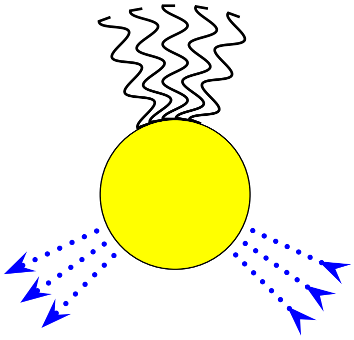
Problem **a**, part (a):

The Ward–Takahashi identities for the scalar QED have the same general form as for the fermionic QED. Take an off-shell amplitude for M incoming charged particles, M outgoing charged particles, and N photons,



$$= \; S_{MN}(p'_1,\ldots,p'_M;\,p_1,\ldots,p_M;\,k_1,\ldots,k_N), \qquad \text{(S.1)}$$

all indices suppressed. In this amplitude, all the photonic external legs are amputated, but the external legs for the charged particles are NOT amputated (the leg bubbles are allowed, and the propagators for the external legs themselves are included). Now take a similar amplitude $S^{\mu}_{M,N+1}$ involving one more photon $(\tilde{k},\mu)$ (other photon's indices suppressed). Contracting the μ index with the momentum $\tilde{k}_\mu$ of the same photon produces a linear combination of the amplitudes S_{MN} without that extra photon, specifically

$$\begin{aligned}
&\tilde{k}_\mu\times S^{\mu}_{M,N+1}(p'_1,\ldots,p'_M;\,p_1,\ldots p_M;\,k_1,\ldots,k_N,\tilde{k})\\
&\quad= \; -q\sum_{j=1}^{M} S_{M,N}(p'_1,\ldots,p'_M;\,p_1,\ldots,p_j+\tilde{k},\ldots,p_M;\,k_1,\ldots,k_N)\\
&\qquad +\; q\sum_{j=1}^{M} S_{M,N-1}(p'_1,\ldots,p'_j-\tilde{k},\ldots,p'_M;\,p_1,\ldots p_M;\,k_1,\ldots,k_N)
\end{aligned} \qquad \text{(S.2)}$$

where q is the electric charge of the scalar field. As written, the WT identities (S.2) relate the complete amplitudes $S_{M,N+1}$ and S_{MN}, which involve large numbers of Feynman diagrams. However, there are similar Ward–Takahashi identities that work on the diagram-by-diagram basis. Indeed, the simplest way to derive the WT identities (S.2) is by proving the following theorem.

Theorem: *let D be any Feynman diagram contributing to the amplitude S_{MN} and let S_D be it's contribution. Now consider all diagrams $D+\gamma$ where one more external photon is connected to one of the charged lines of the diagram D, but all the other external lines, internal lines, and vertices are exactly as in D. Let $S^\mu_{D+\gamma}$ be the net contribution of all these diagrams — but only these diagrams — to the $S^\mu_{N,M+1}$. Then*

$$\begin{aligned}
&\tilde{k}_\mu \times S^\mu_{D+\gamma}(p'_1,\ldots,p'_M;\,p_1,\ldots p_M;\,k_1,\ldots,k_N,\tilde{k}) \\
&\quad= -q\sum_{j=1}^{M} S_D(p'_1,\ldots,p'_M;\,p_1,\ldots,p_j+\tilde{k},\ldots,p_M;\,k_1,\ldots,k_N) \\
&\qquad+ q\sum_{j=1}^{M} S_D(p'_1,\ldots,p'_j-\tilde{k},\ldots,p'_M;\,p_1,\ldots p_M;\,k_1,\ldots,k_N)
\end{aligned} \tag{S.3}$$

The WT identities (S.2) immediately follow from this theorem and the decomposition of the compete amplitudes into sums over the diagrams,

$$S_{MN} = \sum_D S_D \quad\text{while}\quad S^\mu_{M,N+1} = \sum_D^{\text{similar}} S^\mu_{D+\gamma}\,. \tag{S.4}$$

Note: this theorem works equally well for the scalar QED and for the ordinary QED, although I did not prove it in my class notes. In fact, the general outline of the proof works similarly for both version of QED. The basic idea is to organize the diagrams $D+\gamma$ according to the continuous charged lines of D to which the new photon is connected, thus

$$S^\mu_{D+\gamma} = \sum_L^{\text{lines}} S^\mu_{D+\gamma}[L]. \tag{S.5}$$

For each line, we first combine all vertices and propagators *of that line*, sum over all places where the new photon is attached to this line, and only then take care of all the other factors — the other charged lines, if any, all the external attached to those line, all the internal photons connecting the charged lines to each other or to themselves, and all the momentum integrals involving those photons. In obvious notations,

$$S^\mu_{D+\gamma}[L] \;=\; \int\!\cdots\!\int d(\text{momenta})\, R^\mu_{L+\gamma} \times \prod \begin{pmatrix}\text{other}\\ \text{factors}\end{pmatrix}. \tag{S.6}$$

where $R^\mu_{L+\gamma}$ comprises all factors belonging to the line in question. Note that all the other factors — including the momentum integrals — are exactly the same as for the diagram D without the new photon, thus

$$S_D \;=\; \int\!\cdots\!\int d(\text{momenta})\, R_L \times \begin{pmatrix}\text{same other}\\ \text{factors}\end{pmatrix}. \tag{S.7}$$

Consequently, to prove the WT identities (S.3), all we need is to prove them for single charged lines — open or closed — and the corresponding factors $R^\mu_{L+\gamma}$ and R_L. Specifically, we need to prove 2 lemmas:

Lemma 1: *for an open charged line that begins at an incoming particle of momentum p and ends at an outgoing particle of momentum p',*

$$\tilde{k}_\mu \times R^\mu_{L+\gamma}(p', p) \;=\; qR_L(p' - \tilde{k}, p) \;-\; qR_L(p', p + \tilde{k}). \tag{S.8}$$

Lemma 2: *for a closed charged line, once we integrate over the loop momentum of that line (but no other momenta), we get*

$$k_\mu \times R^\mu_{L+\gamma} \;=\; 0. \tag{S.9}$$

Once we prove these two lemmas, we may plug the $R_{L+\gamma}$ and the R_L into eqs. (S.6) and (S.7) to immediately obtain

$$\begin{aligned}
&\text{for an open line } L\\
&\qquad k_\mu \times S^\mu_{D+\gamma}[L](\text{ext. momenta}) \;=\; qS_L(p'_L \to p'_L - \tilde{k}) \;-\; eS_L(p_L \to p_L + \tilde{k}),\\
&\text{for a closed line } L\\
&\qquad k_\mu \times S^\mu_{D+\gamma}[L](\text{ext. momenta}) \;=\; 0.
\end{aligned} \tag{S.10}$$

Finally, we sum over the lines L of the diagram D as in eq. (S.5). There are M open lines — each contributing 2 terms as on the RHS — plus some number of closed line, but they do not contribute anything. Altogether, we end up with $2M$ terms on the RHS,

$$\tilde{k}_\mu \times S^\mu_{D+\gamma}[\text{net}] \;=\; q\sum_{j=1}^{M} S_D(p'_j \to p'_j - \tilde{k}) \;-\; q\sum_{j=1}^{M} S_D(p_j \to p_j + \tilde{k}), \tag{S.11}$$

precisely as in eqs. (S.3).

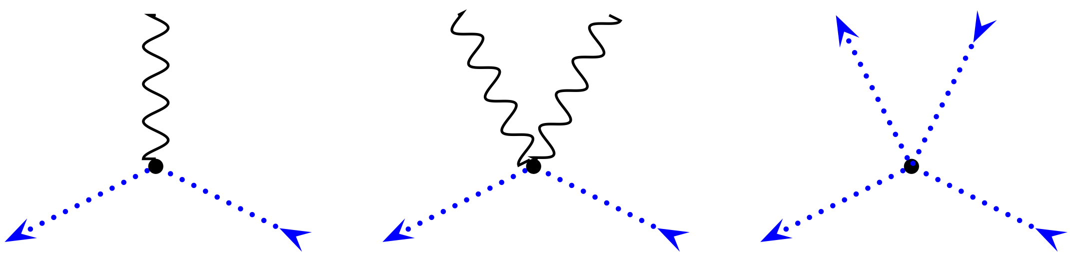
Lemmas for the fermionic QED.

In the fermionic QED, the factors R_L and $R^\mu_{L+\gamma}$ for an open line L are essentially the tree-level amplitudes with one incoming charged particle, one outgoing charged particle, and all the photons attached to this line are treated as external. (If they are internal, their propagators will be includes with the "other factors" in eqs. (S.6) and (S.7)). Consequently, Lemma 1 is equivalent to proving the tree-level Ward–Takahashi identities for the $S_{M=1,N}$ amplitudes (called simply S_N in my notes). The proof works exactly as in my notes, except that here we do not sum over permutations of the N photons we already had before adding the new photon $(\tilde{k}, \mu)$.

Likewise, for a closed line L, the factors R_L and $R^\mu_{L+\gamma}$ (integrated over the fermionic momentum) are the one-loop amplitudes for all the photons attached to L, external or internal. Consequently, Lemma 2 is equivalent to proving the TW identities for the one-loop amplitudes for the N-photon, no-fermion amplitudes $S_{M=0,N} \equiv iV_N$. Again, the proof works exactly as in my notes, except that we do not sum over permutations of the N original photons.

Lemma 1 for the scalar QED.

The scalar QED has two additional complications beyond those we suffer in the ordinary QED. First, scalar QED has 3 types of physical vertices (not counting the counterterms), namely



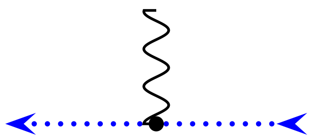
(S.12)

Second, the existance of 4–scalar vertices allows different charged lines to touch each other directly (rather than being connected by a photon propagator). And the R_L for an open line L that touches other charged lines is different from the tree-level $S_{1,N}$ amplitude for one incoming scalar and one outgoing scalar.

Nevertheless, the R_L for any open line L is a tree-level amplitude, and we prove the lemma 1 by induction in the net number of vertices (of any type) on L, just like I did it for the ordinary QED. Let's start with the induction base for $n = 0$ vertices. In this case, the R_L is simply the scalar propagator,

$$R_L(p' = p) \;=\;$$  $$=\; \frac{i}{p^2 - m^2} \qquad \text{(S.13)}$$

and there is only one way to insert a photon,

$$R^\mu_{L+\gamma}(p', p) \;=\;$$  $$=\; \frac{i}{p'^2 - m^2} \times -iq(p' + p)^\mu \times \frac{i}{p^2 - m^2}\,. \qquad \text{(S.14)}$$

Using

$$k_\mu \;=\; p'_\mu - p_\mu \quad\Longrightarrow\quad k_\mu \times (p'+p)^\mu \;=\; (p'-p)_\mu(p'+p)^\mu \;=\; p'^2 - p^2 \;=\; (p'^2 - m^2) - (p^2 - m^2), \qquad \text{(S.15)}$$

we obtain

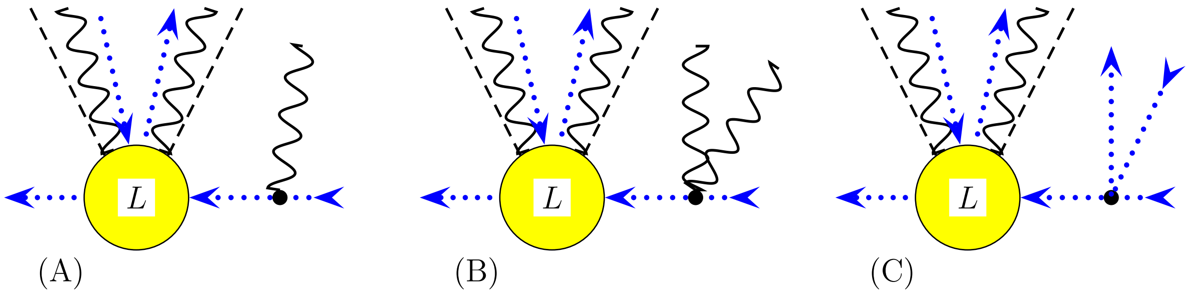
$$\begin{aligned} k_\mu \times R^\mu_{L+\gamma}(p', p) \;&=\; iq\,\frac{1}{p'^2 - m^2} \times k_\mu(p' + p)^\mu \times \frac{1}{p^2 - m^2} \\ &=\; \frac{iq}{p^2 - m^2} \;-\; \frac{iq}{p'^2 - m^2} \\ &=\; qR_L(p) \;-\; qR_L(p'), \end{aligned} \qquad \text{(S.16)}$$

which proves the Lemma 1 for $n = 0$.

Now suppose the lemma holds for open lines L of length n (the induction hypotheses), so let's use that to prove the lemma for an open line L' with one more vertex. Without loss of generality, we may put the new vertex first, so that L' comprises the incoming scalar propagator, followed by the new vertex, followed by an n-vertex line L, thus

$$R_{L'}(p',p) \;=\; R_L(p',p+\Delta p) \times \begin{pmatrix}\text{thefirst}\\ \text{vertex}\end{pmatrix} \times \frac{i}{p^2-m^2} \tag{S.17}$$

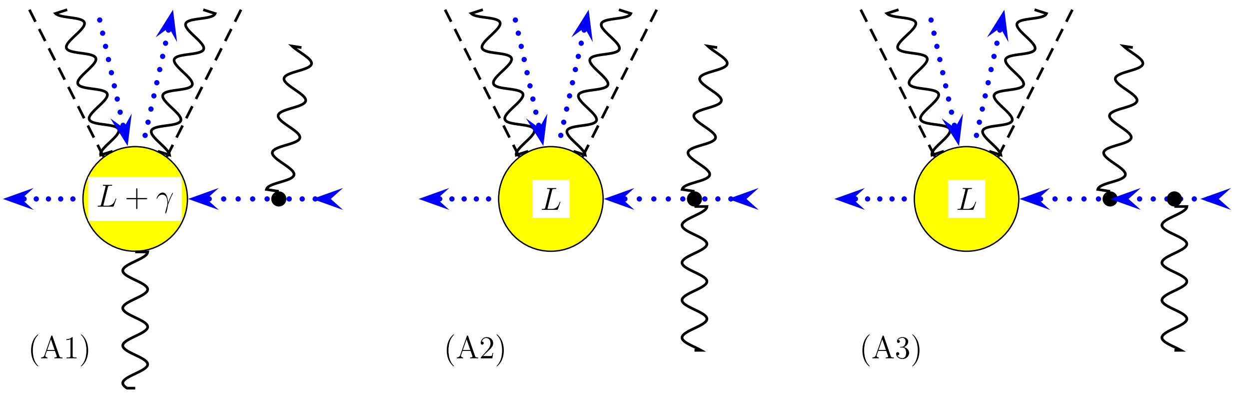
where Δp is the net momentum inflow through the first vertex. Since the scalar QED has 3 vertex types, we have to allow for 3 possibilities:



(S.18)

And for each of these three possibilities, we must consider different places where we may insert the new photon for the $L' + \gamma$.

<u>Case (A)</u>: There are 3 places where we can add one more photon to the L'_A: (1) connect somewhere within L (including the adjacent prpoagators); (2) attach to the one-photon vertex before the L; (3) split the incoming propagator. Diagrammatically,



(S.19)

thus

$$
\begin{aligned}
R^{\mu}_{L'+\gamma}(p',p) \;&=\; R^{\mu}_{L+\gamma}(p',p+k_1)\times -iq(2p+k_1)^{\nu}\times\frac{i}{p^2-m^2}\\
&+\; R_L(p',p+k_1+\tilde{k})\times 2iq^2g^{\mu\nu}\times\frac{i}{p^2-m^2}\\
&+\; R_L(p',p+k_1+\tilde{k})\times -iq(2p+2\tilde{k}+k_1)^{\nu}\times\frac{i}{(p+\tilde{k})^2-m^2}\times\\
&\qquad\qquad\times -iq(2p+\tilde{k})^{\mu}\times\frac{i}{p^2-m^2}\,.
\end{aligned}
\tag{S.20}
$$

Now let's multiply this formla by the k_μ. Using the induction hypotheses for the first term in the above formula and eq. (S.15) for the last term, we obtain

$$
\begin{aligned}
k_\mu\times R^{\mu}_{L'+\gamma}(p',p) \;&=\; \Big(k_\mu\times R^{\mu}_{L+\gamma}(p',p+k_1)\Big)\times -iq(2p+k_1)^{\nu}\times\frac{i}{p^2-m^2}\\
&+\; R_L(p',p+k_1+\tilde{k})\times\Big(2iq^2g^{\mu\nu}\tilde{k}_\mu\Big)\times\frac{i}{p^2-m^2}\\
&+\; R_L(p',p+k_1+\tilde{k})\times -iq(2p+2\tilde{k}+k_1)^{\nu}\times\\
&\qquad\times\left(\frac{i}{(p+\tilde{k})^2-m^2}\times iqk_\mu(2p+\tilde{k})^{\mu}\times\frac{i}{p^2-m^2}\right)\\
&=\; \Big(qR_L(p'-\tilde{k},p+k_1)\,-\,qR_L(p',p+k_1+\tilde{k})\Big)\times\\
&\qquad\times -iq(2p+k_1)^{\nu}\times\frac{i}{p^2-m^2}\\
&+\; R_L(p',p+k_1+\tilde{k})\times 2iq^2\tilde{k}^{\nu}\times\frac{i}{p^2-m^2}\\
&+\; R_L(p',p+k_1+\tilde{k})\times -iq(2p+2\tilde{k}+k_1)^{\nu}\times\\
&\qquad\times\left(\frac{iq}{p^2-m^2}\;-\;\frac{iq}{(p+\tilde{k})^2-m^2}\right)\\
&=\; qR_L(p'-\tilde{k},p+k_1)\times -iq(2p+k_1)^{\nu}\times\frac{i}{p^2-m^2}\\
&-\; R_L(p',p+k_1+\tilde{k})\times -iq(2p+2\tilde{k}+k_1)^{\nu}\times\frac{iq}{(p+\tilde{k})^2-m^2}\\
&\text{while the other terms cancel each other.}
\end{aligned}
\tag{S.21}
$$

On the other hand, without the extra photon

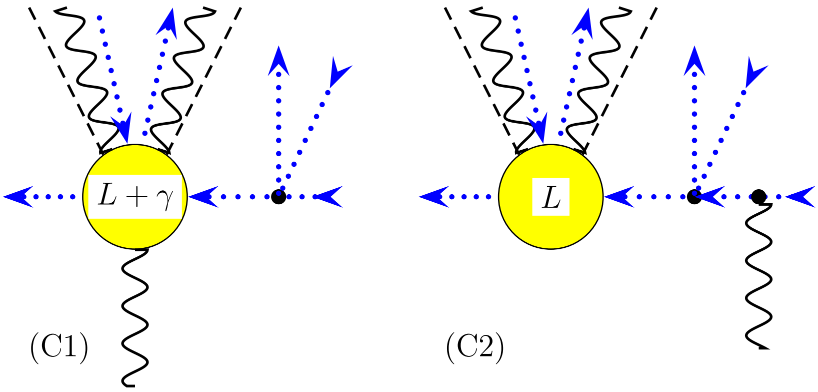
$$
R_{L'}(p',p) \;=\; qR_L(p',p+k_1)\times -iq(2p+k_1)^{\nu}\times\frac{i}{p^2-m^2}\,,
\tag{S.22}
$$

so the bottom lines of eq. (S.21) amount to

$$k_\mu \times R^\mu_{L'+\gamma}(p',p) \;=\; qR_{L'}(p'-\tilde{k},p) \;-\; qR_{L'}(p',p+\tilde{k}). \tag{S.23}$$

This proves the induction step for the case A.

Cases (B) and (C): Fortunately these two cases are simpler that (A) because one cannot attach a new photon to the seagull vertex that already has two photons or to the 4–scalar vertex. This leaves us with only two places for the new photon: (1) within L (including the adjacent propagators), or (2) split the incoming propagator. Thus, for the case (C)



(S.24)

and hence

$$\begin{aligned} R^\mu_{L'_C+\gamma}(p',p) \;&=\; R^\mu_{L+\gamma}(p',p+\Delta p)\times -i\lambda\times\frac{i}{p^2-m^2} \\ &\quad+\; R_L(p',p+\Delta p+\tilde{k})\times -i\lambda\times\frac{i}{(p+\tilde{k})^2-m^2}\times \\ &\qquad\times -iq(2p+\tilde{k})^\mu\times\frac{i}{p^2-m^2}\,. \end{aligned} \tag{S.25}$$

Multiplying this formula by the $\tilde{k}_\mu$ and proceeding as in case (A), we obtain

$$\begin{aligned} k_\mu\times R^\mu_{L'_C+\gamma}(p',p) \;&=\; \Big(k_\mu\times R^\mu_{L+\gamma}(p',p+\Delta p)\Big)\times -i\lambda\times\frac{i}{p^2-m^2} \\ &\quad+\; R_L(p',p+\Delta p+\tilde{k})\times -i\lambda\times \\ &\qquad\times\left(\frac{i}{(p+\tilde{k})^2-m^2}\times -iqk_\mu(2p+k)^\mu\times\frac{i}{p^2-m^2}\right) \end{aligned} \tag{S.26}$$

$$\begin{aligned} &= q\Big(R_L(p'-\tilde{k},p+k_1)\ -\ R_L(p',p+k_1+\tilde{k})\Big)\times -i\lambda\times\frac{i}{p^2-m^2}\\ &\quad +\ R_L(p',p+\Delta p+\tilde{k})\times -i\lambda\times\left(\frac{iq}{p^2-m^2}\ -\ \frac{iq}{(p+\tilde{k})^2-m^2}\right)\\ &= qR_L(p'-\tilde{k},p+k_1)\times -i\lambda\times\frac{i}{p^2-m^2}\\ &\quad -\ qR_L(p',p+\Delta p+\tilde{k})\times -i\lambda\times\frac{iq}{(p+\tilde{k})^2-m^2}\\ &\quad \text{while other terms cancel out.} \end{aligned} \tag{S.27}$$

At the same time, without the extra photon

$$RL'_C(p',p)\ =\ qR_L(p',p+k_1)\times -i\lambda\times\frac{i}{p^2-m^2} \tag{S.28}$$

so the bottom lines in eq. (S.27) amount to

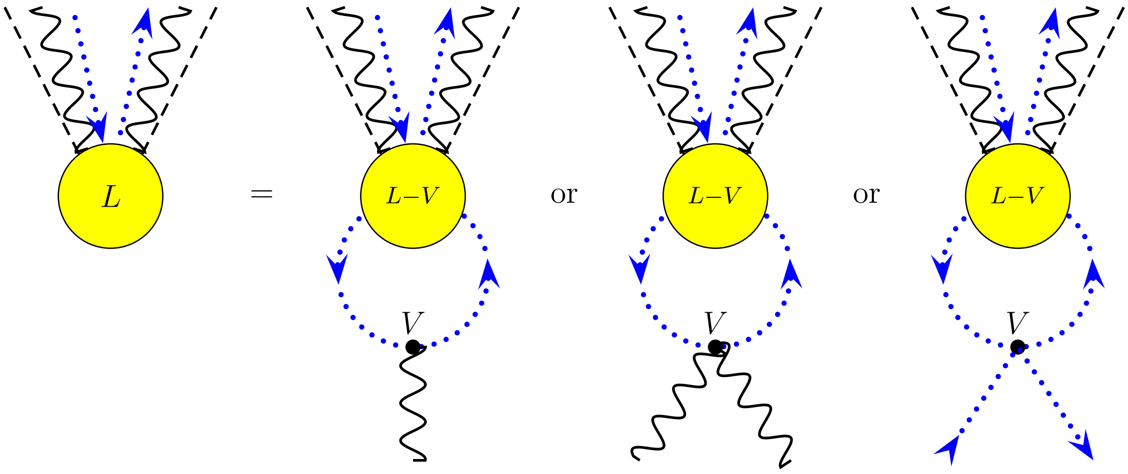
$$k_\mu\times R^\mu_{L'_C+\gamma}(p',p)\ =\ qR_{L'_C}(p'-\tilde{k},p)\ -\ qR_{L'_C}(p',p+\tilde{k}). \tag{S.29}$$

This proves the induction step for the case C.

Finally, the case (B) works similarly to the case (C), the only difference being the seagull 2-photon vertex instead of the 4-scalar vertex. Consequently, all the formulae are just as in case (C), but with the $-i\lambda$ factor replaced with the $+2iq^2g^{\nu\rho}$. But since these are overall factors, they do not affect the proof of the induction step.

<u>The bottom line</u>: We have proven the induction base and the all 3 cases of the induction steps. By induction, this proves the Lemma 1 for all open charged lines L.

Proving Lemma 2 for the scalar QED. Similrly to what I did in class for the ordinary QED, for the scalar QED Lemma 2 also follows from Lemma 1. Let L be any closed loop of charged scalar propagators. Take any vertex V on that loop, remove it, and consider the rest of the line $L-V$. Topologically, $L-V$ is an open charged line: the propagator that immediately follows V in L acts as the incoming propagator of $L-V$, and the propagator immediately preceding V in L acts as the outgoing propagator in $L-V$,



(S.30)

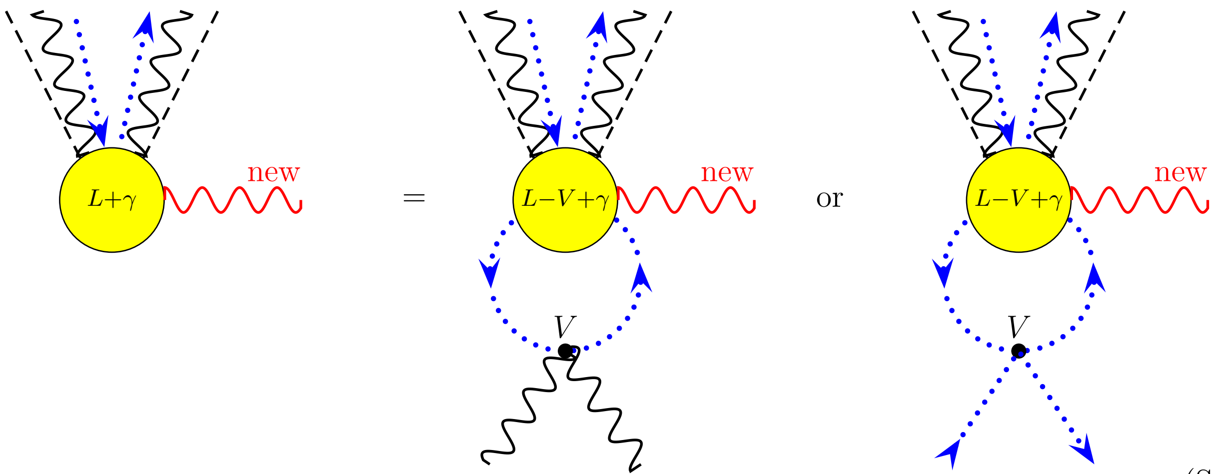
Consequenttly, evaluating the loop L we obtain

$$R_L = \int \frac{d^4p}{(2\pi)^4} R_{L-V}(p, p+\Delta p) \times F_V \qquad \text{(S.31)}$$

where F_V is the vertex factor for the vertex V; depending on the type of that vertex,

$$F_V = -iq(2p+\Delta p)^\nu, \quad \text{or} \quad F_V = +2iq^2 g^{\nu\rho}, \quad \text{or} \quad F_V = -i\lambda. \qquad \text{(S.32)}$$

Now let's attach the new photon γ to the loop L. *If V is a two-photon or 4-scalar vertex,* then it cannot take another photon, so the new photon has to attach to the rest of the loop $L-V$, thus



(S.33)

and hence

$$R^{\mu}_{L+\gamma} \;=\; \int\!\frac{d^4p}{(2\pi)^4}\,R^{\mu}_{L-V+\gamma}(p,p+\Delta p)\times F_V \tag{S.34}$$

Multiplying this loop amplitude by the $\tilde{k}_\mu$ of the new photon and using Lemma 1 for the open line $L-V+\gamma$, we obtain

$$\begin{aligned}
\tilde{k}_\mu\times R^{\mu}_{L+\gamma} \;&=\int\!\frac{d^4p}{(2\pi)^4}\,\tilde{k}_\mu\times R^{\mu}_{L-V+\gamma}(p,p+\Delta p)\times F_V\\
&=\int\!\frac{d^4p}{(2\pi)^4}\Big(qR_{L-V}(p-\tilde{k},p+\Delta p)\;-\;qR_{L-V}(p,p+\Delta p+\tilde{k})\Big)\times F_V\\
&=\;q\int\!\frac{d^4p}{(2\pi)^4}\,R_{L-V}(p-\tilde{k},p+\Delta p)\times F_V\\
&\quad-\;q\int\!\frac{d^4p}{(2\pi)^4}\,R_{L-V}(p,p+\Delta p+\tilde{k})\times F_V\,.
\end{aligned} \tag{S.35}$$

This formula assumes that V is a two-photon seagull vertex or a 4–scalar vertex. The vertex factor F_V for such a vertex does not depend on the momentum p, so the integrals on the last two lines of eq. (S.34) are related by a constant shift of the integration variable, $p\to p+\tilde{k}$. Consequently, the two integrals are equal and cancel each other, leaving us with

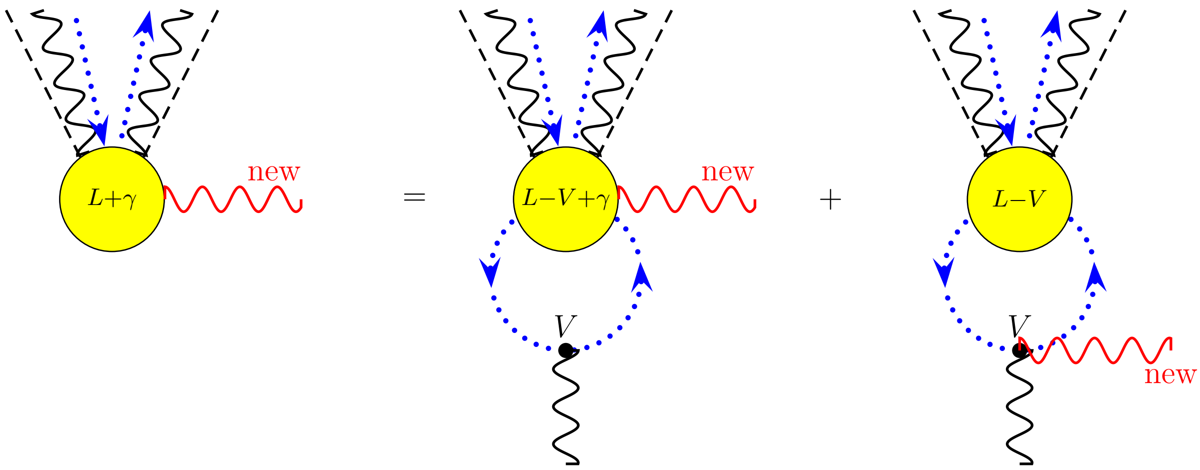
$$\tilde{k}_\mu\times R_{L+\gamma^\mu} \;=\; 0, \tag{S.36}$$

— which is precisely what we want to prove in Lemma 2.

Caveat: the last step of the argument — shifting the integration variable and subtracting the integrals — presumes that that the integrals either converge or may be regulated in a way that: (a) makes the integrals finite (for large but finite UV cutoff Λ); (b) allows shifting of the loop momentum variable; (c) does not change the Feynman rules in a way that breaks Lemma 1. Fortunately, the scalar QED has UV regulators which satisfy all 3 criteria — for example, the dimensional regularization — so the regulated integrals on the last two lines of eq. (S.35) do cancel each other.

Now suppose the bottom vertex V is a single-photon vertex, which differs from the other two vertex types in two important ways: First, the vertex factor $F_V = -iq(2p+\Delta p)^\nu$

depends on the loop momentum p, which spoils the cancelation in eq. (S.35). (Since shifting the momentum changes $F_V(p) \to F_V(-p) \neq F(p)$.) Second, besides attaching the new photon to the $L-V$ "open" line, we may also attach it to the vertex V (making it a two photon vertex), thus



(S.37)

and hence

$$R^{\mu}_{L+\gamma} = \int \frac{d^4p}{(2\pi)^4} R^{\mu}_{L-V+\gamma}(p, p+k_1) \times -iq(2p+k_1)^{\nu} + \int \frac{d^4p}{(2\pi)^4} R_{L-V}(p, p+k_1+\tilde{k}) \times 2iq^2 g^{\mu\nu} \tag{S.38}$$

where (k_1, ν) is the momentum and index of the *old* photon attached to the vertex V.

Now let's multiply both sides of eq. (S.38) by the $\tilde{k}_{\mu}$ of the new photon. Making use of Lemma 1 for the first term on the RHS, we obtain

$$
\begin{aligned}
\tilde{k}\times R^{\mu}_{L+\gamma} \;&=\; \int\!\frac{d^4p}{(2\pi)^4}\,\tilde{k}\times R^{\mu}_{L-V+\gamma}(p,p+k_1)\times -iq(2p+k_1)^{\nu}\\
&\qquad+\int\!\frac{d^4p}{(2\pi)^4}\,R_{L-V}(p,p+k_1+\tilde{k})\times 2iq^2g^{\mu\nu}\times\tilde{k}_{\mu}\\
&=\int\!\frac{d^4p}{(2\pi)^4}\Big(qR_{L-V}(p-\tilde{k},p+k_1)\;-\;qR_{L-V}(p,p+k_1+\tilde{k})\Big)\times -iq(2p+k_1)^{\nu}\\
&\qquad+\int\!\frac{d^4p}{(2\pi)^4}\,R_{L-V}(p,p+k_1+\tilde{k})\times 2iq^2\tilde{k}^{\nu}\\
&=\;-iq^2\!\int\!\frac{d^4p}{(2\pi)^4}\,R_{L-V}(p-\tilde{k},p+k_1)\times(2p+k_1)^{\nu}\\
&\qquad+iq^2\!\int\!\frac{d^4p}{(2\pi)^4}\,R_{L-V}(p,p+k_1+\tilde{k})\times\Big((2p+k_1)^{\nu}\;+\;2\tilde{k}^{\nu}\;=\;(2p+2\tilde{k}+k_1)^{\nu}\Big).
\end{aligned}
\tag{S.39}
$$

Again, the last two integrals here are related by a constant shift of the integration variable $p\to p+\tilde{k}$, so they cancel each other. Thus

$$
\tilde{k}_{\mu}\times R_{L+\gamma^{\mu}}\;=\;0,
\tag{S.40}
$$

which proves Lemma 2 for the single-photon vertex V. Again, the last argument of our proof requires convergent integrals or else a UV regulator that makes them converge while allowing momentum shifts and preserving Lemma 1, but the scalar QED has such regulators, so this is not a problem.

The bottom line is, we have proved that Lemma 1 for the open charged lines implies Lemma 2 for the closed charged loops. And given both lemmas, the Ward–Takahashi identities for the multi-loop diagrams follow in the same way as for the fermionic QED, *cf.* the first $2\frac{1}{2}$ pages of these solutions.