<u>Textbook Problem</u> **4.2**:

The theory in question has two scalar fields $\Phi(x)$ and $\phi(x)$ and the Lagrangian

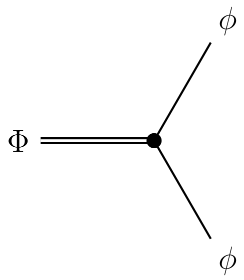
$$\mathcal{L} \;=\; \frac{1}{2}(\partial_\mu\Phi)^2 \;-\; \frac{M^2}{2}\,\Phi^2 \;+\; \frac{1}{2}(\partial_\mu\phi)^2 \;-\; \frac{m^2}{2}\,\phi^2 \;-\; \mu\Phi\phi^2, \tag{S.1}$$

where the first 4 terms on the RHS describe the free fields while the fifth term is the interaction that we treat as a perturbation. In Feynman rules, the propagators follow from the free part of the Lagrangian, so for the theory at hand there are two distinct propagators,

$$\Phi\,\text{=======}\,\Phi \;=\; \frac{i}{q^2 - m^2 + i0} \qquad \text{and} \qquad \phi\,\text{———}\,\phi \;=\; \frac{i}{q^2 - M^2 + i0}\,. \tag{S.2}$$

Likewise, there are two kinds of external lines according to the species of the incoming or outgoing particles for the process in question.

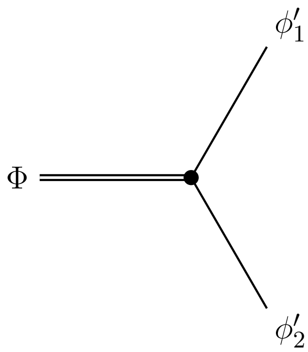
The Feynman vertices follow from the interaction part of the Lagrangian, which for the theory at hand is the cubic potential term $V_3 = \mu\Phi\phi^2$. Consequently, all vertices should be connected to three lines (net valence = 3), one double line for the one $\hat{\Phi}$ field and two single lines for the two $\hat{\phi}$ fields,



$$= \; -2i\mu, \tag{S.3}$$

where the factor $2 = 2!$ comes from the interchangeability of two identical $\hat{\phi}$ fields in the vertex.

Now consider the decay process $\Phi \to \phi + \phi$. To the lowest order of the perturbation theory, the decay amplitude follows from a single tree diagrams



(S.4)

This diagram has one vertex, one incoming double line, two outgoing single lines and no internal lines of either kind, hence

$$\left\langle \phi'_1 + \phi'_2 \right| i\hat{T} \left|\Phi\right\rangle \;\equiv\; i\mathcal{M} \times (2\pi)^4 \delta^{(4)}(p - p'_1 - p'_2) \;=\; -2i\mu \times (2\pi)^4 \delta^{(4)}(p - p'_1 - p'_2), \quad \text{(S.5)}$$

or in other words

$$\mathcal{M}(\Phi \to \phi'_1 + \phi'_2) \;=\; -2\mu. \tag{S.6}$$

This amplitude is related to the $\Phi \to \phi\phi$ decay rate as

$$\Gamma \;=\; \int |\mathcal{M}|^2 \, d\mathcal{P} \tag{S.7}$$

where the phase space factor for 1 particle $\to$ 2 particles decays is

$$\begin{aligned} d\mathcal{P} \;&=\; \frac{1}{2E} \times \frac{d^3\mathbf{p}'_1}{(2\pi)^3\, 2E'_1} \times \frac{d^3\mathbf{p}'_2}{(2\pi)^3\, 2E'_2} \times (2\pi)^4 \delta^{(4)}(p - p'_1 - p'_2) \\ &=\; \frac{1}{32\pi^2\, E E'_1 E'_2} \times d^3\mathbf{p}'_1\, \delta(E - E'_1 - E'_2) \quad \text{for} \quad \mathbf{p}'_2 \;=\; \mathbf{p} - \mathbf{p}'_1 \quad \text{and on-shell energies,} \end{aligned} \tag{S.8}$$

cf. §4.5 of the textbook. In the rest frame of the decaying Φ particle $E = M$, $\mathbf{p}'_2 = -\mathbf{p}'_1$, and $E'_1 = E'_2 = \sqrt{m^2 + \mathbf{p}'^2}$ (for equal masses of the two final-state particles), hence

$$d\mathcal{P} \;=\; \frac{d^3\mathbf{p}'}{32\pi^2\, ME'^2} \times \delta(M - 2E'(\mathbf{p}')) \;=\; \frac{p'^2\, dp'\, d^2\Omega}{32\pi^2\, ME'^2} \times \delta(M - 2E'(\mathbf{p}')). \tag{S.9}$$

To remove the remaining δ function, we integrate over the $p' = |\mathbf{p}'|$:

$$\int dp'\, \delta(M - 2E'(\mathbf{p}')) \;=\; \frac{dp'}{2dE'(\mathbf{p})}\bigg|_{\text{on shell}} \;=\; \frac{E'}{2p'}\,, \tag{S.10}$$

hence

$$d\mathcal{P} \;=\; \frac{p'}{E'} \times \frac{d^2\Omega}{64\pi^2 M} \tag{S.11}$$

where

$$\frac{p'}{E'} \;=\; \sqrt{1 \,-\, \frac{m^2}{E'^2}} \;=\; \sqrt{1 \,-\, \frac{4m^2}{M^2}} \tag{S.12}$$

since $2E' = M$ by energy conservation.

Altogether, the partial decay rate of a heavy particle of mass M into two lighter particles of equal masses $m < \frac{1}{2}M$ is

$$\frac{d\Gamma}{d^2\Omega} \;=\; \sqrt{1 \,-\, \frac{4m^2}{M^2}} \times \frac{|\mathcal{M}|^2}{64\pi^2 M}\,. \tag{S.13}$$

For the problem at hand, $\mathcal{M} = -2\mu$ regardless of directions of final particles, hence

$$\frac{d\Gamma}{d^2\Omega} \;=\; \sqrt{1 \,-\, \frac{4m^2}{M^2}} \times \frac{\mu^2}{16\pi^2 M}\,. \tag{S.14}$$

Integrating this partial decay rate over the directions of $\mathbf{p}'$ we must remember that the two final particles are identical bosons, so we cannot tell $\mathbf{p}'_1$ from $\mathbf{p}'_2 = -\mathbf{p}'_1$. Consequently, $\int d^2\Omega = 4\pi/2$ and therefore

$$\Gamma \;=\; \sqrt{1 \,-\, \frac{4m^2}{M^2}} \times \frac{\mu^2}{8\pi M}\,. \tag{S.15}$$

Textbook Problem **4.3**(a):

Similar to the previous problem, the Feynman propagators of a theory follow from the free part of its Lagrangian. This time, we have N scalar fields $\Phi^i(x)$ of similar mass m, hence in momentum space

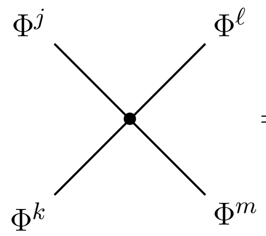
$$\Phi^j \text{———————} \Phi^k \;=\; \frac{i\delta^{jk}}{q^2 - m^2 + i0}\,. \tag{S.16}$$

Note the δ^{jk} factor — the two fields connected by a propagator must be of the same species. Graphically, this means that both ends of the propagator carry the same species label $j = k$. Likewise, the external lines should also carry a species label of the incoming or outgoing particle in question. For the external lines, these labels are fixed (for a particular process), while for the internal lines we sum over $j = 1, 2, \ldots, N$.

The Feynman vertices follow from the interactions between the fields; for the theory in question, they come form the quartic potential

$$V_4 \;=\; \frac{\lambda}{4}\Big(\Phi\cdot\Phi = \sum_j \Phi^j\Phi^j\Big)^2 = \;\sum_j \frac{\lambda}{4}\left(\hat{\Phi}^j\right)^4 \;+\; \sum_{j<k}\frac{\lambda}{2}\left(\hat{\Phi}^j\right)^2\left(\hat{\Phi}^k\right)^2. \tag{S.17}$$

Consequently, all vertices have net valence = 4, but there are two vertex types with different indexologies: (1) a vertex involving 4 lines of the same field species Φ^j, with the vertex factor $-i(\lambda/4)\times 4! = -6i\lambda$; and (2) a vertex involving 2 lines of one field species Φ^j and 2 lines of a different species Φ^k, with the vertex factor $-i(\lambda/2)\times(2!)^2 = -2i\lambda$. (The combinatorial factors arise from the interchanges of the identical fields in the same vertex, thus 4! for the first vertex type and $(2!)^2$ for the second type.) Equivalently, we may use a single vertex type involving 4 fields of whatever species, with the species-dependent vertex factor



$$= \;-2i\lambda\big(\delta^{jk}\delta^{\ell m} + \delta^{j\ell}\delta^{km} + \delta^{jm}\delta^{k\ell}\big). \tag{S.18}$$

Now consider the scattering process $\Phi^j + \Phi^k \to \Phi^\ell + \Phi^m$. At the lowest order of the perturbation theory, there is just one Feynman diagram for this process; it has one vertex, 4 external legs and no internal lines. Consequently, at the lowest order,

$$\mathcal{M}(\Phi^j + \Phi^k \to \Phi^\ell + \Phi^m) \;=\; -2\lambda\big(\delta^{jk}\delta^{\ell m} + \delta^{j\ell}\delta^{km} + \delta^{jm}\delta^{k\ell}\big) \tag{S.19}$$

independent of the particles' momenta. Specifically,

$$\begin{aligned}
\mathcal{M}(\Phi^1 + \Phi^2 \to \Phi^1 + \Phi^2) &= -2\lambda,\\
\mathcal{M}(\Phi^1 + \Phi^1 \to \Phi^2 + \Phi^2) &= -2\lambda,\\
\mathcal{M}(\Phi^1 + \Phi^1 \to \Phi^1 + \Phi^1) &= -6\lambda,
\end{aligned} \tag{S.20}$$

and consequently (using eq. (4.85) of the textbook)

$$\begin{aligned}
\frac{d\sigma(\Phi^1 + \Phi^2 \to \Phi^1 + \Phi^2)}{d\Omega_{\text{c.m.}}} &= \frac{\lambda^2}{16\pi^2 E^2_{\text{c.m.}}}\,,\\
\frac{d\sigma(\Phi^1 + \Phi^1 \to \Phi^2 + \Phi^2)}{d\Omega_{\text{c.m.}}} &= \frac{\lambda^2}{16\pi^2 E^2_{\text{c.m.}}}\,,\\
\frac{d\sigma(\Phi^1 + \Phi^1 \to \Phi^1 + \Phi^1)}{d\Omega_{\text{c.m.}}} &= \frac{9\lambda^2}{16\pi^2 E^2_{\text{c.m.}}}\,.
\end{aligned} \tag{S.21}$$

These are *partial* cross sections. To calculate the total cross sections, we integrate over $d\Omega$, which gives the factor of 4π when the two final particles are of distinct species, but for the same species, we only get 2π because of Bose statistics. Hence,

$$\begin{aligned}
\sigma_{\text{tot}}(\Phi^1 + \Phi^2 \to \Phi^1 + \Phi^2) &= \frac{\lambda^2}{4\pi E^2_{\text{c.m.}}}\,,\\
\sigma_{\text{tot}}(\Phi^1 + \Phi^1 \to \Phi^2 + \Phi^2) &= \frac{\lambda^2}{8\pi E^2_{\text{c.m.}}}\,,\\
\sigma_{\text{tot}}(\Phi^1 + \Phi^1 \to \Phi^1 + \Phi^1) &= \frac{9\lambda^2}{8\pi E^2_{\text{c.m.}}}\,.
\end{aligned} \tag{S.22}$$

Textbook Problem **4.3**(b):

The linear sigma model was discussed earlier in class. The classical potential

$$V(\Phi^2) \;=\; -\tfrac{1}{2}\mu^2(\Phi^2) \;+\; \tfrac{1}{4}\lambda(\Phi^2)^2 \tag{S.23}$$

with a negative mass term $m^2 = -\mu^2 < 0$ has a minimum (or rather a spherical shell of minima) for

$$\Phi^2 \equiv \Phi\cdot\Phi \;=\; v^2 \;=\; \frac{\mu^2}{\lambda} \;>\; 0. \tag{S.24}$$

Semi-classically, we expect a non-zero vacuum expectation value of the scalar fields, $\langle\Phi\rangle \neq 0$ with $\langle\Phi\rangle^2 = v^2$, or equivalently, $\langle\Phi^j\rangle = v\delta^{jN}$ modulo the $O(N)$ symmetry of the problem. Shifting the fields according to

$$\Phi^N(x) \;=\; v \;+\; \sigma(x), \quad \Phi^j(x) \;=\; \pi^j(x)\ (j < N), \tag{S.25}$$

and re-writing the Lagrangian in terms of the shifted fields, we obtain

$$\mathcal{L} \;=\; \tfrac{1}{2}(\partial\sigma)^2 \;-\; \mu^2\sigma^2 \;+\; \tfrac{1}{2}(\partial\underset{\sim}{\pi})^2 \;-\; \lambda v\sigma(\sigma^2+\underset{\sim}{\pi}^2) \;-\; \tfrac{1}{4}\lambda(\sigma^2+\underset{\sim}{\pi}^2)^2 +\ \text{const} \tag{S.26}$$

where $\underset{\sim}{\pi}$ stands for the $(N-1)$–plet of the π^j fields, thus $\underset{\sim}{\pi}^2 = \sum_j(\pi^j)^2$.

The first three terms on the RHS of eq. (S.26) comprise the free Lagrangian for one massive real scalar field $\sigma(x)$ of mass $m_\sigma = \sqrt{2}\mu$ and $(N-1)$ massless real scalar fields $\pi^j(x)$. (They are massless because they are Goldstone bosons of the $O(N)$ symmetry spontaneously broken down to the $O(N-1)$. There are $N-1$ broken symmetry generators, hence $N-1$ Goldstone bosons $\pi^j(x)$.) Consequently, the Feynman rules have two different propagator types

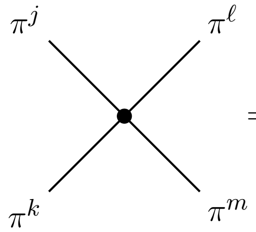
$$\sigma \;=\!=\!=\!=\!=\!=\; \sigma \quad = \quad \frac{i}{q^2-2\mu^2+i0} \qquad \text{and} \qquad \pi^j \;\text{———————}\; \pi^k \quad = \quad \frac{i\delta^{jk}}{q^2+i0}\,, \tag{S.27}$$

and the $\pi\pi$ propagator carries a label $j = k = 1, 2\ldots, (N-1)$ specifying a particular species of the pion field.

The Feynman vertices follow from the interaction terms in the Lagrangian (S.26), namely the cubic and quartic potential terms

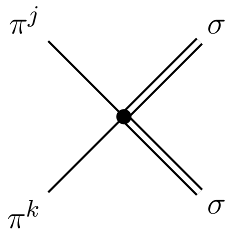
$$V_{\rm int} \;=\; \lambda v \times \hat\sigma^3 \;+\; \lambda v \times \hat\sigma \underset{\sim}{\hat\pi}^2 \;+\; \frac{\lambda}{4} \times \hat\sigma^4 \;+\; \frac{\lambda}{4} \times \hat\sigma^2 \underset{\sim}{\hat\pi}^2 \;+\; \frac{\lambda}{4} \times \big(\underset{\sim}{\hat\pi}^2\big)^2. \tag{S.28}$$

The five terms here give rise to five types of Feynman vertices, two types of valence = 3 and three types of valence = 4. The 4-valent types — which follow from the quartic terms in the potential (S.28) — are just as in part (a) of this problem modulo renaming of the fields,

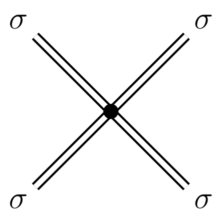


$$= \; -2i\lambda\big(\delta^{jk}\delta^{\ell m} + \delta^{j\ell}\delta^{km} + \delta^{jm}\delta^{k\ell}\big) \tag{S.29}$$

and similarly

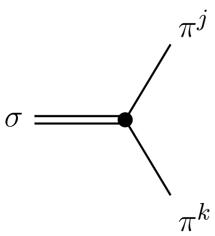


$$= \; -2i\lambda\delta^{ik} \qquad \text{and}$$

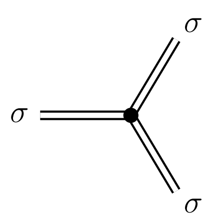


$$= \; -6i\lambda. \tag{S.30}$$

Likewise, the 3-valent vertices follow from the cubic terms in the potential (S.28); proceeding just as we did in the previous problem, we obtain



$$= \; -2i\lambda v\delta^{jk} \qquad \text{and}$$

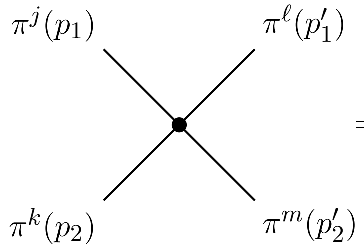


$$= \; -6i\lambda v. \tag{S.31}$$

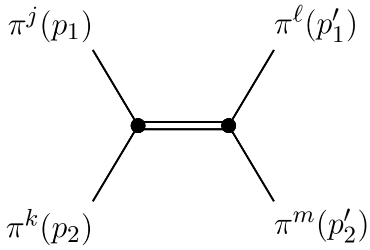
This completes the Feynman rules of the linear sigma model.

Textbook Problem **4.3**(c):

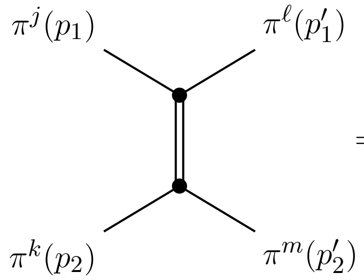
In this part of the problem, we use the Feynman rules we have just derived to calculate the tree-level $\pi\pi \to \pi\pi$ scattering amplitudes. As explained in class, a tree diagram ($L = 0$) with $E = 4$ external legs has either one valence $= 4$ vertex and no propagators, or else two valence $= 3$ vertices and one propagator. Altogether, there are four such diagrams contributing to the tree-level $i\mathcal{M}\bigl(\pi^j(p_1) + \pi^k(p_2) \to \pi^\ell(p_1') + \pi^m(p_2')\bigr)$ — they are shown in the textbook. The diagrams evaluate to:



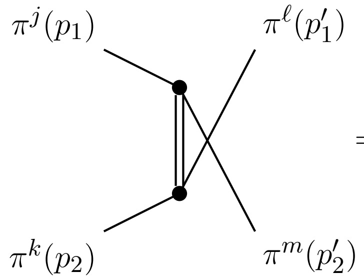
$$= -2i\lambda\bigl(\delta^{jk}\delta^{\ell m} + \delta^{j\ell}\delta^{km} + \delta^{jm}\delta^{k\ell}\bigr),$$



$$= (-2i\lambda v\delta^{jk})\,\frac{i}{s - 2\mu^2}\,(-2i\lambda v\delta^{\ell m}),$$



$$= (-2i\lambda v\delta^{j\ell})\,\frac{i}{t - 2\mu^2}\,(-2i\lambda v\delta^{km}),$$



$$= (-2i\lambda v\delta^{jm})\,\frac{i}{u - 2\mu^2}\,(-2i\lambda v\delta^{k\ell}),$$

(S.32)

where s, t, u are the Mandelstam variables

$$\begin{aligned} s &\stackrel{\text{def}}{=} (p_1+p_2)^2 \equiv (p_1'+p_2')^2, \\ t &\stackrel{\text{def}}{=} (p_1'-p_1)^2 \equiv (p_2'-p_2)^2, \\ u &\stackrel{\text{def}}{=} (p_1'-p_2)^2 \equiv (p_2'-p_1')^2. \end{aligned} \tag{S.33}$$

Each of the three 2-vertex diagrams (S.32) comes with a different combination of Kronecker δ's for the pion indices i, j, k, ℓ, while the 1-vertex diagram comprises all three combinations. Thus, arranging the net tree-level scattering amplitude by the δ's, we obtain

$$\begin{aligned} \mathcal{M}\big(\pi^j(p_1)+\pi^k(p_2)\to\pi^\ell(p_1')+\pi^m(p_2')\big) \;=\; & -2\delta^{jk}\delta^{\ell m}\left(\lambda+\frac{2\lambda^2v^2}{s-2\mu^2}\right) \\ & -2\delta^{j\ell}\delta^{km}\left(\lambda+\frac{2\lambda^2v^2}{t-2\mu^2}\right) \\ & -2\delta^{jm}\delta^{k\ell}\left(\lambda+\frac{2\lambda^2v^2}{u-2\mu^2}\right). \end{aligned} \tag{S.34}$$

Moreover, each of the three terms on the RHS may be simplifies using a relation between the cubic and quartic couplings of the shifted fields and the σ-particle's mass$^2 = 2\mu^2$. Indeed, the quartic coupling is λ and the cubic coupling is $\lambda\times v$ for $v^2=\mu^2/\lambda$, *cf.* eq. (S.24), hence

$$2\mu^2\times\lambda \;=\; 2(\lambda v)^2 \tag{S.35}$$

Thanks to this relation,

$$\lambda+\frac{2\lambda^2v^2}{s-2\mu^2} \;=\; \frac{\lambda s-2\mu^2\lambda+2\lambda^2v^2}{s-2\mu^2} \;=\; \frac{\lambda s}{s-2\mu^2} \tag{S.36}$$

and likewise

$$\lambda+\frac{2\lambda^2v^2}{t-2\mu^2} \;=\; \frac{\lambda t}{t-2\mu^2} \quad\text{and}\quad \lambda+\frac{2\lambda^2v^2}{u-2\mu^2} \;=\; \frac{\lambda u}{u-2\mu^2}\,. \tag{S.37}$$

Consequently, the amplitude (S.34) simplifies to

$$\mathcal{M} \;=\; -2\lambda\Big(\delta^{jk}\delta^{\ell m}\times\frac{s}{s-2\mu^2} \;+\; \delta^{j\ell}\delta^{km}\times\frac{t}{t-2\mu^2} \;+\; \delta^{jm}\delta^{k\ell}\times\frac{u}{u-2\mu^2}\Big). \tag{S.38}$$

Note that this amplitude vanishes in the zero-momentum limit for *any one of the four pions*, initial or final. Indeed, for the massless pions with $(p_1)^2=(p_2)^2=(p_1')^2=(p_2')^2=0$

we have

$$\begin{aligned} s &\stackrel{\text{def}}{=} (p_1+p_2)^2 \equiv (p_1'+p_2')^2 = +2(p_1p_2) = +2(p_1'p_2'),\\ t &\stackrel{\text{def}}{=} (p_1'-p_1)^2 \equiv (p_2'-p_2)^2 = -2(p_1'p_1) = -2(p_2'p_2),\\ u &\stackrel{\text{def}}{=} (p_1'-p_2)^2 \equiv (p_2'-p_1')^2 = -2(p_1p_2') = -2(p_1'p_2), \end{aligned} \tag{S.39}$$

so whenever any one of the four momenta becomes small, all three numerators in the amplitude (S.38) become small $\implies \mathcal{M} = O(\text{small } p)$.

The reason for this behavior is the **Goldstone theorem:** Among other things, it says that *all scattering amplitudes involving Goldstone particles — such as the pions in this problem — become small as $O(p_\pi)$ when* **any** *Goldstone particle's momentum p_π becomes small.* A few lines above we saw how this works for the tree-level $\langle\pi,\pi|\,\mathcal{M}\,|\pi,\pi\rangle$ amplitude (S.38); the same behavior persists at all the higher orders of the perturbation theory, but seeing how *that* works is waaay beyond the scope of this exercise.

To complete this part of the problem, let us now assume that all four pions' momenta are small compared to the σ-particle's mass $m_\sigma = \sqrt{2}\mu$. In this limit, all three denominators in eq. (S.38) are dominated by the $-2\mu^2$ term, hence

$$\mathcal{M} = \left(\frac{\lambda}{\mu^2} = \frac{1}{v^2}\right) \times \left(\delta^{jk}\delta^{\ell m} \times s + \delta^{j\ell}\delta^{km} \times t + \delta^{jm}\delta^{k\ell} \times u + O\left(\frac{p^4}{m_\sigma^2}\right)\right). \tag{S.40}$$

For generic species of the four pions, this amplitude is of the order $O(p^2/v^2)$, but there is a cancellation when all four pions belong to the same species (which is unavoidable for $N=2$). Indeed, for $j=k=\ell=m$

$$\delta^{jk}\delta^{\ell m} \times s + \delta^{j\ell}\delta^{km} \times t + \delta^{jm}\delta^{k\ell} \times u = s + t + u = 4m_\pi^2 = 0, \tag{S.41}$$

hence

$$\mathcal{M}(\pi^j + \pi^j \to \pi^j + \pi^j) = \frac{1}{v^2}\left(0 + O\left(\frac{p^4}{m_\sigma^2}\right)\right). \tag{S.42}$$

$\mathcal{Q.E.D.}$

Finally, let us translate the amplitudes (S.40) into the low-energy scattering cross sections:

$$\begin{aligned}
\frac{d\sigma(\pi^1+\pi^2\to\pi^1+\pi^2)}{d\Omega_{\text{c.m.}}} &= \frac{t^2}{64\pi^2v^2s} = \frac{E^2_{\text{c.m.}}}{64\pi^2v^4}\times\sin^4\frac{\theta_{\text{c.m.}}}{2}\,,\\
\sigma_{\text{tot}}(\pi^1+\pi^2\to\pi^1+\pi^2) &= \frac{E^2_{\text{c.m.}}}{48\pi v^4}\,,\\
\frac{d\sigma(\pi^1+\pi^1\to\pi^2+\pi^2)}{d\Omega_{\text{c.m.}}} &= \frac{s^2}{64\pi^2v^2s} = \frac{E^2_{\text{c.m.}}}{64\pi^2v^4}\,,\\
\sigma_{\text{tot}}(\pi^1+\pi^1\to\pi^2+\pi^2) &= \frac{E^2_{\text{c.m.}}}{32\pi v^4}\,,\\
\sigma(\pi^1+\pi^1\to\pi^1+\pi^1) &= \frac{O(p^8/m^4_\sigma)}{64\pi^2v^2s} = O\left(\frac{E^6_{\text{c.m.}}}{v^4m^4_\sigma}\right).
\end{aligned} \tag{S.43}$$

Textbook Problem **4.3**(d1):

Adding a linear term $\Delta V=-a\Phi^{(N)}$ to the classical potential for the N scalar fields *explicitly* breaks the $O(N)$ symmetry of the theory. Before we do anything else, we must find how this term affects the vacuum states of the theory and the masses of the σ and π fields.

Fortunately, we have already done this calculation back in homework set 6, problem 1 (solutions), so let me simply summarize the results: Instead of a spherical shell of degenerate minima, the modified potential

$$V = \frac{\lambda}{4}\times\left(\Phi^j\Phi^j\right)^2 - \frac{\mu^2}{2}\times\left(\Phi^j\Phi^j\right) - a\times\Phi^N \tag{S.44}$$

has a unique minimum at

$$\Phi = \begin{pmatrix}0\\ \vdots\\ 0\\ v\end{pmatrix} \quad \text{for} \quad v\approx\frac{\mu^2}{\lambda}+\frac{a}{2\mu^2}\,. \tag{S.45}$$

Shifting the fields as in eq. (S.25) — but for the modified VEV (S.45) — we arrive at the

Lagrangian

$$\mathcal{L} \;=\; \frac{1}{2}(\partial\sigma)^2 - \frac{m_\sigma^2}{2}\,\sigma^2 + \frac{1}{2}(\partial\underset{\sim}{\pi})^2 - \frac{m_\pi^2}{2}\,\underset{\sim}{\pi}^2 - \lambda v\times\sigma\big(\sigma^2+\underset{\sim}{\pi}^2\big) - \frac{\lambda}{4}\times\big(\sigma^2+\underset{\sim}{\pi}^2\big)^2 + \text{const} \tag{S.46}$$

which looks just like the old (S.26), except for the modified masses

$$m_\pi^2 \;=\; \lambda v^2 \;-\; \mu^2 \;=\; \frac{a}{v} \;\approx\; a\times\frac{\sqrt{\lambda}}{\mu}\,, \tag{S.47}$$

$$m_\sigma^2 \;=\; 3\lambda v^2 \;-\; \mu^2 \;=\; 2\lambda v^2 \;+\; m_\pi^2\,. \tag{S.48}$$

In particular, the pions are no longer massless Goldstone bosons, but they are still much lighter than the σ particle.

<u>Textbook Problem **4.3**(d2):</u>

Now consider the Feynman rules of the modified theory. Since the interaction terms in the Lagrangian (S.46) are exactly similar to what we had in eq. (S.26) of part (b), the Feynman vertices of the modified sigma model are exactly as in eqs. (S.29), (S.30) and (S.31), without any modification except for the slightly different value of v. On the other hand, the Feynman propagators need adjustment to accommodate the new masses (S.47) and (S.48), thus

$$\begin{aligned} \sigma \;\overline{\underline{\hspace{5em}}}\; \sigma \;&=\; \frac{i}{q^2 - m_\sigma^2 + i0}\,, \\ \pi^j \;\rule[0.5ex]{5em}{0.4pt}\; \pi^k \;&=\; \frac{i\delta^{jk}}{q^2 - m_\pi^2 + i0}\,. \end{aligned} \tag{S.49}$$

The tree-level $\pi + \pi \to \pi + \pi$ scattering amplitude is governed by the same four Feynman diagrams as before, thus

$$\begin{aligned} \mathcal{M}\big(\pi^j(p_1) + \pi^k(p_2) \to \pi^\ell(p_1') + \pi^m(p_2')\big) \;=\; &-2\delta^{jk}\delta^{\ell m}\left(\lambda \;+\; \frac{2\lambda^2 v^2}{s - m_\sigma^2}\right) \\ &-2\delta^{j\ell}\delta^{km}\left(\lambda \;+\; \frac{2\lambda^2 v^2}{t - m_\sigma^2}\right) \\ &-2\delta^{jm}\delta^{k\ell}\left(\lambda \;+\; \frac{2\lambda^2 v^2}{u - m_\sigma^2}\right), \end{aligned} \tag{S.50}$$

exactly as in eq. (S.34), except for the new v and new m_σ^2. However, instead of $m_\sigma^2 = 2\lambda v^2$

we now have

$$m_\sigma^2 \;-\; 2\lambda v^2 \;=\; \lambda v^2 \;-\; \mu^2 \;=\; m_\pi^2 \;>\; 0, \tag{S.51}$$

hence

$$\lambda + \frac{2\lambda^2 v^2}{s - m_\sigma^2} \;=\; \lambda \times \frac{s - m_\sigma^2 + \lambda v^2}{s - m_\sigma^2} \;=\; \lambda \times \frac{s - m_\pi^2}{s - m_\sigma^2} \tag{S.52}$$

and likewise

$$\lambda + \frac{2\lambda^2 v^2}{t - m_\sigma^2} \;=\; \lambda \times \frac{t - m_\pi^2}{t - m_\sigma^2}\,, \quad \lambda + \frac{2\lambda^2 v^2}{u - m_\sigma^2} \;=\; \lambda \times \frac{u - m_\pi^2}{u - m_\sigma^2}\,. \tag{S.53}$$

Therefore, instead of eq. (S.38) we now have

$$\mathcal{M} \;=\; -2\lambda\Big(\delta^{jk}\delta^{\ell m} \times \frac{s - m_\pi^2}{s - m_\sigma^2} \;+\; \delta^{j\ell}\delta^{km} \times \frac{t - m_\pi^2}{t - m_\sigma^2} \;+\; \delta^{jm}\delta^{k\ell} \times \frac{u - m_\pi^2}{u - m_\sigma^2}\Big). \tag{S.54}$$

In the low energy-momentum limit $p_i^\mu \ll m_\sigma$, this amplitude simplifies to

$$\begin{aligned}\mathcal{M} \;=\; \left(\frac{2\lambda}{m_\sigma^2} \approx \frac{1}{v^2}\right) \times \Bigg(&\delta^{jk}\delta^{\ell m}\left(s - m_\pi^2\right) \;+\; \delta^{j\ell}\delta^{km}\left(t - m_\pi^2\right) \\ &+\; \delta^{jm}\delta^{k\ell}\left(u^2 - m_\pi^2\right) \;+\; O\Big(\frac{p^4}{m_\sigma^2}\Big)\Bigg).\end{aligned} \tag{S.55}$$

In particular, for the slow pions with $p_i^0 \approx m$ while $\mathbf{p}_i \ll m_\pi$, we have $s = (E_{\rm cm} \approx 2m_\pi)^2 \approx 4m_\pi^2$ while $t, u = O(\mathbf{p}^2) \ll m_\pi^2$, so the amplitude (S.55) becomes

$$\mathcal{M} \;\approx\; \frac{m_\pi^2}{v^2} \times \left(3\delta^{jk}\delta^{\ell m} \;-\; \delta^{j\ell}\delta^{km} \;-\; \delta^{jm}\delta^{k\ell}\right). \tag{S.56}$$

This threshold amplitude does not vanish. Instead,

$$\mathcal{M} \;\sim\; \frac{m_\pi^2}{v^2} \;\approx\; \frac{a}{v^3}\,. \tag{S.57}$$

$\mathcal{Q.E.D.}$