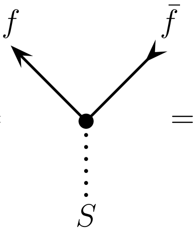
Problem **2**(a):

At the tree level, the scalar decay amplitude is simply

$$i\mathcal{M}(S\to f+\bar f) \;=\;$$  $$=\; \bar u_f(-ig)v_{\bar f}\,. \qquad \text{(S.1)}$$

Summing over spins of the outgoing fermions, we get

$$\sum|\mathcal{M}|^2 \;=\; g^2\times \mathrm{tr}\left[(\not p_1+m_f)(\not p_2-m_f)\right] \;=\; g^2\times(4p_1p_2-4m_f^2) \;=\; 2g^2\times(M_s^2-4m_f^2), \quad \text{(S.2)}$$

where the last equality follows from $2p_1p_2-2m_f^2=(p_1+p_2)^2-p_1^2-p_2^2-2m_f^2=M_s^2-4m_f^2$.

The phase space factor for one particle decaying into two — in the frame of the initial particle where the momenta of the final particles are $\pm\mathbf{p}$ — is

$$\begin{aligned} d\mathcal{P} \;&=\; \frac{1}{(2M_s)(2E_1)(2E_2)}\times\frac{d^3\mathbf{p}}{(2\pi)^3}\times(2\pi)\delta(E_1+E_2-M_s)\\ &=\; \frac{\mathbf{p}^2\,d\Omega_{\mathbf{p}}}{32\pi^2M_sE_1E_2}\times\left(\frac{d(E_1+E_2)}{d|\mathbf{p}|}\;=\;\frac{|\mathbf{p}|M_s}{E_1E_2}\right), \\ &\Downarrow\\ \mathcal{P}_{\rm net} \;&=\; \frac{|\mathbf{p}|}{8\pi M_s^2}\,. \end{aligned} \qquad \text{(S.3)}$$

Consequently, the net tree-level decay rate is

$$\Gamma(S\to f+\bar f) \;=\; \mathcal{P}\times\sum|\mathcal{M}|^2 \;=\; \frac{g^2}{4\pi}\times\frac{M_s^2-4m_f^2}{M_s^2}\times|\mathbf{p}|. \qquad \text{(S.4)}$$

By energy conservation,

$$|\mathbf{p}| \;=\; \sqrt{(\tfrac12 M_s)^2-m_f^2} \;=\; \frac{\beta M_s}{2} \quad\text{where}\quad \beta \;=\; \sqrt{1-\frac{4m_f^2}{M_s^2}}\,, \qquad \text{(S.5)}$$

so in terms of the fermions' speed β,

$$\Gamma^{\rm tree}(S\to f+\bar{f})\ =\ \frac{g^2}{8\pi}\times\beta^3 M_s\,. \tag{S.6}$$

Note that for weak Yukawa coupling $\frac{g^2}{16\pi}\ll 1$, the decay rate is small compared to the scalar's mass, $\Gamma\ll M_s$, so the resonance due to the unstable scalar should be narrow.

Problem **2**(b):

For real p^2, everything under the integral in eq. (1) is real — except for the logarithm when $\Delta(\xi)$ happens to be negative, in which case $\log=\text{real}\pm\pi i$. To determine the sign, we let $p^2=\text{real}+i\epsilon$, hence

$$\Delta\ =\ m_f^2\ -\ \xi(1-\xi)\times p^2\ =\ \text{real}\ -\ i\epsilon \tag{S.7}$$

and therefore

$$\operatorname{Im}\log\frac{4\pi m^2}{\Delta}\ =\ -\operatorname{Im}\log(\Delta-i\epsilon)\ =\ +\pi\times\Theta(\Delta<0). \tag{S.8}$$

Consequently, the imaginary part of Σ_ϕ is given by

$$\operatorname{Im}\Sigma_\phi^{\rm 1\,loop}(p^2+i\epsilon)\ =\ \frac{12g^2}{16\pi}\times\int_0^1 d\xi\,(m_f^2-\xi(1-\xi)p^2)\times\Theta(m_f^2-\xi(1-\xi)p^2<0). \tag{S.9}$$

Technically, the m_f here is the *bare* fermion mass, but at the $O(g^2)$ level of accuracy we may neglect the difference between $m_f^{\rm bare}$ and $m_f^{\rm phys}$. Consequently, the threshold for the imaginary part (S.8) lies at $p^2_{\rm min}=(2m_f^{\rm phys})^2$ — which is precisely the lowest scalar mass $(M_s^{\rm phys})^2$ that allows for decay $S\to f+\bar{f}$.

Letting $p^2=M_s^2>4m_f^2$, we have

$$\frac{m_f^2}{p^2}\ =\ \frac{1-\beta^2}{4}\qquad\Longrightarrow\qquad\Delta(\xi)\ =\ \frac{M_s^2}{4}\times\bigl(1-\beta^2-4\xi(1-\xi)\bigr)\ =\ \frac{M_s^2}{4}\times\bigl((1-2x)^2-\beta^2\bigr) \tag{S.10}$$

— which becomes negative for $\frac{1-\beta}{2}<x<\frac{1+\beta}{2}$. Consequently, the integral in eq. (S.9) evaluates to

$$\frac{M_s^2}{4}\times\int\limits_{\frac{1}{2}(1-\beta)}^{\frac{1}{2}(1+\beta)} d\xi\left[(1-2\xi)^2\,-\,\beta^2\right]\;=\;-\frac{M_s^2}{8}\times\int\limits_{-\beta}^{+\beta}d(1-2\xi)\left[\beta^2-(1-2\xi)^2\right]\;=\;-\frac{M_s^2}{8}\times\frac{4\beta^3}{3}\quad\text{(S.11)}$$

and therefore

$$\operatorname{Im}\Sigma_\phi^{1\,\text{loop}}(M_s^2+i\epsilon)\;=\;-\frac{g^2}{8\pi}\times\beta^3M_s^2\,.\tag{S.12}$$

<u>Problem **1**(c)</u>:

By inspection of eqs. (S.6) and (S.12), eq. (3) holds true:

$$\operatorname{Im}\Sigma_\phi^{1\,\text{loop}}(p^2=M_s^2+i\epsilon)\;=\;-\frac{g^2}{8\pi}\times\beta^3M_s^2\;=\;-M_s\times\Gamma^{\text{tree}}(S\to f+\bar{f}).\tag{1}$$

Higher-loop imaginary parts are similarly related to the decay rates calculated to higher orders. In the bare perturbation theory (using the bare λ_b and M_b^2 parameters and Z factors instead of the counterterms),

$$\operatorname{Im}\Sigma_\phi^{\text{bare pert. theory}}(p^2=(M_s^{\text{phys}})^2+i\epsilon)\;=\;-M_s^{\text{phys}}\times\Gamma_{\text{total}}(S\to\text{anything})\times Z_\phi;\tag{S.13}$$

in the perturbation theory using counterterms, the $\Sigma_\phi(p^2)$ amplitude has a different normalization by a $1/Z_\phi$ factor, so we have simply

$$\operatorname{Im}\Sigma_\phi^{\text{counterterm pert. theory}}(p^2=(M_s^{\text{phys}})^2+i\epsilon)\;=\;-M_s^{\text{phys}}\times\Gamma_{\text{total}}(S\to\text{anything}).\tag{S.14}$$

Eqs. (S.13) and (S.14) work in all quantum field theories. For any field $\hat{\phi}(x)$ which can create an unstable particle U of physical mass M_U and lifetime $1/\Gamma_U\gg 1/M_U$, the imaginary part of Σ_ϕ for that field satisfies

$$\begin{aligned}\operatorname{Im}\Sigma_\phi^{\text{bare pert. theory}}(p^2=(M_U^{\text{phys}})^2+i\epsilon)\;&=\;-M_U^{\text{phys}}\times\Gamma_{\text{total}}(U\to\text{anything})\times Z_\phi\,,\\ \operatorname{Im}\Sigma_\phi^{\text{counterterm pert. theory}}(p^2=(M_U^{\text{phys}})^2+i\epsilon)\;&=\;-M_U^{\text{phys}}\times\Gamma_{\text{total}}(U\to\text{anything}).\end{aligned}\tag{S.15}$$

The relation (S.15) follows from the optical theorem, which makes a narrow resonance out of any slowly-decaying particle. Consequently, the propagator of the field creating such particles should have form

$$\mathcal{F}_{\phi\phi}(p^2+i\epsilon) \;=\; \frac{iZ}{p^2\,-\,(M_U^{\rm phys})^2\,+\,iM_U^{\rm phys}\times\Gamma_{\rm tot}(U\to\text{anything})}\;+\;\text{finite} \tag{S.16}$$

for p^2 near $(M_U^{\rm phys})^2$. The perturbation theory gives this propagator as

$$\mathcal{F}_{\phi\phi}(p^2) \;=\; \frac{i}{p^2\,-\,m^2_{\rm bare}\,-\,\Sigma_\phi(p^2)}\,, \tag{S.17}$$

so to make a Breit–Wigner resonance (S.16) out of this formula, we need

$$(M_U^{\rm phys})^2\;-\;(m_\phi^{\rm bare})^2 \;=\; \mathrm{Re}\,\Sigma_\phi(p^2=(M_U^{\rm phys})^2+i\epsilon), \tag{S.18}$$

$$\frac{1}{Z_\phi} \;=\; 1\;-\;\mathrm{Re}\,\frac{d\Sigma_\phi}{dp^2}\bigg|_{p^2=(M_U^{\rm phys})^2+i\epsilon}\,, \tag{S.19}$$

$$\mathrm{Im}\,\Sigma_\phi(p^2=(M_U^{\rm phys})^2+i\epsilon) \;<\; 0 \quad \text{(this is essential!)}, \tag{S.20}$$

$$M_U^{\rm phys}\times\Gamma_{\rm tot}(U\to\text{anything})\times Z_\phi \;=\; -\,\mathrm{Im}\,\Sigma_\phi(p^2=(M_U^{\rm phys})^2+i\epsilon). \tag{S.21}$$

In addition, we also assume that $\Gamma_{\rm tot}(U)\ll M_U^{\rm phys}$ and that the imaginary part $\mathrm{Im}\,\Sigma_\phi(p^2+i\epsilon)$ does not change much for $p^2=(M_U^{\rm phys})^2\pm O(M_U^{\rm phys}\times\Gamma_{\rm tot}(U))$. If these assumptions fail, the resonance looks wide and/or deformed rather than a nice Breit–Wigner peak (S.16).

<u>Problem **3**(a):</u>

Feynman rules for the diagram (4) evaluate to

$$-i\Sigma(p^2) \;=\; \frac{(-i\lambda)^2}{3!}\int\!\frac{d^4q_1}{(2\pi)^4}\!\int\!\frac{d^4q_2}{(2\pi)^4}\,\frac{i}{q_1^2-m_i^2\epsilon}\times\frac{i}{q_2^2-m_i^2\epsilon}\times\frac{i}{q_3=p-q_1-q_2)^2-m_i^2\epsilon} \tag{S.22}$$

where the overall 1/3! factor comes from the permutation symmetry between the 3 propagators. Using Feynman's parameter tricks — specifically, eq. (F.d) from the homework set 13 — we may combine the denominators of the three propagators into a complete cube,

$$\prod_{i=1}^{3} \frac{1}{q_i^2 - m^2 + i\epsilon} \;=\; \int\limits_{\triangle} d(FP)\, \frac{2}{\mathcal{D}^3} \;\stackrel{\rm def}{=}\; \iiint\limits_{\xi,\eta,\zeta\geq 0} d\xi\, d\eta\, d\zeta\, \delta(\xi+\eta+\zeta-1) \times \frac{2}{\mathcal{D}^3} \tag{13.F.d}$$

where

$$\mathcal{D}(\xi,\eta,\zeta) \;=\; \xi\times q_1^2 \;+\; \eta\times q_2^2 \;+\; \zeta\times(q_3 = p-q_1-q_2)^2 \;-\; m^2 \;+\; i\epsilon. \tag{S.23}$$

Consequently, we may rewrite eq. (S.22) as

$$\Sigma(p^2) \;=\; -\frac{\lambda^2}{3} \int\limits_{\triangle} d(FP) \int\frac{d^4q_1}{(2\pi)^4} \int\frac{d^4q_2}{(2\pi)^4}\, \frac{1}{\mathcal{D}^3}\,. \tag{S.24}$$

Our next step is to shift the independent momentum variables from q_1 and q_2 to some k_1 and k_2 so that $\mathcal{D}$ takes sum-of-squares form (6). So let us expand the $\zeta(q_3 = p - q_1 - q_2)^2$ term in eq. (S.23) and then collect all the terms containing the q_1 momentum into a full square,

$$\begin{aligned} \mathcal{D} \;+\; m^2 \;&=\; \xi\times q_1^2 \;+\; \eta\times q_2^2 \;+\; \zeta\times(p-q_1-q_2)^2 \\ &=\; (\xi+\zeta)\times q_1^2 \;+\; 2\zeta\times q_1^\mu(q_2-p)_\mu \;+\; \zeta\times(q_2-p)^2 \;+\; \eta\times q_2^2 \\ &=\; (\xi+\zeta)\times\left(q_1 \;+\; \frac{\zeta}{\xi+\zeta}\,(q_2-p)\right)^2 \;+\; \frac{\xi\zeta}{\xi+\zeta}\times(q_2-p)^2 \;+\; \eta\times q_2^2\,. \end{aligned} \tag{S.25}$$

Naturally, we interpret the first term on the last line as $\alpha\times k_1^2$, thus

$$\alpha \;=\; (\xi+\zeta), \qquad k_1 \;=\; q_1 \;+\; \frac{\zeta}{\xi+\zeta}\times(q_2-p). \tag{S.26}$$

For the other two terms on the last line of (S.25), we expand $(q_2 - p)^2$ and collect all terms containing the q_2 momentum into another full square, thus

$$\frac{\xi\zeta}{\xi+\zeta}\times(q_2-p)^2 \;+\; \eta\times q_2^2 \;=\; \frac{\xi\zeta+\eta(\xi+\zeta)}{\xi+\zeta}\times\left(q_2\;-\;\frac{\xi\zeta}{\xi\zeta+\eta(\xi+\zeta)}\,p\right)^2 \;+\; \frac{\xi\zeta\eta}{\xi\zeta+\eta(\xi+\zeta)}\,p^2. \tag{S.27}$$

Consequently, we define

$$\beta \;=\; \frac{\xi\eta+\xi\zeta+\eta\zeta}{\xi+\zeta}\,, \qquad \gamma \;=\; \frac{\xi\eta\zeta}{\xi\eta+\xi\zeta+\eta\zeta}\,, \qquad k_2 \;=\; q_2\;-\;\frac{\xi\zeta}{\xi\eta+\xi\zeta+\eta\zeta}\times p, \tag{S.28}$$

which makes the right hand side of eq. (S.27) into $\beta\times k_2^2+\gamma\times p^2$. Altogether, we arrive at

$$\xi\times q_1^2 \;+\; \eta\times q_2^2 \;+\; \zeta\times q_3^2 \;=\; \alpha\times k_1^2 \;+\; \beta\times k_2^2 \;+\; \gamma\times p^2 \tag{S.29}$$

and hence eq. (6).

Note: the Feynman-parameter-dependent coefficients α, β, and γ in eqs. (S.26) and (S.28) are precisely as in eq. (7).

Finally, we need to check the Jacobian of replacing the original independent loop momenta q_1 and q_2 with k_1 and k_2. In light of eqs. (S.26) and (S.28), it is easy to see that

$$\frac{\partial(k_1,k_2)}{\partial(q_1,q_2)} \;=\; \det\begin{pmatrix}1 & \frac{\zeta}{\xi+\zeta}\\ 0 & 1\end{pmatrix} \;=\; 1, \tag{S.30}$$

and therefore $dk_1\,dk_2 = dq_1\,dq_2$, dimension by dimension. In other words, for fixed Feynman parameters

$$\int\!\frac{d^4q_1}{(2\pi)^4}\int\!\frac{d^4q_2}{(2\pi)^4} \;=\; \int\!\frac{d^4k_1}{(2\pi)^4}\int\!\frac{d^4k_2}{(2\pi)^4}\,, \tag{S.31}$$

and therefore

$$\Sigma^{2\,\mathrm{loop}}(p^2) \;=\; -\frac{\lambda^2}{3}\int\limits_{\triangle}\!d(FP)\iint\!\frac{d^4k_1\,d^4k_2}{(2\pi)^8}\,\frac{1}{[\mathcal{D}\;=\;\alpha k_1^2+\beta k_2^2+\gamma p^2-m^2+i\epsilon]^3}\,. \tag{S.32}$$

Problem **3**(b):

The momentum integral in eq. (S.32) has form

$$\int \frac{d^8k}{[k^2+\cdots]^3}\,, \tag{S.33}$$

which is quadratically divergent for $k\to\infty$. However, the quadratic divergence here is a p–independent constant, so it does not affect the derivative $d\Sigma/dp^2$ and hence the field strength renormalization factor Z. Instead, the derivative is only logarithmically divergent.

To see how this works, let's take d/dp^2 derivatives of both sides of eq. (S.32). On the right hand side, the only p-dependent thing is the γp^2 term in $\mathcal{D}$, hence

$$\frac{\partial\mathcal{D}}{\partial p^2} = \gamma \quad\Longrightarrow\quad \frac{\partial}{\partial p^2}\frac{1}{\mathcal{D}^3} = \frac{-3\gamma}{\mathcal{D}^4} \tag{S.34}$$

and therefore

$$\frac{d\Sigma}{dp^2} = +\lambda^2 \int\limits_{\triangle} d(FP)\,\gamma\times\iint\frac{d^4k_1\,d^4k_2}{(2\pi)^8}\,\frac{1}{\mathcal{D}^4} \tag{S.35}$$

cf. eq. (7). Here, the momentum integral has form

$$\int \frac{d^8k}{[k^2+\cdots]^4}\,, \tag{S.36}$$

so its UV divergence for $k\to\infty$ is logarithmic rather than quadratic.

Problem **3**(c–d):

Rotating both loop momenta k_1 and k_2 into the Euclidean momentum space, we have $d^4k_1 \to id^4k_1^E$, $d^4k_2\to id^4k_2^E$, and

$$\mathcal{D} \to -\alpha\times(k_1^E)^2 \;-\; \beta\times(k_2^E)^2 \;+\; \gamma\times p^2 \;-\; m^2 \tag{S.37}$$

hence

$$\frac{d\Sigma}{dp^2} = -\lambda^2\int\limits_{\triangle} d(FP)\,\gamma\times\int\frac{d^4k_1^E}{(2\pi)^4}\int\frac{d^4k_2^E}{(2\pi)^4}\,\frac{1}{[\alpha(k_1^E)^2\,+\,\beta(k_2^E)^2\,+\,m^2\,-\,\gamma p^2]^4}\,. \tag{S.38}$$

Next, we need dimensional regularization to actually perform the momentum integrals.

Changing

$$\int\frac{d^4k_1}{(2\pi)^4}\int\frac{d^4k_2}{(2\pi)^4}\ \to\ \mu^{2(4-D)}\int\frac{d^Dk_1}{(2\pi)^D}\int\frac{d^Dk_2}{(2\pi)^D} \tag{S.39}$$

(Euclidean signature for all dimensions), we have

$$\begin{aligned}
&\mu^{8-2D}\int\frac{d^Dk_1}{(2\pi)^D}\int\frac{d^Dk_2}{(2\pi)^D}\,\frac{1}{[\alpha(k_1^E)^2\,+\,\beta(k_2^E)^2\,+\,m^2\,-\,\gamma p^2]^4}\ =\\
&\qquad\langle\!\langle\text{using eq. (9)}\rangle\!\rangle\\
&\qquad=\ \frac{\mu^{8-2D}}{6}\int\frac{d^Dk_1}{(2\pi)^D}\int\frac{d^Dk_2}{(2\pi)^D}\int_0^\infty\!dt\,t^3\exp\Big(-t\times\big[\alpha(k_1^E)^2\,+\,\beta(k_2^E)^2\,+\,m^2\,-\,\gamma p^2\big]\Big)\\
&\qquad=\ \frac{\mu^{8-2D}}{6}\int_0^\infty\!dt\,t^3e^{-t(m^2-\gamma p^2)}\int\frac{d^Dk_1}{(2\pi)^D}\int\frac{d^Dk_2}{(2\pi)^D}\,e^{-t\alpha k_1^2}e^{-t\beta k_2^2}\\
&\qquad\langle\!\langle\text{using eq. (10)}\rangle\!\rangle\\
&\qquad=\ \frac{\mu^{8-2D}}{6}\int_0^\infty\!dt\,t^3e^{-t(m^2-\gamma p^2)}\times(4\pi\alpha t)^{-D/2}(4\pi\beta t)^{-D/2}\\
&\qquad=\ \frac{\mu^{8-2D}}{6(4\pi)^D(\alpha\beta)^{D/2}}\times\int_0^\infty\!dt\,t^{3-D}e^{-t(m^2-\gamma p^2)}\\
&\qquad=\ \frac{\mu^{8-2D}}{6(4\pi)^D(\alpha\beta)^{D/2}}\times\Gamma(4-D)(m^2\,-\,\gamma p^2)^{D-4}.
\end{aligned} \tag{S.40}$$

Note the $\Gamma(4-D)$ factor: It has a pole at $D=4$ but no poles at $D<4$. This is dimensional regularization's way to show that the momentum integrals diverge, but only logarithmically.

At this point, we may take $D=4-2\epsilon$ for an infinitesimally small ϵ. Hence, the last line of eq. (S.40) becomes

$$\frac{1}{6(4\pi)^4(\alpha\beta)^2}\times\Gamma(2\epsilon)\times\left(\frac{4\pi\mu^2\sqrt{\alpha\beta}}{m^2-\gamma p^2}\right)^{2\epsilon}\ \underset{\epsilon\to0}{\longrightarrow}\ \frac{1}{6(4\pi)^4(\alpha\beta)^2}\times\left(\frac{1}{2\epsilon}\,-\,\gamma_E\,+\,\log\frac{4\pi\mu^2\sqrt{\alpha\beta}}{m^2-\gamma p^2}\right). \tag{S.41}$$

Plugging this formula back into eq. (S.38) and assembling all the factors, we finally arrive at

$$\frac{d\Sigma}{dp^2} \;=\; -\frac{\lambda^2}{12(4\pi)^4}\int\limits_{\triangle} d(FP)\,\frac{\gamma}{(\alpha\beta)^2}\times\left\{\frac{1}{\epsilon}\;-\;2\gamma_E\;+\;2\log\frac{4\pi\mu^2}{m^2}\;+\;\log\frac{\alpha\beta}{[1-(p^2/m^2)\gamma]^2}\right\} \tag{S.42}$$

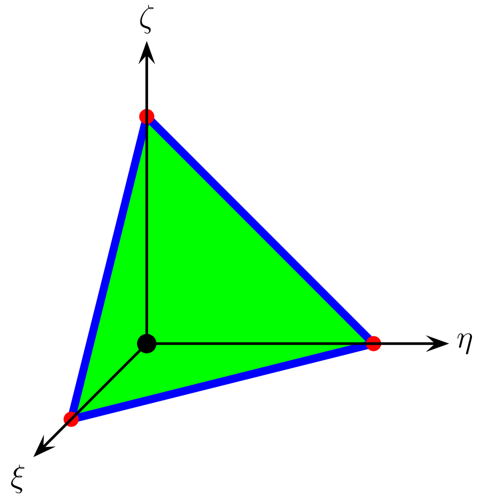
where α, β, and γ depend on the Feynman parameters ξ,η,ζ according to eq. (7). Plugging in their explicit form — and also the explicit form of the Feynman parameter integral — we immediately obtain eq. (12). $\mathcal{Q.E.D.}$

Problem **3**(e):

When a divergent diagram is regularized using DR (dimensional regularization), the $1/\epsilon$ poles could come from several places. Most commonly, they appear as $\Gamma(\epsilon)$ or $\Gamma(2\epsilon)$ factors from integrals over t–like parameters introduced to make the momentum integral Gaussian, for example see the last couple of lines of eq. (S.40). But for some diagrams — especially with nested or overlapping divergences, see §10.5 of the textbook for an example — there are additional singularities for $\epsilon\to 0$ coming from divergent integrals over the Feynman parameters.

Fortunately, this does not happen for the two-loop amplitude in question, and that's what we need to verify in this part of the problem.

We have 3 Feynman parameters ξ,η,ζ satisfying $\xi+\eta+\zeta=1$ and $\xi,\eta\zeta\geq 0$; together, they span a 2D area (since only 2 are independent) in the shape of an equilateral triangle



(S.43)

We are to verify that the functions

$$F(\xi,\eta,\zeta) \;=\; \frac{\xi\eta\zeta}{[\xi\eta+\xi\zeta+\eta\zeta]^3} \tag{S.44}$$

and

$$\begin{aligned} H(\xi,\eta,\zeta) \;&=\; F(\xi,\eta,\zeta)\times\log G(\xi,\eta,\zeta) \\ \text{for } G \;&=\; \frac{[\xi\eta+\xi\zeta+\eta\zeta]^3}{[\xi\eta+\xi\zeta+\eta\zeta\;-\;\xi\eta\zeta(p^2/m^2)]^2} \end{aligned} \tag{S.45}$$

maybe safely integrated over that area, so let's start with the $F(\xi,\eta,\zeta)$ and check it for singularities. The denominator $[\xi\eta+\xi\zeta+\eta\zeta]^3$ stays positive in the interior of the triangle (green area in fig. (S.43) where all three of ξ,η,ζ are positive) and also along the edges (blue lines where precisely one of the ξ,η,ζ becomes zero), but it vanishes in the vertices (red dots where two variables go to zero at the same time). So as far as the first integral (13) is concerned, the only potentially dangerous parts of the triangle are the vertices, all other places are completely safe.

Let's take a closer look at any one vertex (they are related by symmetry), say $\xi,\eta\to 0$ while $\zeta\approx 1$. Near this vertex

$$F \;\approx\; \frac{\xi\eta}{(\xi+\eta)^3}\,, \tag{S.46}$$

and if we approach this vertex along a line $\eta=\xi\times$ a constant, then

$$F \;\propto\; \frac{1}{\xi} \;\to\; \infty \quad \text{as } \xi\to 0. \tag{S.47}$$

This behavior would create a divergence in one-dimensional integral $\int d\xi$, but not for the 2D integral we are interested in. Indeed, let's change our coordinates according to eq. (17) and consider what happens for $w\to 0$. In this limit — which corresponds to the corner $\xi,\zeta\to 0$ — we have

$$F \;\approx\; \frac{x(1-x)}{w} \tag{S.48}$$

but the differential

$$F\,d\xi\,d\eta \;=\; F\times w\,dw\,dx \;\approx\; x(1-x)\,dx\times dw \tag{S.49}$$

remains perfectly finite for $w\to 0$, so the integral converges just fine.

Now consider the other integral (13) where we have an extra $\log G(\xi,\eta,\zeta)$ factor in the integrand. Since G is a rational function, $\log G$ does not have any singularities worse that logarithmic, and log singularities may be safely integrated over. The only potential danger comes from singularities of the $\log G$ coinciding with singularities of the F factor, so the net singularity becomes worse.

Since F's singularities lie at the 3 corner of the triangle, let's see how the G function and its log behave hear the corners. Going back to the $\xi,\eta\to 0$, $\zeta\approx 1$ corner, we have

$$G\;\approx\;\frac{(\xi+\eta)^3}{[\xi+\eta\,-\,\xi\eta(p^2/m^2)]^2}\approx\;(\xi+\eta) \tag{S.50}$$

so $\log G$ has a logarithmic singularity on top of the "pole" of F. However, in terms of the w,x coordinates, the differential

$$F\times\log G\times d\xi\,d\eta\;\approx\;x(1-x)\,dx\times\log(w)\,dw \tag{S.51}$$

has only a mild logarithmic singularity at $w\to 0$ and the integral converges.

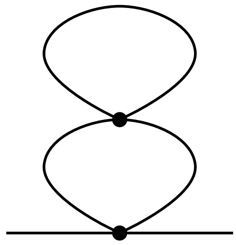
Optional problem **3**($\star$): my Mathematics code; my numeric code.

Problem **3**(f):

Having verified that the integral (12) over the Feynman parameters converges, we now face the daunting task of actually evaluating the integral. Fortunately, we do not need to evaluate its as an analytic function of the external momentum p^2 — for the purpose of calculating the field strength renormalization factor Z we are interested in only one value of p^2, namely p^2 = physical mass^2. Moreover, since we are working at the leading order of perturbation theory which contributes to the $d\Sigma/dp^2$, we may neglect the difference between the physical and the bare masses as higher-order correction and set $p^2=m^2$. Consequently, the integral (12) reduces to a combination of the integrals (16), thus

$$\frac{d\Sigma^{2\,\text{loops}}}{dp^2}\bigg|_{p^2=m^2}\;=\;-\frac{\lambda^2}{24(4\pi)^4}\times\left\{\frac{1}{\epsilon}\,-\,2\gamma_E\,+\,2\log\frac{4\pi\mu^2}{m^2}\,-\,\frac{3}{3}\right\}. \tag{S.52}$$

Note: there are two two-loop 1PI diagrams for the $\Sigma(p^2)$, namely (4) and also



(S.53)

However, the diagram (S.51) produces a p–independent Σ, so it does not contribute to the $d\Sigma/dp^2$. This means that eq. (S.52) is the entire two-loop contribution to the derivative. Also, this two-loop contribution is leading (in the power series in λ) because the one-loop contribution happens to vanish in the $\lambda\phi^4$ theory, thus

$$\left.\frac{d\Sigma^{\rm net}}{dp^2}\right|_{p^2=M^2} = -\frac{\lambda^2}{24(4\pi)^4}\times\left\{\frac{1}{\epsilon} - 2\gamma_E + 2\log\frac{4\pi\mu^2}{m^2} - \frac{3}{3}\right\} + O(\lambda^3). \tag{S.54}$$

Consequently, the field strength renormalization factor is

$$Z = \left.\frac{1}{1-\frac{d\Sigma}{dp^2}}\right|_{p^2=M^2} = 1 + \frac{\lambda^2}{6144\pi^4}\left\{\frac{1}{\epsilon} + 2\log\frac{\mu^2}{m^2} + C - \tfrac{3}{2}\right\} + O(\lambda^3). \tag{S.55}$$