbf{x}_1,\ldots,\mathbf{x}_N)\right]^*
\end{aligned} \tag{S.31}$$

where $\widetilde{\psi}'(\mathbf{x}_1,\ldots,\mathbf{x}_N)$ is the wave function of the state $\left|N,\widetilde{\psi}'\right\rangle = \hat{a}^\dagger_\alpha\,|N-1,\widetilde{\psi}\rangle$. Applying eq. (5.7) of part (a) to this wave-function, we obtain

$$\begin{aligned}
&\int d^3\mathbf{x}_1 \cdots \int d^3\mathbf{x}_{N-1}\, \widetilde{\psi}^*(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1}) \times \psi''(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1}) \;= \\
&= \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \int d^3\mathbf{x}_1 \cdots \int d^3\mathbf{x}_N\, \psi(\mathbf{x}_1,\ldots,\mathbf{x}_N) \times \phi^*_\alpha(\mathbf{x}_i) \times \widetilde{\psi}^*(\mathbf{x}_1,\ldots,\not{\mathbf{x}}_i,\ldots,\mathbf{x}_N) \\
&\qquad \langle\!\langle \text{ by permutational symmetry } \rangle\!\rangle \\
&= \frac{N}{\sqrt{N}} \int d^3\mathbf{x}_1 \cdots \int d^3\mathbf{x}_N\, \psi(\mathbf{x}_1,\ldots,\mathbf{x}_N) \times \phi^*_\alpha(\mathbf{x}_N) \times \widetilde{\psi}^*(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1}) \\
&= \int d^3\mathbf{x}_1 \cdots \int d^3\mathbf{x}_{N-1}\, \widetilde{\psi}^*(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1}) \times \sqrt{N} \int d^3\mathbf{x}_N \phi^*_\alpha(\mathbf{x}_N) \times \psi(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1},\mathbf{x}_N).
\end{aligned} \tag{S.32}$$

This formula holds true for any totally symmetric wave-function $\widetilde{\psi}(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1})$, and this is possible only when

$$\psi''(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1}) \;=\; \sqrt{N}\int d^3\mathbf{x}_N \phi^*_\alpha(\mathbf{x}_N) \times \psi(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1},\mathbf{x}_N), \tag{5.8}$$

or rather when the totally symmetric part of the left hand side here equals to the totally symmetric part of the right hand side. But for bosonic wave functions ψ and $\widetilde{\psi}$ both sides must be already totally symmetric in $(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1})$ as they are, so eq. (5.8) must apply exactly as written. $\mathcal{Q.E.D.}$

Problem **2**(c):

Let $A_{\alpha\beta} = \langle\alpha|\,\hat{A}_1\,|\beta\rangle$. Since states $|\alpha\rangle$ make a complete basis of the 1-particle Hilbert space, for any 1-particle states $\langle\widetilde{\psi}|$ and $|\psi\rangle$

$$\langle\widetilde{\psi}|\,\hat{A}_1\,|\psi\rangle \;=\; \sum_{\alpha,\beta} A_{\alpha\beta}\,\langle\widetilde{\psi}|\alpha\rangle\,\langle\beta|\psi\rangle \;=\; \sum_{\alpha,\beta} A_{\alpha\beta} \times \int d^3\tilde{\mathbf{x}}\,\widetilde{\psi}^*(\tilde{\mathbf{x}})\phi_\alpha(\tilde{\mathbf{x}}) \times \int d^3\mathbf{x}\,\phi^*_\beta(\mathbf{x})\psi(\mathbf{x}). \tag{S.33}$$

This is undergraduate-level QM.

In the N–particle Hilbert space we have a similar formula for the matrix elements of the $\hat{A}_1$ acting on particle $\#i$, the only modification being integrals over the coordinates of the other particles,

$$\begin{aligned}
\langle N,\widetilde{\psi}|\,\hat{A}_1(i^{\rm th})\,|N,\psi\rangle\ &= \\
&= \int\!\cdots\!\int d^3\mathbf{x}_1\cdots\cancel{d^3\mathbf{x}_i}\cdots d^3\mathbf{x}_N\,\sum_{\alpha,\beta}A_{\alpha\beta}\times\left(\int d^3\tilde{\mathbf{x}}_i\,\widetilde{\psi}^*(\mathbf{x}_1,\ldots,\tilde{\mathbf{x}}_i,\ldots,\mathbf{x}_N)\phi_\alpha(\tilde{\mathbf{x}}_i)\right)\\
&\qquad\times\left(\int d^3\mathbf{x}_i\,\phi^*_\beta(\mathbf{x}_i)\psi(\mathbf{x}_1,\ldots,\mathbf{x}_i,\ldots,\mathbf{x}_N)\right)\\
&= \sum_{\alpha,\beta}A_{\alpha\beta}\times\int\!\cdots\!\int d^3\mathbf{x}_1\cdots d^3\mathbf{x}_N\,d^3\tilde{\mathbf{x}}_i\,\widetilde{\psi}^*(\mathbf{x}_1,\ldots,\tilde{\mathbf{x}}_i,\ldots,\mathbf{x}_N)\times\phi_\alpha(\tilde{\mathbf{x}}_i)\\
&\qquad\times\phi^*_\beta(\mathbf{x}_i)\times\psi(\mathbf{x}_1,\ldots,\mathbf{x}_i,\ldots,\mathbf{x}_N).
\end{aligned}\tag{S.34}$$

For symmetric wave-functions of identical bosons, we get the same matrix element regardless of which particle $\#i$ we are acting on with the operator $\hat{A}_1$, hence for the *net* A operator (5.9),

$$\begin{aligned}
\langle N,\widetilde{\psi}|\,\hat{A}^{(1)}_{\rm net}\,|N,\psi\rangle\ &=\ N\times\sum_{\alpha,\beta}A_{\alpha\beta}\times\int\!\cdots\!\int d^3\mathbf{x}_1\cdots d^3\mathbf{x}_{N-1}\,d^3\mathbf{x}_N d^3\tilde{\mathbf{x}}_N\\
&\qquad\widetilde{\psi}^*(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1},\tilde{\mathbf{x}}_N)\times\phi_\alpha(\tilde{\mathbf{x}}_N)\\
&\qquad\times\phi^*_\beta(\mathbf{x}_N)\times\psi(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1},\mathbf{x}_N).
\end{aligned}\tag{S.35}$$

Now consider matrix elements of the Fock-space operator (5.10). According to eq. (5.8) of part (b), the state $|N-1,\psi''\rangle=\hat{a}_\beta\,|N,\psi\rangle$ has wave-function

$$\psi''(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1})\ =\ \sqrt{N}\int d^3\mathbf{x}_N\,\phi^*_\beta(\mathbf{x}_N)\times\psi(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1},\mathbf{x}_N).\tag{S.36}$$

Likewise, according to eq. (5.7) of part (a), the state $\big|N-1,\widetilde{\psi}''\big\rangle=\hat{a}_\alpha\,|N,\widetilde{\psi}\rangle$ has wave-function

$$\widetilde{\psi}''(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1})\ =\ \sqrt{N}\int d^3\tilde{\mathbf{x}}_N\,\phi^*_\alpha(\tilde{\mathbf{x}}_N)\times\widetilde{\psi}(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1},\tilde{\mathbf{x}}_N).\tag{S.37}$$

Consequently,

$$
\begin{aligned}
\langle N,\widetilde{\psi}|\,\hat{a}^\dagger_\alpha\hat{a}_\beta\,|N,\psi\rangle \;&=\; \left\langle N-1,\widetilde{\psi}''\right|\left|N-1,\psi''\right\rangle \\
&= \int\!\cdots\!\int d^3\mathbf{x}_1\cdots\mathbf{x}_{N-1}\,\widetilde{\psi}''^*(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1})\times\psi''(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1}) \\
&= \int\!\cdots\!\int d^3\mathbf{x}_1\cdots\mathbf{x}_{N-1}\,\sqrt{N}\int d^3\tilde{\mathbf{x}}_N\,\phi_\alpha(\tilde{\mathbf{x}}_N)\times\widetilde{\psi}^*(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1},\tilde{\mathbf{x}}_N)\times \\
&\qquad\times\sqrt{N}\int d^3\mathbf{x}_N\,\phi^*_\beta(\mathbf{x}_N)\times\psi(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1},\mathbf{x}_N).
\end{aligned}
\tag{S.38}
$$

Comparing this formula to the integrals in eq. (S.35), we see that

$$
\langle N,\widetilde{\psi}|\,\hat{A}^{(1)}_{\rm net}\,|N,\psi\rangle \;=\; \sum_{\alpha,\beta}A_{\alpha\beta}\times\langle N,\widetilde{\psi}|\,\hat{a}^\dagger_\alpha\hat{a}_\beta\,|N,\psi\rangle \;=\; \langle N,\widetilde{\psi}|\,\hat{A}^{(2)}_{\rm net}\,|N,\psi\rangle\,. \tag{S.39}
$$

$\mathcal{Q.E.D.}$

<u>Problem **2**(d):</u>

This part follows from the second commutator in problem **1**(a). Indeed, Given

$$
\hat{A}^{(2)}_{\rm tot} \;=\sum_{\alpha,\beta}\langle\alpha|\,\hat{A}_1\,|\beta\rangle\;\hat{a}^\dagger_\alpha\hat{a}_\beta \tag{S.40}
$$

and

$$
\hat{B}^{(2)}_{\rm tot} \;=\sum_{\gamma,\delta}\langle\gamma|\,\hat{B}_1\,|\delta\rangle\;\hat{a}^\dagger_\gamma\hat{a}_\delta\,, \tag{S.41}
$$

we immediately have

$$
\begin{aligned}
\left[\hat{A}^{(2)}_{\rm tot}, \hat{B}^{(2)}_{\rm tot}\right] &= \sum_{\alpha,\beta,\gamma,\delta} \langle\alpha|\,\hat{A}_1\,|\beta\rangle\;\langle\gamma|\,\hat{B}_1\,|\delta\rangle\;[\hat{a}^\dagger_\alpha\hat{a}_\beta,\hat{a}^\dagger_\gamma\hat{a}_\delta] \\
&\qquad\langle\!\langle\text{ using (S.3) }\rangle\!\rangle \\
&= \sum_{\alpha,\beta,\gamma,\delta} \langle\alpha|\,\hat{A}_1\,|\beta\rangle\;\langle\gamma|\,\hat{B}_1\,|\delta\rangle\;\Big(\delta_{\beta,\gamma}\hat{a}^\dagger_\alpha\hat{a}_\delta\;-\;\delta_{\alpha,\delta}\hat{a}^\dagger_\gamma\hat{a}_\beta\Big) \\
&= \sum_{\alpha,\delta}\hat{a}^\dagger_\alpha\hat{a}_\delta\times\sum_{\beta=\gamma}\langle\alpha|\,\hat{A}_1\,|\gamma\rangle\,\langle\gamma|\,\hat{B}_1\,|\delta\rangle\;-\sum_{\beta,\gamma}\hat{a}^\dagger_\gamma\hat{a}_\beta\times\sum_{\alpha=\delta}\langle\gamma|\,\hat{B}_1\,|\alpha\rangle\,\langle\alpha|\,\hat{A}_1\,|\beta\rangle \\
&= \sum_{\alpha,\delta}\hat{a}^\dagger_\alpha\hat{a}_\delta\;\langle\alpha|\,\hat{A}_1\hat{B}_1\,|\delta\rangle\;-\sum_{\beta,\gamma}\hat{a}^\dagger_\gamma\hat{a}_\beta\;\langle\gamma|\,\hat{B}_1\hat{A}_1\,|\beta\rangle \\
&\qquad\langle\!\langle\text{ renaming summation indices }\rangle\!\rangle \\
&= \sum_{\alpha,\beta}\hat{a}^\dagger_\alpha\hat{a}_\beta\times\Big(\langle\alpha|\,\hat{A}_1\hat{B}_1\,|\beta\rangle\;-\;\langle\alpha|\,\hat{B}_1\hat{A}_1\,|\beta\rangle\Big) \\
&= \sum_{\alpha,\beta}\hat{a}^\dagger_\alpha\hat{a}_\beta\times\langle\alpha|\,\Big([\hat{A}_1,\hat{B}_1]=\hat{C}_1\Big)\,|\beta\rangle\;\equiv\;\hat{C}^{(2)}_{\rm tot}.
\end{aligned}
\tag{S.42}
$$

Problem **2**(e):

This works similarly to part (c), except for more integrals. Let

$$
B_{\alpha\beta,\gamma\delta} \;=\; \big(\langle\alpha|\otimes\langle\beta|\big)\hat{B}_2\big(|\gamma\rangle\otimes|\delta\rangle\big) \tag{S.43}
$$

be matrix elements of a two-body operator $\hat{B}_2$ between *un-symmetrized* two-particle states. Then for generic two-particle states $\langle\widetilde{\psi}|$ and $|\psi\rangle$ — symmetric or not — we have

$$
\begin{aligned}
\langle\widetilde{\psi}|\,\hat{B}_2\,|\psi\rangle &= \sum_{\alpha,\beta,\gamma,\delta} B_{\alpha\beta,\gamma\delta}\times\langle\widetilde{\psi}|\,\big(|\alpha\rangle\otimes|\beta\rangle\big)\times\big(\langle\gamma|\otimes\langle\delta|\big)\,|\psi\rangle \\
&= \sum_{\alpha,\beta,\gamma,\delta} B_{\alpha\beta,\gamma\delta}\times\iint d^3\tilde{\mathbf{x}}_1\,d^3\tilde{\mathbf{x}}_2\,\widetilde{\psi}^*(\tilde{\mathbf{x}}_1,\tilde{\mathbf{x}}_2)\phi_\alpha(\tilde{\mathbf{x}}_1)\phi_\beta(\tilde{\mathbf{x}}_2) \\
&\qquad\times\iint d^3\mathbf{x}_1\,d^3\mathbf{x}_2\,\phi^*_\gamma(\mathbf{x}_1)\phi^*_\delta(\mathbf{x}_2)\psi(\mathbf{x}_1,\mathbf{x}_2).
\end{aligned}
\tag{S.44}
$$

Similarly, in the Hilbert space of $N>2$ particles — identical bosons or not — the operator $\hat{B}_2$

acting on particles $\#i$ and $\#j$ has matrix elements

$$\begin{aligned}
\langle N,\widetilde{\psi}|\,\hat{B}_2(i^{\rm th},j^{\rm th})\,|N,\psi\rangle\ &= \\
= \sum_{\alpha,\beta,\gamma,\delta} B_{\alpha\beta,\gamma\delta}\times\int\!\cdots\!\int & d^3\mathbf{x}_1\cdots\cancel{d^3\mathbf{x}_i}\cdots\cancel{d^3\mathbf{x}_j}\cdots d^3\mathbf{x}_N \\
& \iint d^3\tilde{\mathbf{x}}_i\,d^3\tilde{\mathbf{x}}_j\,\widetilde{\psi}^*(\mathbf{x}_1,\ldots,\tilde{\mathbf{x}}_i,\ldots,\tilde{\mathbf{x}}_j,\ldots,\mathbf{x}_N)\phi_\alpha(\tilde{\mathbf{x}}_i)\phi_\beta(\tilde{\mathbf{x}}_j) \\
& \times\iint d^3\mathbf{x}_i\,d^3\mathbf{x}_j\,\phi^*_\gamma(\mathbf{x}_i)\phi^*_\delta(\mathbf{x}_j)\psi(\mathbf{x}_1,\ldots,\mathbf{x}_i,\ldots,\mathbf{x}_j,\ldots,\mathbf{x}_N)
\end{aligned} \tag{S.45}$$

For identical bosons — and hence totally symmetric wave-functions ψ and $\widetilde{\psi}$ — such matrix elements do not depend on the choice of particles on which $\hat{B}_2$ acts, so we may just as well let $i=N-1$ and $j=N$. Consequently, the *net* $\hat{B}$ operator (5.11) has matrix elements

$$\begin{aligned}
\langle N,\widetilde{\psi}|\,\hat{B}^{(1)}_{\rm net}\,|N,\psi\rangle\ &=\ \frac{N(N-1)}{2}\times\langle N,\widetilde{\psi}|\,\hat{B}_2(N-1,N)\,|N,\psi\rangle \\
&=\ \frac{N(N-1)}{2}\sum_{\alpha,\beta,\gamma,\delta} B_{\alpha\beta,\gamma\delta}\times I_{\alpha\beta,\gamma\delta}
\end{aligned} \tag{S.46}$$

where

$$\begin{aligned}
I_{\alpha\beta,\gamma\delta}\ =\int\!\cdots\!\int d^3\mathbf{x}_1\cdots d^3\mathbf{x}_{N-2}\iint & d^3\tilde{\mathbf{x}}_{N-1}\,d^3\tilde{\mathbf{x}}_N\,\widetilde{\psi}^*(\mathbf{x}_1,\ldots,\mathbf{x}_{N-2},\tilde{\mathbf{x}}_{N-1},\tilde{\mathbf{x}}_N)\phi_\alpha(\tilde{\mathbf{x}}_{N-1})\phi_\beta(\tilde{\mathbf{x}}_N) \\
& \times\iint d^3\mathbf{x}_{N-1}\,d^3\mathbf{x}_N\,\phi^*_\gamma(\mathbf{x}_{N-1})\phi^*_\delta(\mathbf{x}_N)\psi(\mathbf{x}_1,\ldots,\mathbf{x}_{N-2},\mathbf{x}_{N-1},\mathbf{x}_N)
\end{aligned}$$

Now let's compare this to the Fock space formalism. Applying eq. (5.8) of part (b) *twice*, we find that the $(N-2)$–particle state

$$\left|N-2,\psi'''\right\rangle\ =\ \hat{a}_\delta\hat{a}_\gamma\,|N,\psi\rangle \tag{S.47}$$

has wave function

$$\begin{aligned}
\psi'''(\mathbf{x}_1,\ldots,\mathbf{x}_{N-2})\ =\ \sqrt{N(N-1)}\iint & d^3\mathbf{x}_{N-1}\,d^3\mathbf{x}_N\,\phi^*_\gamma(\mathbf{x}_{N-1})\phi^*_\delta(\mathbf{x}_N) \\
& \times\psi(\mathbf{x}_1,\ldots,\mathbf{x}_{N-2},\mathbf{x}_{N-1},\mathbf{x}_N).
\end{aligned} \tag{S.48}$$

Likewise, applying eq. (5.7) of part (a) *twice*, we find that the $(N-2)$–particle state

$$\left|N-2,\widetilde{\psi}'''\right\rangle \;=\; \hat{a}_\beta\hat{a}_\alpha\left|N,\widetilde{\psi}\right\rangle \tag{S.49}$$

has wave function

$$\begin{aligned}\widetilde{\psi}'''(\mathbf{x}_1,\ldots,\mathbf{x}_{N-2}) \;=\; \sqrt{N(N-1)}\iint d^3\mathbf{x}_{N-1}\,d^3\mathbf{x}_N\,\phi^*_\beta(\tilde{\mathbf{x}}_{N-1})\phi^*_\alpha(\tilde{\mathbf{x}}_N)&\\ \times\,\widetilde{\psi}(\mathbf{x}_1,\ldots,\mathbf{x}_{N-2},\tilde{\mathbf{x}}_{N-1},\tilde{\mathbf{x}}_N).&\end{aligned} \tag{S.50}$$

Taking Dirac product of these two states, we obtain

$$\begin{aligned}\langle N,\widetilde{\psi}|\,\hat{a}^\dagger_\alpha\hat{a}^\dagger_\beta\hat{a}_\delta\hat{a}_\gamma\,|N,\psi\rangle \;&=\; \left\langle N-2,\widetilde{\psi}'''\right|\left|N-2,\psi'''\right\rangle\\ &=\int\!\cdots\!\int d^3\mathbf{x}_1\cdots d^3\mathbf{x}_{N-2}\,\widetilde{\psi}'''^*(\mathbf{x}_1,\ldots,\mathbf{x}_{N-2})\times\psi'''(\mathbf{x}_1,\ldots,\mathbf{x}_{N-2})\\ &=\;N(N-1)\times I_{\alpha\beta,\gamma\delta}\end{aligned} \tag{S.51}$$

where $I_{\alpha\beta,\gamma\delta}$ is exactly the same mega-integral as in eq. (S.46). Therefore,

$$\langle N,\widetilde{\psi}|\,\hat{B}^{(1)}_{\rm net}\,|N,\psi\rangle \;=\; \tfrac{1}{2}\sum_{\alpha,\beta,\gamma,\delta}B_{\alpha\beta,\gamma\delta}\times\langle N,\widetilde{\psi}|\,\hat{a}^\dagger_\alpha\hat{a}^\dagger_\beta\hat{a}_\delta\hat{a}_\gamma\,|N,\psi\rangle \;=\; \langle N,\widetilde{\psi}|\,\hat{B}^{(2)}_{\rm net}\,|N,\psi\rangle \tag{S.52}$$

where the second equality follows from eq. (5.12). $\mathcal{Q.E.D.}$

Problem **2**(f):

In the Fock space,

$$\hat{A}^{(2)}_{\rm tot} \;=\; \sum_{\mu\,nu}\langle\mu|\,\hat{A}_1\,|\nu\rangle\;\hat{a}^\dagger_\mu\hat{a}_\nu \tag{5.10}$$

and

$$\hat{B}^{(2)}_{\rm tot} \;=\; \tfrac{1}{2}\sum_{\alpha,\beta,\gamma,\delta}\langle\alpha\otimes\beta|\,\hat{B}_2\,|\gamma\otimes\delta\rangle\;\hat{a}^\dagger_\alpha\hat{a}^\dagger_\beta\hat{a}_\gamma\hat{a}_\delta\,, \tag{5.12}$$

where $\langle\alpha\otimes\beta|$ is a short-hand for the un-symmetrized two-particle wave function $(\langle\alpha|\otimes\langle\beta|)$

and likewise $|\gamma\otimes\delta\rangle = (|\gamma\rangle\otimes|\delta\rangle)$. Therefore,

$$\begin{aligned}
[\hat{A}^{(2)}_{\rm tot},\hat{B}^{(2)}_{\rm tot}] &= \tfrac{1}{2}\sum_{\mu,\nu,\alpha,\beta,\gamma,\delta} \langle\mu|\,\hat{A}_1\,|\nu\rangle\ \langle\alpha\otimes\beta|\,\hat{B}_2\,|\gamma\otimes\delta\rangle\ [\hat{a}^\dagger_\mu\hat{a}_\nu,\hat{a}^\dagger_\alpha\hat{a}^\dagger_\beta\hat{a}_\gamma\hat{a}_\delta] \\
&\qquad \langle\!\langle \text{ using eq. (S.4) } \rangle\!\rangle \\
&= \tfrac{1}{2}\sum_{\mu,\beta,\gamma,\delta} \hat{a}^\dagger_\mu\hat{a}^\dagger_\beta\hat{a}_\gamma\hat{a}_\delta \times \sum_\nu \langle\mu|\,\hat{A}_1\,|\nu\rangle\,\langle\nu\otimes\beta|\,\hat{B}_2\,|\gamma\otimes\delta\rangle \\
&\quad + \tfrac{1}{2}\sum_{\alpha,\mu,\gamma,\delta} \hat{a}^\dagger_\alpha\hat{a}^\dagger_\mu\hat{a}_\gamma\hat{a}_\delta \times \sum_\nu \langle\mu|\,\hat{A}_1\,|\nu\rangle\,\langle\alpha\otimes\nu|\,\hat{B}_2\,|\gamma\otimes\delta\rangle \\
&\quad - \tfrac{1}{2}\sum_{\alpha,\beta,\nu,\delta} \hat{a}^\dagger_\alpha\hat{a}^\dagger_\beta\hat{a}_\nu\hat{a}_\delta \times \sum_\mu \langle\alpha\otimes\beta|\,\hat{B}_2\,|\mu\otimes\delta\rangle\,\langle\mu|\,\hat{A}_1\,|\nu\rangle \\
&\quad - \tfrac{1}{2}\sum_{\alpha,\beta,\gamma,\nu} \hat{a}^\dagger_\alpha\hat{a}^\dagger_\beta\hat{a}_\gamma\hat{a}_\nu \times \sum_\mu \langle\alpha\otimes\beta|\,\hat{B}_2\,|\gamma\otimes\mu\rangle\,\langle\mu|\,\hat{A}_1\,|\nu\rangle \\
&\qquad \langle\!\langle \text{ renaming summation indices } \rangle\!\rangle \\
&= \tfrac{1}{2}\sum_{\alpha,\beta,\gamma,\delta} \hat{a}^\dagger_\alpha\hat{a}^\dagger_\beta\hat{a}_\gamma\hat{a}_\delta \times C_{\alpha,\beta,\gamma,\delta}\,,
\end{aligned} \tag{S.53}$$

where

$$\begin{aligned}
C_{\alpha,\beta,\gamma,\delta} &= \sum_\lambda \langle\alpha|\,\hat{A}_1\,|\lambda\rangle\,\langle\lambda\otimes\beta|\,\hat{B}_2\,|\gamma\otimes\delta\rangle\ +\ \sum_\lambda \langle\beta|\,\hat{A}_1\,|\lambda\rangle\,\langle\alpha\otimes\lambda|\,\hat{B}_2\,|\gamma\otimes\delta\rangle \\
&\quad -\ \sum_\lambda \langle\alpha\otimes\beta|\,\hat{B}_2\,|\lambda\otimes\delta\rangle\ \langle\lambda|\,\hat{A}_1\,|\gamma\rangle\ -\ \sum_\lambda \langle\alpha\otimes\beta|\,\hat{B}_2\,|\gamma\otimes\lambda\rangle\ \langle\lambda|\,\hat{A}_1\,|\delta\rangle \\
&= \langle\alpha\otimes\beta|\left(\hat{A}_1(1^{\rm st})\hat{B}_2\ +\ \hat{A}_1(2^{\rm nd})\hat{B}_2\ -\ \hat{B}_2\hat{A}_1(1^{\rm st})\ -\ \hat{B}_2\hat{A}_1(2^{\rm nd})\right)|\gamma\otimes\delta\rangle \\
&= \langle\alpha\otimes\beta|\left[\left(\hat{A}_1(1^{\rm st})+\hat{A}_1(2^{\rm nd})\right),\hat{B}_2\right]|\gamma\otimes\delta\rangle\ \equiv\ \langle\alpha\otimes\beta|\,\hat{C}_2\,|\gamma\otimes\delta\rangle\,.
\end{aligned} \tag{S.54}$$

Consequently, $[\hat{A}^{(2)}_{\rm tot},\hat{B}^{(2)}_{\rm tot}] = \hat{C}^{(2)}_{\rm tot}$. $\mathcal{Q.E.D.}$

Problem **3**(a):

To simplify the $\exp(\xi\hat{a}^\dagger - \xi^*\hat{a})$ in the definition of a coherent state $|\xi\rangle$, we use the product-of-exponentials formula

$$\forall\hat{A},\hat{B}:\quad e^{\hat{A}}e^{\hat{B}}\ =\ \exp\left(\hat{A}+\hat{B}\ +\ \tfrac{1}{2}[\hat{A},\hat{B}]\ +\ \tfrac{1}{12}[(\hat{A}-\hat{B}),[\hat{A},\hat{B}]]\ +\ \cdots\right). \tag{S.55}$$

In particular, for $\hat{A}=\xi\hat{a}^\dagger$, $\hat{B}=-\xi^*\hat{a}$ and $[\hat{A},\hat{B}]=\xi\xi^*$ being a c-number, all the multiple commutators vanish and

$$e^{\xi\hat{a}^\dagger}e^{-\xi^*\hat{a}} \;=\; \exp\left(\xi\hat{a}^\dagger-\xi^*\hat{a}+\tfrac{1}{2}\xi\xi^*\right), \quad \text{exactly.} \tag{S.56}$$

Consequently

$$|\xi\rangle \;\stackrel{\text{def}}{=}\; e^{\xi\hat{a}^\dagger-\xi^*\hat{a}}\,|0\rangle \;=\; e^{-|\xi|^2/2}e^{\xi\hat{a}^\dagger}e^{-\xi^*\hat{a}}\,|0\rangle \;=\; e^{-|\xi|^2/2}e^{\xi\hat{a}^\dagger}\,|0\rangle\,, \tag{S.57}$$

where the last equality follows from $\hat{a}\,|0\rangle=0$ and hence $\exp(-\xi^*\hat{a})\,|0\rangle=|0\rangle$.

Next, we saw in problem **1**(c) that $\hat{a}-\xi=e^{\xi\hat{a}^\dagger}\hat{a}e^{-\xi\hat{a}^\dagger}$. Consequently,

$$(\hat{a}-\xi)\,|\xi\rangle \;=\; e^{\xi\hat{a}^\dagger}\hat{a}e^{-\xi\hat{a}^\dagger}\times e^{-|\xi|^2/2}e^{\xi\hat{a}^\dagger}\,|0\rangle \;=\; e^{-|\xi|^2/2}e^{\xi\hat{a}^\dagger}\times\hat{a}\,|0\rangle \;=\; 0 \tag{S.58}$$

where the last equality follows from $\hat{a}\,|0\rangle=0$, and hence $\hat{a}\,|\xi\rangle=\xi\,|\xi\rangle$.

BTW, the coherent states are often *defined* by the condition $\hat{a}\,|\xi\rangle=\xi\,|\xi\rangle$. We may formally prove the existence of such a coherent state for any complex number ξ by noting that the operators $\hat{a}'=\hat{a}-\xi$ and $\hat{a}'^\dagger=\hat{a}^\dagger-\xi^*$ satisfy the same commutation relation $[\hat{a}',\hat{a}'^\dagger]=1$ as the $\hat{a}$ and $\hat{a}^\dagger$ operators. Consequently, the same formal argument that proves the existence of the ground state $|0\rangle$ annihilated by the $\hat{a}$ operator also prove the existence of the state $|\xi\rangle$ annihilated by the $\hat{a}'=\hat{a}-\xi$. Alas, the formal proof of existence does not tell us what that state looks like, so the explicit construction used in this problem provides the missing description.

Problem **3**(b):

In the coordinate basis, the annihilation operator $\hat{a}$ acts as

$$\hat{a} \;=\; \frac{\omega m\hat{x}\,+\,i\hat{p}}{\sqrt{2\hbar\omega m}} = \frac{\omega m\hat{x}\,+\,\hbar\partial_x}{\sqrt{2\hbar\omega m}} \tag{S.59}$$

and the condition $\hat{a}\,|\xi\rangle=\xi\,|\xi\rangle$ becomes a first-order differential equation

$$\left(\hbar\frac{d}{dx}\;+\;\omega m\times x\;-\;\sqrt{2\hbar\omega m}\times\xi\right)\psi_\xi(x) \;=\; 0 \tag{S.60}$$

for the wave-function $\psi_\xi(x)$ of the coherent state. This equation has a unique solution (up to an overall normalization), namely

$$\psi_\xi(x) \;=\; \text{const}\times\exp\left(\xi\sqrt{\frac{2m\omega}{\hbar}}\times x \;-\; \frac{m\omega}{2\hbar}\times x^2\right), \tag{S.61}$$

or equivalently,

$$\psi_\xi(x) \;=\; \text{const}\times e^{i\bar{p}x/\hbar}\times e^{-m\omega(x-\bar{x})^2/2\hbar}, \tag{S.62}$$

a Gaussian wave-packet with

$$\bar{x} \;=\; \sqrt{\frac{2\hbar}{\omega m}}\times\operatorname{Re}\xi \quad\text{and}\quad \bar{p} \;=\; \sqrt{2\hbar\omega m}\times\operatorname{Im}\xi. \tag{S.63}$$

Note that the width of the wave packet (S.62) does not depend on ξ, so all coherent states have the same Δx. In particular, since $|\xi=0\rangle$ is the oscillator's ground state, all coherent states have the same width as the ground state.

Problem **3**(c):

For any *normal-ordered* product of creation and annihilation operators — *i.e.*, a product in which all creation operators are to the right of all annihilation operators — one has

$$\langle\xi|\,(\hat{a}^\dagger)^k(\hat{a})^\ell\,|\xi\rangle \;=\; (\xi^*)^k\xi^\ell, \tag{S.64}$$

simply because $\hat{a}\,|\xi\rangle=\xi\,|\xi\rangle \implies (\hat{a})^\ell\,|\xi\rangle=\xi^\ell\,|\xi\rangle$ and $\langle\xi|\,\hat{a}^\dagger=\xi^*\,\langle\xi| \implies \langle\xi|\,(\hat{a}^\dagger)^k=(\xi^*)^k\,\langle\xi|$. In particular, $\langle\xi|\,(\hat{n}=\hat{a}^\dagger\hat{a})\,|\xi\rangle=\xi^*\xi$. On the other hand,

$$\hat{n}^2 \;=\; \hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a} \;=\; \hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a} \;+\; \hat{a}^\dagger\hat{a} \implies \langle\xi|\,\hat{n}^2\,|\xi\rangle \;=\; (\xi^*)^2\xi^2 \;+\; \xi^*\xi \;=\; \bar{n}^2 \;+\; \bar{n} \tag{S.65}$$

hence $\Delta n=\sqrt{\langle\hat{n}^2\rangle-\bar{n}^2}=\sqrt{\bar{n}}$.

In a similar manner,

$$\hat{x} \;=\; \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}+\hat{a}^\dagger), \qquad \hat{x}^2 \;=\; \frac{\hbar}{2m\omega}\left((\hat{a})^2+(\hat{a}^\dagger)^2+2\hat{a}^\dagger\hat{a}+1\right), \tag{S.66}$$

hence

$$\langle\xi|\,\hat{x}^2\,|\xi\rangle \;=\; \frac{\hbar}{2m\omega}\bigl((\xi+\xi^*)^2+1\bigr) \;=\; \langle\xi|\,\hat{x}\,|\xi\rangle^2 \;+\; \frac{\hbar}{2m\omega}\,. \tag{S.67}$$

Likewise,

$$\langle\xi|\,\hat{p}^2\,|\xi\rangle \;=\; \frac{m\omega\hbar}{2}\bigl((-i\xi+i\xi^*)^2+1\bigr) \;=\; \langle\xi|\,\hat{p}\,|\xi\rangle^2 \;+\; \frac{m\omega\hbar}{2}\,.$$

Altogether, this gives us for any coherent state

$$\Delta x \;=\; \sqrt{\frac{\hbar}{2m\omega}}\,, \quad \Delta p \;=\; \sqrt{\frac{m\omega\hbar}{2}}\,, \quad \Delta x\times\Delta p \;=\; \frac{\hbar}{2}\,. \tag{S.68}$$

$\mathcal{Q.E.D.}$

Problem **3**(d):
In a classical harmonic oscillator, the position $\bar{x}(t)$ and the momentum $\bar{p}(t)$ oscillate as

$$\begin{aligned} \bar{x}(t)\ \bar{x}(0)\times\cos(\omega t) \;&+\; \frac{\bar{p}(0)}{m\omega}\times\sin(\omega t),\\ \bar{p}(t)\ \bar{p}(0)\times\cos(\omega t) \;&-\; m\omega\bar{x}(0)\times\sin(\omega t). \end{aligned} \tag{S.69}$$

Consequently,

$$\begin{aligned} \xi(t) \;&=\; \frac{m\omega\bar{x}(t)\;+\;i\bar{p}(t)}{\sqrt{2\hbar\omega m}} \;=\; \frac{m\omega\bar{x}(0)+i\bar{p}(0)}{\sqrt{2\hbar\omega m}}\times\cos(\omega t) \;+\; \frac{-im\omega\bar{x}(0)+\bar{p}(0)}{\sqrt{2\hbar\omega m}}\times\sin(\omega t)\\ &=\; \frac{m\omega\bar{x}(0)+i\bar{p}(0)}{\sqrt{2\hbar\omega m}}\times e^{-i\omega t} \;=\; \xi(0)\times e^{-i\omega t}. \end{aligned} \tag{S.70}$$

Now consider the quantum state $|\xi(t)\rangle$ for the classically oscillating $\xi(t)=\xi_0\times e^{-i\omega t}$. In light of eq. (5.13),

$$|\xi(t)\rangle \;=\; e^{-|\xi|^2/2}\,e^{\xi(t)\hat{a}^\dagger}\,|0\rangle\,, \tag{S.71}$$

and only the second factor here depends on time. Indeed, $|\xi(t)|^2=|\xi_0|^2=\text{const}\implies e^{-|\xi|^2/2}=$ const, while $|0\rangle$ is time-independent because we work in the Schrödinger picture. In this picture, the $\hat{a}^\dagger$ operator is also time independent, hence

$$\frac{d}{dt}\,e^{\xi\hat{a}^\dagger} \;=\; \frac{d\xi}{dt}\,\hat{a}^\dagger\times e^{\xi\hat{a}^\dagger} \;=\; -i\omega\xi\,\hat{a}^\dagger\times e^{\xi\hat{a}^\dagger}, \tag{S.72}$$

and therefore

$$\frac{d}{dt}\,|\xi\rangle \;=\; -i\omega\xi\,\hat{a}^\dagger\,|\xi\rangle \;=\; -i\omega\,\hat{a}^\dagger\hat{a}\,|\xi\rangle \tag{S.73}$$

where the second equality follows from $\xi\,|\xi\rangle = \hat{a}\,|\xi\rangle$. Consequently,

$$i\hbar\frac{d}{dt}\,|\xi(t)\rangle \;=\; \hbar\omega\hat{a}^\dagger\hat{a}\,|\xi(t)\rangle \;\equiv\; \hat{H}\,|\xi(t)\rangle \tag{S.74}$$

— the time-dependent coherent state $|\xi(t)\rangle$ obeys the Schrödinger equation. $\mathcal{Q.E.D.}$

Problem **3**(e):

In problem **1**(c) we saw that $e^{\xi\hat{a}^\dagger}f(\hat{a}) = f(\hat{a}-\xi)e^{\xi\hat{a}^\dagger}$ for any function $f(\hat{a})$ of the annihilation operator, and in particular

$$\exp(\xi\hat{a}^\dagger)\times\exp(\eta^*\hat{a}) \;=\; \exp(\eta^*(\hat{a}-\xi))\times\exp(\xi\hat{a}^\dagger) \;=\; \exp(-\eta^*\xi)\times\exp(\eta^*\hat{a})\times\exp(\xi\hat{a}^\dagger). \tag{S.75}$$

Consequently, the quantum overlap of the coherent states $|\xi\rangle$ and $\langle\eta|$ is

$$\begin{aligned}\langle\eta|\xi\rangle \;&=\; e^{-|\eta|^2/2}e^{-|\xi|^2/2}\times\langle 0|\exp(\eta^*\hat{a})\,\exp(\xi\hat{a}^\dagger)\,|0\rangle\\ &=\; e^{-|\eta|^2/2}e^{-|\xi|^2/2}e^{+\eta^*\xi}\,\langle 0|\exp(\xi\hat{a}^\dagger)\,\exp(\eta^*\hat{a})\,|0\rangle\\ &=\; \exp\left(-\tfrac{1}{2}|\eta|^2-\tfrac{1}{2}|\xi|^2+\eta^*\xi\right)\times 1\end{aligned} \tag{S.76}$$

because $e^{\eta^*\hat{a}}\,|0\rangle = |0\rangle$, $\langle 0|\,e^{\xi\hat{a}^\dagger} = \langle 0|$, and $\langle 0|0\rangle = 1$. In terms of the probability overlap,

$$|\langle\eta|\xi\rangle|^2 \;=\; \exp(-|\eta-\xi|^2). \tag{S.77}$$

Problem **3**(f):

Generalization of coherent states to multi-oscillatory systems and further to the creation / annihilation fields is completely straightforward:

$$|\text{coherent}\rangle \stackrel{\text{def}}{=} \exp\bigl(\hat{F}^\dagger - \hat{F}\bigr)|0\rangle = e^{-\bar{N}/2}\, e^{\hat{F}^\dagger}|0\rangle \tag{S.78}$$

where

$$\hat{F}^\dagger = \xi\hat{a}^\dagger \to \sum_\alpha \xi_\alpha \hat{a}^\dagger_\alpha \to \int\! d^3\mathbf{x}\, \Phi(\mathbf{x})\hat{\Psi}^\dagger(\mathbf{x}). \tag{S.79}$$

Similar to the single-oscillator theory, $\bigl(\hat{\Psi}(\mathbf{x}) - \Phi(\mathbf{x})\bigr)e^{\hat{F}^\dagger} = e^{\hat{F}^\dagger}\hat{\Psi}^\dagger(\mathbf{x})$, hence

$$\hat{\Psi}(\mathbf{x})|\Phi\rangle = \Phi(\mathbf{x})|\Phi\rangle . \tag{S.80}$$

Problem **3**(g):

Using eq. (S.80) and its hermitian conjugate, we have

$$\langle\Phi|\,\hat{\Psi}^\dagger(\mathbf{x}_1)\cdots\hat{\Psi}^\dagger(\mathbf{x}_k)\hat{\Psi}(\mathbf{y}_1)\cdots\hat{\Psi}(\mathbf{y}_\ell)\,|\Phi\rangle = \Phi^*(\mathbf{x}_1)\cdots\Phi^*(\mathbf{x}_k)\Phi(\mathbf{y}_1)\cdots\Phi(\mathbf{y}_\ell) \tag{S.81}$$

for any *normal-ordered* product of the quantum fields. Specifically, for the particle-number operator $\hat{N}$ we have eq. (5.16), while for its square — whose normal-ordered form

$$\hat{N}^2 = \iint\! d^3\mathbf{x}\, d^3\mathbf{y}\; \hat{\Psi}^\dagger(\mathbf{x})\hat{\Psi}^\dagger(\mathbf{y})\hat{\Psi}(\mathbf{x})\hat{\Psi}(\mathbf{y}) + \int\! d^3\mathbf{x}\; \hat{\Psi}^\dagger(\mathbf{x})\hat{\Psi}(\mathbf{x}) \tag{S.82}$$

generalizes eq. (S.65) — we have

$$\langle\Phi|\,\hat{N}^2\,|\Phi\rangle = \iint\! d^3\mathbf{x}\, d^3\mathbf{y}\; \Phi^*(\mathbf{x})\Phi^*(\mathbf{y})\Phi(\mathbf{x})\Phi(\mathbf{y}) + \int\! d^3\mathbf{x}\; \Phi^*(\mathbf{x})\Phi(\mathbf{x}) = \langle\Phi|\,\hat{N}\,|\Phi\rangle^2 + \langle\Phi|\,\hat{N}\,|\Phi\rangle , \tag{S.83}$$

and hence $\Delta N = \sqrt{\bar{N}}$. $\mathcal{Q.E.D.}$

Problem **3**(h):

First of all, if $\Phi(\mathbf{x},t)$ satisfies the classical field equation (5.18) — which looks exactly like a one-particle Schrödinger equation — then $\bar{N}$ remains constant. (This is undergraduate-level QM.) Also, in the Schrödinger picture of the QFT,

$$\frac{d}{dt}\,e^{\hat{F}^\dagger} \;=\; \frac{d\hat{F}^\dagger}{dt}\,e^{\hat{F}^\dagger} \;=\; \left[\int\! d^3\mathbf{x}\,\frac{\partial\Phi(\mathbf{x},t)}{\partial t}\,\hat{\Psi}^\dagger(\mathbf{x})\right]\,e^{\hat{F}^\dagger} \tag{S.84}$$

thanks to mutual commutativity of the creation fields. Consequently, exactly as in part (e),

$$\begin{aligned} i\hbar\frac{d}{dt}\left(|\Phi\rangle = e^{-\bar{N}/2}e^{\hat{F}^\dagger}\,|0\rangle\right) \;&=\; \left[\int\! d^3\mathbf{x}\,i\hbar\frac{\partial\Phi(\mathbf{x},t)}{\partial t}\,\hat{\Psi}^\dagger(\mathbf{x})\right]|\Phi\rangle \\ \langle\!\langle\text{ using eq. (5.18) }\rangle\!\rangle & \\ &=\; \left[\int\! d^3\mathbf{x}\,\left(\left(\frac{-\hbar^2}{2M}\nabla^2+V(\mathbf{x})\right)\Phi(\mathbf{x})\right)\,\hat{\Psi}^\dagger(\mathbf{x})\right]|\Phi\rangle \\ \langle\!\langle\text{ using eq. (5.15) }\rangle\!\rangle & \\ &=\; \left[\int\! d^3\mathbf{x}\,\hat{\Psi}^\dagger(\mathbf{x})\left(\frac{-\hbar^2}{2M}\nabla^2+V(\mathbf{x})\right)\hat{\Psi}(\mathbf{x})\right]|\Phi\rangle \\ &=\; \hat{H}\,|\Phi\rangle\,. \end{aligned} \tag{S.85}$$

$\mathcal{Q.E.D.}$

Problem **3**(i):

Generalizing (e) to multi-oscillatory systems is completely straightforward:

$$|\langle\eta|\xi\rangle|^2 \;=\; \prod_\alpha e^{-|\xi_\alpha-\eta_\alpha|^2} \;=\; \exp\left(-\sum_\alpha|\xi_\alpha-\eta_\alpha|^2\right)$$

or for the field theory,

$$|\langle\Phi_1|\Phi_2\rangle|^2 \;=\; \exp\left(\int\! d^3\mathbf{x}\;|\Phi_1(\mathbf{x})-\Phi_2(\mathbf{x})|^2\right), \tag{S.86}$$

which is exponentially small for any macroscopic $\delta\Phi(\mathbf{x})=\Phi_1(x)-\Phi_2(x)$. Indeed, a *macroscopic* difference between two coherent states means (by definition) that $\delta\Phi$ affects a large number of particles, $\int|\delta\Phi|^2\gg 1$, which makes for an *exponentially* tiny overlap (S.86).