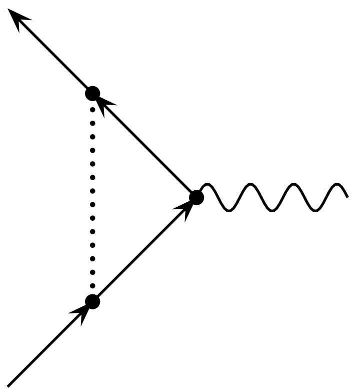
Problem **1**(a):

The form factor $F_2(q^2)$ of the muon — and hence the anomalous magnetic moment $g = 2 + 2F_2(0)$ — is a part of the 'dressed QED vertex' $ie\Gamma^\mu$, which is *net 1PI amplitude* for two on-shell muons and one off-shell photons; additional fields (besides the EM and the muon's Ψ) affect this amplitude through loop diagrams containing their propagators. For a neutral scalar Φ with a Yukawa coupling to the muon, there is a single one-loop diagram for the $ie\Gamma^\mu$(muon), namely



(S.1)

At the two-loop and higher-loop levels there are many more diagrams involving the Φ, but they are beyond the scope of this exercise.

Evaluating the one-loop diagram (S.1), we obtain

$$\begin{aligned}\Delta_S[ie\Gamma^\mu(p',p)] &= \int\limits_{\text{reg}} \frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2 - M_s^2 + i0} \times (-ig)\frac{i}{\not{p}' + \not{k} - m + i0}\,(ie\gamma^\mu)\,\frac{i}{\not{p} + \not{k} - m + i0}(-ig) \\ &= -eg^2 \int\limits_{\text{reg}} \frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2 - M_s^2 + i0} \times \frac{\not{p}' + \not{k} + m}{(p'+k)^2 - m^2 + i0}\,(ie\gamma^\mu)\,\frac{\not{p} + \not{k} + m}{(p+k)^2 - m^2 + i0} \\ &= -eg^2 \int\limits_{\text{reg}} \frac{d^4k}{(2\pi)^4}\,\frac{\mathcal{N}^\mu}{\mathcal{D}}\end{aligned} \tag{S.2}$$

where in the numerator

$$\mathcal{N}^\mu \;=\; (\not p' + \not k + m) \times \gamma^\mu \times (\not p + \not k + m) \tag{S.3}$$

and in the denominator

$$\frac{1}{\mathcal{D}} \;=\; \frac{1}{k^2 - M_s^2 + i0} \times \frac{1}{(k+p')^2 - m^2 + i0} \times \frac{1}{(k+p)^2 - m^2 + i0}\,. \tag{S.4}$$

As usual, we may combine the 3 denominator factors using the Feynman parameter trick. Proceeding similarly to the QED calculation of the Γ^μ in class — *cf.* eqs. (9) through (14) of my notes — we have

$$\begin{aligned} \frac{1}{\mathcal{D}} \;&=\; \frac{1}{k^2 - M_s^2 + i0} \times \frac{1}{(k+p')^2 - m^2 + i0} \times \frac{1}{(k+p)^2 - m^2 + i0} \\ &=\; \iiint\limits_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\,\frac{2}{(\ell^2 - \Delta + i0)^3}\,. \end{aligned} \tag{S.5}$$

where

$$\ell^2 \;-\; \Delta \;=\; z(k^2 - M_s^2) \;+\; x((k+p)^2 - m^2) \;+\; y((k+p')^2 - m^2) \tag{S.6}$$

and hence

$$\ell \;=\; k \;+\; xp \;+\; yp', \tag{S.7}$$

$$\Delta \;=\; zM_s^2 \;+\; (1-z)^2 m^2 \;-\; xyq^2, \tag{S.8}$$

assuming the on-shell muon momenta, $p'^2 = p^2 = m^2$.

Altogether, we now have

$$\Delta_S\Gamma^\mu(p',p) \;=\; 2ig^2 \iiint\limits_0^1 dx\,dy\,dz\,\delta(x+y+z-1) \int\limits_{\mathrm{reg}} \frac{d^4\ell}{(2\pi)^4}\,\frac{\mathcal{N}^\mu}{(\ell^2 - \Delta + i0)^3}\,, \tag{S.9}$$

and our next task is to simplify the numerator (S.3) in the present context. That is, we re-express $\mathcal{N}^\mu$ in terms of the shifted loop momentum ℓ, discard terms which integrate to zero (because they are odd with respect to $\ell \to -\ell$ or $x \leftrightarrow y$ symmetries), and also make use of the $\bar{u}(p')\,\Gamma^\mu\,u(p)$ context which allows us to substitute $\not{p} \to m$ in the rightmost factor and $\not{p}' \to m$ in the leftmost factor. Thus, proceeding similarly to eqs. (18) through (26) of my notes, we obtain

$$\begin{aligned}
\mathcal{N}^\mu \;&=\; (\not{k} + \not{p}' + m)\,\gamma^\mu\,(\not{k} + \not{p} + m) \\
&=\; \big((\not{\ell} - x\,\not{p} - y\,\not{p}') + \not{p}' + m\big)\,\gamma^\mu\,\big((\not{\ell} - x\,\not{p} - y\,\not{p}') + \not{p} + m\big) \\
&\quad \langle\!\langle \text{ disregarding odd powers of } \ell \,\rangle\!\rangle \\
&\cong \not{\ell}\gamma^\mu\,\not{\ell} \;+\; (\not{p}' - x\,\not{p} - y\,\not{p}' + m)\,\gamma^\mu(\not{p} - x\,\not{p} - y\,\not{p}' + m) \\
&= \not{\ell}\gamma^\mu\,\not{\ell} + \; (z\,\not{p}' + x\,\not{q} + m)\,\gamma^\mu\,(z\,\not{p} - y\,\not{q} + m) \\
&\quad \langle\!\langle \text{ between } \bar{u}(p') \text{ and } u(p) \,\rangle\!\rangle \\
&\cong \not{\ell}\gamma^\mu\,\not{\ell} \;+\; ((z+1)m + x\,\not{q})\,\gamma^\mu\,((z+1)m - y\,\not{q}) \\
&= \not{\ell}\gamma^\mu\,\not{\ell} \;+\; (z+1)^2 m^2\,\gamma^\mu \;-\; xy\,\not{q}\gamma^\mu\,\not{q} \\
&\qquad +\; (z+1)m\Big(x\,\not{q}\gamma^\mu \;-\; y\gamma^\mu\,\not{q} \;=\; (x-y)q^\mu \;+\; (x+y)i\sigma^{\mu\nu}q_\nu\Big) \\
&\cong \not{\ell}\gamma^\mu\,\not{\ell} \;+\; (z+1)^2 m^2\,\gamma^\mu \;+\; xyq^2\,\gamma^\mu + \; m(1-z)^2 \times i\sigma^{\mu\nu}q_\nu \;+\; (x-y)\times(z+1)mq^\mu
\end{aligned} \tag{S.10}$$

where the last term $(x-y)\times\cdots$ may be disregarded thanks to the $x \leftrightarrow$ symmetry of the integral over the Feynman parameters. Finally, thanks to the Lorentz symmetry of the $\int d^4\ell$ and of the denominator,

$$\not{\ell}\gamma^\mu\,\not{\ell} \;=\; \gamma^\lambda\gamma^\mu\gamma^\nu \times \ell^\lambda\ell^\nu \;\cong\; \gamma^\lambda\gamma^\mu\gamma^\nu \times g^{\lambda\nu}\,\frac{\ell^2}{D} \;=\; (2-D)\gamma^\mu \times \frac{\ell^2}{D}\,. \tag{S.11}$$

Plugging this formula into eq. (S.10), we arrive at

$$\begin{aligned}
\mathcal{N}^\mu \;&\cong\; (2-D)\gamma^\mu \times \frac{\ell^2}{D} \;+\; (z+1)^2 m^2\,\gamma^\mu \;+\; xyq^2\,\gamma^\mu + \; m(1-z)^2 \times i\sigma^{\mu\nu}q_\nu \\
&\equiv\; \mathcal{N}_1 \times \gamma^\mu \;+\; \mathcal{N}_2 \times \frac{i\sigma^{\mu\nu}q_\nu}{2m}
\end{aligned} \tag{S.12}$$

where

$$\mathcal{N}_1 \;=\; -\frac{D-2}{D}\,\ell^2 \;+\; (1+z)^2 m^2 \;+\; xyq^2 \tag{S.13}$$

$$\text{and} \quad \mathcal{N}_2 \;=\; 2(1-z^2)m^2. \tag{S.14}$$

In light of the Dirac-matrix structure of the last line of eq. (S.12), the $\mathcal{N}_1$ contributes to the F_1 form-factor of the muon while the $\mathcal{N}_2$ contributes to the F_2 form factor,

$$\begin{aligned} \Delta_S F_1(q^2) \;&=\; 2ig^2 \iiint_0^1 dx\,dy\,dz\,\delta(x+y+z-1) \int_{\text{reg}} \frac{d^4\ell}{(2\pi)^4}\,\frac{\mathcal{N}_1}{(\ell^2-\Delta+i0)^3} \\ &\qquad -\ \text{similar integral for } q^2=0 \text{ because of } \Delta_S\delta_1\,, \end{aligned} \tag{S.15}$$

$$\Delta_S F_2(q^2) \;=\; 2ig^2 \iiint_0^1 dx\,dy\,dz\,\delta(x+y+z-1) \int_{\text{reg}} \frac{d^4\ell}{(2\pi)^4}\,\frac{\mathcal{N}_2}{(\ell^2-\Delta+i0)^3}\,. \tag{S.16}$$

In this exercise, we are interested in the anomalous magnetic moment of the muon, so all we need is the F_2 for $q^2=0$, and we do not need to worry about the counterterm δ_1 because it affects only the other form factor F_1. In eq. (S.16)for the F_2, the numerator $\mathcal{N}_2$ does not depend on the loop momentum ℓ, so the $\int d^4\ell$ integral converges without any regularization, UV or IR,

$$\begin{aligned} i\int \frac{d^4\ell}{(2\pi)^4}\,\frac{1}{(\ell^2-\Delta+i0)^3} \;&=\; \int \frac{d^4\ell_E}{(2\pi)^4}\,\frac{1}{(\ell_E^2+\Delta)^3} \\ &=\; \frac{1}{16\pi^2}\int_0^2 d\ell_E^2\,\frac{\ell_E^2}{(\ell_E^2+\Delta)^3} \\ &=\; \frac{1}{16\pi^2}\times\frac{1}{2\Delta}\,. \end{aligned} \tag{S.17}$$

Consequently,

$$\Delta_S F_2(q^2) \;=\; \frac{g^2}{16\pi^2} \iiint_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\,\frac{2m^2(1-z^2)}{\Delta} \tag{S.18}$$

where Δ is as in eq. (S.8). In particular, for $q^2 = 0$ $\Delta = zM_s^2 + (1-z)^2 m^2$, hence

$$\begin{aligned}
\Delta_S \left(\frac{g_\mu - 2}{2}\right) &= \Delta_S F_2(q^2 = 0) \\
&= \frac{g^2}{16\pi^2} \iiint_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\,\frac{2m^2(1-z^2)}{zM_s^2 + (1-z)^2 m^2} \\
&= \frac{g^2}{16\pi^2} \int_0^1 dz\,(1-z) \times \frac{2m^2(1-z^2)}{zM_s^2 + (1-z)^2 m^2}\,.
\end{aligned} \tag{S.19}$$

The last integral here is a complicated function of the muon-to-scalar mass ratio m/M_s, but for the problem at hand, the scalar is much heavier than the muon. Hence, we approximate the denominator according to

$$\begin{aligned}
zM_s^2 \;+\; (1-z)^2 m^2 \;&\approx \begin{cases} zM_s^2 \;+\; 0 & \text{except when } z \approx 0 \\ zM_s^2 \;+\; m^2 & \text{for } z \approx 0 \end{cases} \\
&\approx\; zM_s^2 \;+\; m^2 \quad \text{for all } z,
\end{aligned} \tag{S.20}$$

and consequently evaluate

$$\begin{aligned}
\int_0^1 dz\,\frac{2m^2(1-z^2)(1-z)}{zM_s^2 + m^2} \;&\approx\; \frac{2m^2}{M_s^2}\int_0^1 dz\left(\frac{1}{z + (m^2/M_S^2)} \;-\; 1 \;-\; z \;+\; z^2 \;+\; O(m^2/M_s^2)\right) \\
&=\; 2\,\frac{m^2}{M_s^2}\left(\log\frac{M_s^2}{m^2} \;-\; \frac{7}{6}\right) +\; O\left(\frac{m^4}{M_s^4}\right).
\end{aligned} \tag{S.21}$$

Thus, to the leading orders in the Yukawa coupling g and in the m/M_s mass ration, the scalar's effect on the anomalous magnetic moment of the muon amounts to

$$\Delta_S g_\mu \;\approx\; \frac{g^2}{4\pi^2}\,\frac{m^2}{M_s^2}\left(\log\frac{M_s^2}{m^2} \;-\; \frac{7}{6}\right). \tag{S.22}$$

Experimentally, muon's anomalous magnetic moment agrees with the MSM (Minimal Standard Model) to 8 significant digits; beyond that, we have eqs. (1) and (2). Interpreting these equations as limits on contributions from outside the MSM — *i.e.*, as limit on the $\Delta_S g_\mu$, we have

$$\Delta_S g_\mu \ < \ 93 \cdot 10^{-10} \tag{S.23}$$

at 95% confidence level.[⋆] In light of eq. (S.22), this limit amounts to a limit on the Yukawa coupling of the scalar Φ to the muon field,

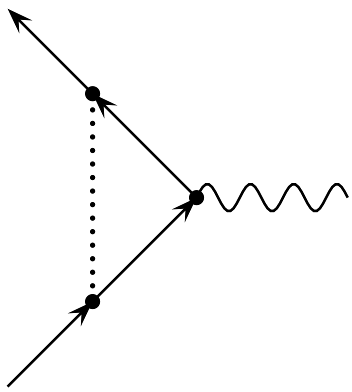
$$g \ < \ 0.3 \quad \text{for } M_S = 200\,\text{GeV} \tag{S.24}$$

or more generally

$$g \ < \ 0.3 \times \left(\frac{M_S}{200\,\text{GeV}}\right). \tag{S.25}$$

Problem **1**(b):

At the one-loop level, the axion's effect on the QED vertex of the muon follows from a single diagram



(S.26)

which looks exactly like (S.1) but evaluates differently because of different Yukawa vertices:

$$\frac{2m_\mu}{f_a}\gamma^5 \ \equiv \ g\gamma^5 \quad \text{instead of } \ -ig. \tag{S.27}$$

Also, the axion is lighter than the muon, $M_a \ll m_\mu$.

⋆ The RHS here is the central value from eq. (1) plus two sigmas, statistical and systematic errors being added in quadrature. The central value is taken from eq. (1) rather than eq. (2) because it allows for a bigger effect of the scalar Φ.

Consequently,

$$\begin{aligned}\Delta_a[ie\Gamma^\mu(p',p)] &= \int\limits_{\text{reg}}\frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-M_a^2+i0}\times(g\gamma^5)\frac{i}{\not p'+\not k-m+i0}(ie\gamma^\mu)\frac{i}{\not p+\not k-m+i0}(g\gamma^5)\\ &= -2eg^2\iiint_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\int\limits_{\text{reg}}\frac{d^4\ell}{(2\pi)^4}\,\frac{\mathcal{N}^\mu}{(\ell^2-\Delta+i0)^3}\end{aligned} \tag{S.28}$$

where the denominator is exactly as in part (a) of the problems (*cf.* eqs. (S.7) and (S.8)) except for $M_s^2\to M_a^2$, but the numerator is now

$$\begin{aligned}\mathcal{N}^\mu &= -\gamma^5\times(\not k+\not p'+m)\times\gamma^\mu\times(\not k+\not p+m)\times\gamma^5\\ &= +(\not k+\not p'-m)\times\gamma^\mu\times(\not k+\not p-m).\end{aligned} \tag{S.29}$$

As in part (a), we need to re-express this numerator in terms of the shifted loop momentum ℓ and then discard terms which integrate to zero or vanish on-shell (in the context of $\bar u'\Gamma^\mu u$). Proceeding similarly to eq. (S.12), we obtain

$$\mathcal{N}^\mu \cong \mathcal{N}_1\times\gamma^\mu + \mathcal{N}_2\times\frac{i\sigma^{\mu\nu}q_\nu}{2m} \tag{S.30}$$

$$\text{where}\quad \mathcal{N}_1 = -\frac{D-2}{D}\,\ell^2 + (1-z)^2m^2 + xyq^2 \tag{S.31}$$

$$\text{and}\quad \mathcal{N}_2 = -2(1-z)^2m^2. \tag{S.32}$$

Again, the $\mathcal{N}_1$ affects the F_1 form factor of the muon while the $\mathcal{N}_2$ affects the F_2 form factor. The anomalous magnetic moment follows from the latter, which is given by

$$\begin{aligned}\Delta_aF_2(q^2) &= 2ig^2\iiint_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\int\frac{d^4\ell}{(2\pi)^4}\,\frac{\mathcal{N}_2}{(\ell^2-\Delta+i0)^3}\\ &= \frac{g^2}{16\pi^2}\iiint_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\times\frac{\mathcal{N}_2}{\Delta},\end{aligned} \tag{S.33}$$

where the momentum integral is exactly as in eq. (S.17) because the numerator $\mathcal{N}_2$ does not depend on the loop momentum ℓ. It also depends on only one Feynman parameter — z but not x or y — and for $q^2 = 0$ so does the denominator $\Delta \to M_a^2 + (1-z)^2 m^2$. Therefore,

$$\begin{aligned}
\Delta_a \left(\frac{g_\mu - 2}{2}\right) &= \Delta_a F_2(q^2 = 0) \\
&= \frac{g^2}{16\pi^2} \iiint_0^1 dx\, dy\, dz\, \delta(x+y+z-1)\, \frac{-2m^2(1-z)^2}{zM_a^2 + (1-z)^2 m^2} \\
&= \frac{g^2}{16\pi^2} \int_0^1 dz\, (1-z) \times \frac{-2m^2(1-z)^2}{zM_a^2 + (1-z)^2 m^2}\,.
\end{aligned} \tag{S.34}$$

Unlike the scalar field in part (a) of the problem, the axion is light compared to the muon, so the approximation (S.20) does not apply here. Instead, for $M_a \ll m_\mu$ we simply neglect the axion's mass in the denominator of eq. (S.34),

$$\int_0^1 dz\, (1-z) \times \frac{-2m^2(1-z)^2}{zM_a^2 + (1-z)^2 m^2} \approx \int_0^1 dz\, (1-z) \times \frac{-2m^2(1-z)^2}{(1-z)^2 m^2} = -1 \tag{S.35}$$

and hence

$$\Delta_a g_\mu \approx -\frac{g^2}{8\pi^2} = -\frac{m_\mu^2}{2\pi^2 f_a^2}\,. \tag{S.36}$$

Because the negative sign of the axion's effect on the muon's magnetic moment is opposite from the sign of discrepancy (1) between the experiment and the Minimal Standard Model, a theory made out of MSM plus an axion plus nothing else seems to be ruled out. However, if we use the alternative method for calculating the hadronic loops in MSM which leads to eq. (2) instead of eq. (1), then we have a little room for a negative contribution of the axion as long as

$$\Delta_a g_\mu > -20 \cdot 10^{-10}. \tag{S.37}$$

In light of eq. (S.36), this leads to an upper limit on the axion's Yukawa coupling and hence to a lower limit on the axion's scale,

$$g < 2 \cdot 10^{-4} \implies f_a > 10^4\, m_\mu \approx 1000\,\text{GeV}. \tag{S.38}$$

Problem **2**:

The δ_2 and δ_m counterterms of QED are related to the electron's self-energy correction

$$\Sigma^{\rm tot}(\not{p}) \;=\; \Sigma^{\rm loops}(\not{p}) \;+\; \delta_m \;-\; \delta_2\times\not{p} \tag{S.39}$$

which satisfies renormalization conditions

$$\textbf{for}\quad \not{p}=m,\quad \textbf{both}\quad \Sigma^{\rm tot}=0 \quad\textbf{and}\quad \frac{d\Sigma^{\rm tot}}{d\not{p}}=0. \tag{S.40}$$

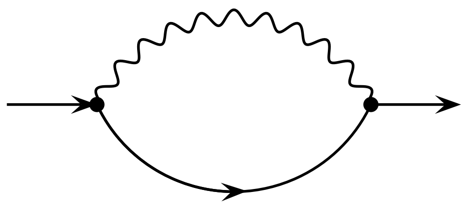
Thanks to these conditions,

$$\delta_2 \;=\; \frac{d\Sigma^{\rm loops}}{d\not{p}}\bigg|_{\not{p}\,=\,m} \tag{S.41}$$

and also

$$\delta_m \;=\; m\times\delta_2 \;-\; \Sigma^{\rm loops}\Big|_{\not{p}\,=\,m}\,. \tag{S.42}$$

At the one loop level of analysis, there is only one diagram



(S.43)

for the electron's self energy, which yields

$$-i\Sigma^{1\,\rm loop}(\not{p}) \;=\; \int\limits_{\rm reg}\frac{d^4k}{(2\pi)^4}\,\frac{-ig^{\lambda\nu}}{k^2+i0}\times ie\gamma_\lambda\,\frac{i}{\not{k}+\not{p}-m_e+i0}\times ie\gamma_\nu\,. \tag{S.44}$$

For consistency with the calculation of the δ_1 counterterm in the notes I distributed in class, we need to use the same regulators here: a tiny photon's mass m_γ to regulate the infrared divergence, and dimension $D < 4$ to regulate the UV divergence. Thus,

$$\begin{aligned}\Sigma^{1\,\mathrm{loop}}(\not{p}) &= i\mu^{4-D}\int\frac{d^Dk}{(2\pi)^D}\,\frac{-ig^{\lambda\nu}}{k^2-m_\gamma^2+i0}\times ie\gamma_\lambda\,\frac{i}{\not{k}+\not{p}-m_e+i0}\times ie\gamma_\nu\\ &= -ie^2\mu^{4-D}\int\frac{d^Dk}{(2\pi)^D}\,\frac{\mathcal{N}}{\mathcal{D}}\end{aligned}\tag{S.45}$$

where the numerator is

$$\mathcal{N} \;=\; \gamma^\nu(\not{k}+\not{p}+m_e)\gamma_\nu \;=\; Dm_e \;-\; (D-2)(\not{p}+\not{k})\tag{S.46}$$

and the denominator is

$$\frac{1}{\mathcal{D}} \;=\; \frac{1}{k^2-m_\gamma^2+i0}\times\frac{1}{(k+p)^2-m_e^2+i0} \;=\int\limits_0^1 dx\,\frac{1}{(\ell^2-\Delta+i0)^2}\tag{S.47}$$

for

$$\ell^2 \;-\; \Delta \;=\; (1-x)(k^2-m_\gamma^2) \;+\; x((k+p)^2-m_e^2)\tag{S.48}$$

and hence

$$\ell \;=\; k \;+\; xp,\tag{S.49}$$

$$\Delta \;=\; xm_e^2 \;-\; x(1-x)p^2 \;+\; (1-x)m_\gamma^2\,.\tag{S.50}$$

As usual, we re-express the numerator (S.46) in terms of the shifted loop momentum ℓ and then discard odd powers of ℓ, thus

$$\mathcal{N} \;=\; Dm_e \;-\; (D-2)(\not{\ell}-x\,\not{p}+\not{p}) \;\cong\; Dm_e \;-\; (D-2)(1-x)\,\not{p},\tag{S.51}$$

and therefore

$$\Sigma^{1\,\mathrm{loop}}(\not{p}) \;=\; e^2\int\limits_0^1 dx\,\Big(Dm_e \,-\, (D-2)(1-x)\,\not{p}\Big)\times\int\limits_{\mathrm{reg}}\frac{d^4\ell}{(2\pi)^4}\,\frac{-i\mu^{4-D}}{(\ell^2-\Delta+i0)^2}\,.\tag{S.52}$$

The momentum integral here should be familiar to you by now, so let me simply write it down,

$$\frac{d^4\ell}{(2\pi)^4}\,\frac{-i\mu^{4-D}}{(\ell^2-\Delta+i0)^2} \;=\; \frac{\mu^{4-D}}{(4\pi)^{D/2}}\,\Gamma\left(2-\tfrac{D}{2}\right)\Delta^{\frac{D}{2}-2} \;=\; \frac{1}{16\pi^2}\,\Gamma(\epsilon)\,\left(\frac{4\pi\mu^2}{\Delta}\right)^{\epsilon}. \tag{S.53}$$

Consequently,

$$\Sigma^{1\,\text{loop}}(\not{p}) \;=\; \frac{\alpha}{2\pi}\,\Gamma(\epsilon)(4\pi\mu^2)^{\epsilon}\int\limits_0^1 dx\,\frac{(2-\epsilon)m_e\;-\;(1-\epsilon)(1-x)\,\not{p}}{\Delta^{\epsilon}(z)}\,. \tag{S.54}$$

In the absence of the IR regulator (*i.e.*, for $m_\gamma^2 = 0$), the integral (S.54) converges for $\epsilon < 1$ (*i.e.*, for $D > 2$) and off-shell momenta $p^2 < m_e^2$. Consequently, $\Sigma(\not{p})$ can be analytically continued to any complex D and $\not{p}$ and the IR regulator seems un-necessary. Unfortunately, this continuation has a mild (for small ϵ) singularity for $p^2 = m_e^2$: The $\Sigma(\not{p})$ is continuous but the derivative $d\Sigma/d\not{p}$ becomes infinite. Indeed, taking the derivative of eq. (S.54) with respect to $\not{p}$, we have

$$\frac{d\Sigma^{1\,\text{loop}}}{d\not{p}} \;=\; \frac{\alpha}{2\pi}\,\Gamma(\epsilon)(4\pi\mu^2)^{\epsilon}\int\limits_0^1 dx\left(\frac{-(1-\epsilon)(1-x)}{\Delta^{\epsilon}} \;-\; \epsilon\frac{(2-\epsilon)m_e-(1-\epsilon)(1-x)\not{p}}{\Delta^{1+\epsilon}}\times \times\left[\frac{\partial\Delta}{\partial\not{p}} \;=\; -2x(1-x)\,\not{p}\right]\right). \tag{S.55}$$

Let us neglect the IR regulator for a moment and take $m_\gamma^2 = 0$ so $\Delta = x\times(m^2-p^2+xp^2)$. Then for $x\to 0$, the integrand of eq. (S.55) behaves as

$$\frac{1}{x^{\epsilon}}\times\left[-\frac{1-\epsilon}{[m^2-p^2+xp^2]^{\epsilon}} \;+\; \frac{2\epsilon\,\not{p}((2-\epsilon)m-(1-\epsilon)\,\not{p})}{[m^2-p^2+xp^2]^{1+\epsilon}}\right]. \tag{S.56}$$

For off-shell momenta $p^2 < m^2$, the expression in the square brackets here is finite and the $\int dx\,x^{-\epsilon}$ is perfectly finite as long as $\epsilon < 1$ *i.e.*, $D > 2$. But for the on-shell momentum $p^2 = m^2$, the second term in the square brackets blows up at $x\to 0$ and we end up with

$$\int\limits_0^1 dx\,\frac{\text{finite}}{x^{1+2\epsilon}}\,, \tag{S.57}$$

which diverges for any $\epsilon \geq 0$ *i.e.*, $D \leq 4$. And that's why we need the IR regulator $m_\gamma^2 > 0$.

So let us put the IR regulator back where its belongs and calculate the derivative (S.55) for the on-shell momentum. For $\not p = m_e$, the integrand on the RHS of eq. (S.55) simplifies to

$$-\frac{(1-\epsilon)(1-x)}{\Delta^\epsilon} + \frac{2\epsilon x(1-x)[1+(1-\epsilon)x]\times m_e^2}{\Delta^{1+\epsilon}} \tag{S.58}$$

where

$$\Delta = x^2 m_e^2 + (1-x)m_\gamma^2 \approx x^2 m_e^2 + m_\gamma^2 . \tag{S.59}$$

The approximation here follows from the IR regulator being important only for $z \to 0$, and it allows us to extract a total derivative from the integrand:

$$(\text{S.58}) = -\frac{d}{dx}\left(\frac{(1-x)[1+(1-\epsilon)x]}{\Delta^\epsilon}\right) - \frac{1+(1-\epsilon)x}{\Delta^\epsilon} . \tag{S.60}$$

For $\epsilon < \frac{1}{2}$ — *i.e.*, for $D > 3$ — the second term here can be integrated without the photon's mass,

$$\int_0^1 dx\, \frac{1+(1-\epsilon)x}{[\Delta = x^2 m_e^2]^\epsilon} = \frac{1}{m_e^{2\epsilon}} \times \left(\frac{1}{1-2\epsilon} + \frac{1-\epsilon}{2-2\epsilon}\right), \tag{S.61}$$

while the first term yields

$$-\left.\frac{(1-x)[1+(1-\epsilon)x]}{\Delta^\epsilon}\right|_0^1 = \frac{+1}{\Delta(x=0)} = \frac{+1}{m_\gamma^{2\epsilon}} . \tag{S.62}$$

Altogether

$$\int_0^1 dx\,(\text{S.58}) = \frac{1}{m_\gamma^{2\epsilon}} - \frac{1}{m_e^{2\epsilon}} \times \left(\frac{1}{1-2\epsilon} + \frac{1}{2}\right) \tag{S.63}$$

and therefore

$$\begin{aligned}
\delta_2^{\text{order}\,\alpha} &= \left.\frac{d\Sigma^{1\,\text{loop}}}{d\not p}\right|_{\not p = m} \\
&= \frac{\alpha}{2\pi}\Gamma(\epsilon)\left(\frac{4\pi\mu^2}{m_e^2}\right)^\epsilon \times \left[\left(\frac{m_e^2}{m_\gamma^2}\right)^\epsilon - \frac{1}{1-2\epsilon} - \frac{1}{2}\right] \\
&= -\frac{\alpha}{4\pi}\Gamma(\epsilon)\left(\frac{4\pi\mu^2}{m_e^2}\right)^\epsilon \times \left[1 + \frac{2}{1-2\epsilon} - 2\left(\frac{4\pi\mu^2}{m_e^2}\right)\right]
\end{aligned} \tag{S.64}$$

In the $D \to 4$ limit, this counterterm becomes

$$\delta_2^{\text{order}\,\alpha} = -\frac{\alpha}{4\pi}\left[\frac{1}{\epsilon} - \gamma_E + \log\frac{4\pi\mu^2}{m_e^2} + 4 - 2\log\frac{m_e^2}{m_\gamma^2}\right]. \tag{S.65}$$

By comparison, in class I have calculated

$$\begin{aligned}\delta_1^{\text{order}\,\alpha} &= -\frac{\alpha}{4\pi}\Gamma(\epsilon)\left(\frac{4\pi\mu^2}{m_e^2}\right)^{\epsilon} \times \left[1 + \frac{2}{1-2\epsilon} - 2\left(\frac{4\pi\mu^2}{m_e^2}\right)\right] \\ &\xrightarrow[D\to 4]{} -\frac{\alpha}{4\pi}\left[\frac{1}{\epsilon} - \gamma_E + \log\frac{4\pi\mu^2}{m_e^2} + 4 - 2\log\frac{m_e^2}{m_\gamma^2}\right],\end{aligned} \tag{S.66}$$

cf. eq. (80) of my notes. Thus, $\delta_2 = \delta_1$ (to order α), and this equality holds for any dimension between 3 and 4. *Quod erat demonstrandum.*

In fact, the identity $\delta_1 = \delta_2$ holds in any spacetime dimension and even for a finite photon mass $m_\gamma \not\ll m_e$, but proving *that* goes beyond the scope of this exercise.

<u>Problem **3**</u>:

In class, I have proved a similar equality $\delta_1 = \delta_2$ for the ordinary QED. My starting point was the Ward–Takahashi identity for the un-amputated amplitudes

$$k_\mu S^\mu(k;p',p) = eS(p') - eS(p). \tag{S.67}$$

I used it to prove the identity

$$k_\mu \Gamma^\mu(p',p) = \big(\not{p}' - m - \Sigma(\not{p}')\big) - \big(\not{p} - m - \Sigma(\not{p})\big). \tag{S.68}$$

for the 1PI amplitudes, and the used *that* identity to prove $\delta_1 = \delta_2$.

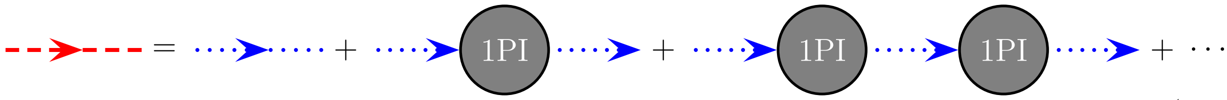
In this homework, our task is to prove the similar counterterm relations (5) for the scalar QED, and in this solutions I am going to proceed just as I did in class. My starting point are the Ward–Takahashi identities for the two-charged-scalar un-amputated amplitudes you should have proved in your previous homework. (And if you did not, please read the solutions.) Specifically, we need the identities for the no-photon, one-photon and two-photon amplitudes,

$$k_\mu S^\mu(k;p',p) \;=\; -qS(p') \;+\; qS(p), \tag{S.69}$$

$$k_{1\mu} S^{\mu\nu}(p',p;k_1,k_2) \;=\; -qS^\nu(p',p+k_1;k_2) \;+\; qS^\nu(p'-k_1,p;k_2). \tag{S.70}$$

These amplitudes have amputated photon legs, but the scalar legs are not amputated: the external leg bubbles are allowed, and the propagator for the scalar legs themselves are included in the amplitudes. Am going to convert eqs. (S.69) and (S.70) into Ward identities for the 1PI amplitudes $\Sigma(p)$, $G^\mu(p',p)$, and $G^{\mu\nu}(p',p;k_1,k_2)$, then I shall use those identities to prove the counterterm relations (5).

Let's start with the diagrammatic analysis of the un-amputated amplitudes S and S^μ. The no-photon amplitude $S(p)$ is nothing but the dressed scalar propagator,

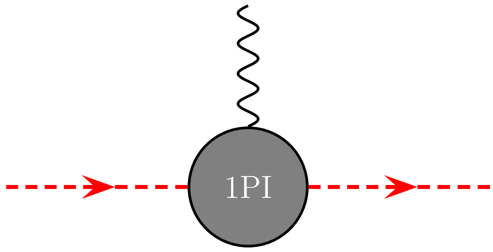


(S.71)

in other words,

$$S(p) \;=\; \frac{i}{p^2-M^2-\Sigma(p^2)}\,. \tag{S.72}$$

The one-photon amplitude $S^\mu(k;p',p)$ is amputated with respect to the photon but not the charged scalars; summing over the leg bubbles in those un-amputated scalar legs gives us two dressed propagators, thus

$S^\mu(k;p',p) \;=$  $=\; S(p')\times ieG^\mu(k;p',p)\times S(p).$ (S.73)

Applying the WT identity (S.69) to this amplitude gives us

$$S(p')\times k_\mu\times G^\mu(p',p)\times S(p) \;=\; -qS(p') \;+\; qS(p) \tag{S.74}$$

and hence

$$k_\mu \times G^\mu(p',p) \;=\; -\frac{q}{S(p)} \;+\; \frac{q}{S(p')}$$
$$=\; iq\big(p^2 - M^2 - \Sigma(p^2)\big) \;-\; iq\big(p'^2 - M^2 - \Sigma(p'^2)\big). \tag{S.75}$$

Or rather

$$k_\mu \times G^\mu_{\text{tree+loops}}(p',p) \;=\; -iq\big(p'^2 - M^2 - \Sigma_{\text{loops}}(p'^2)\big) \;+\; iq\big(p^2 - M^2 - \Sigma_{\text{loops}}(p^2)\big) \tag{S.76}$$

because the identity (S.69) was derived for the bare theory without accounting for the counterterms.

In order to relate this identity to the counterterms and the renormalization conditions for their finite parts, let's take the limit $k \to 0$ and hence $p' \to p$. In this limit

$$\big(p'^2 - M^2 - \Sigma_{\text{loops}}(p'^2)\big) \;-\; \big(p^2 - M^2 - \Sigma_{\text{loops}}(p^2)\big) \;=$$
$$=\; (p'^2 - p^2) \times \left(1 \;-\; \frac{d\Sigma_{\text{loops}}}{dp^2}\right) \;+\; O\big((p'^2 - p^2)^2\big)$$
$$=\; k_\mu \times (p'+p)^\mu \left(1 \;-\; \frac{d\Sigma_{\text{loops}}}{dp^2}\right) \;+\; O(|k|^2), \tag{S.77}$$

so eq. (S.76) becomes

$$G^\mu_{\text{tree+loops}}(p'=p) \;=\; -iq(p'+p)^\mu \times \left(1 \;-\; \frac{d\Sigma_{\text{loops}}}{dp^2}\right). \tag{S.78}$$

In terms of the first eq. (6), this formula gives us

$$A_{\text{tree+loops}}(p^2) \;=\; 1 \;-\; \frac{d\Sigma_{\text{loops}}}{dp^2}\,. \tag{S.79}$$

where 1 on the RHS here corresponds to the tree-level one-photon vertex of the scalar QED,

$G^\mu_{\rm tree} = -iq(p'+p)^\mu$, thus $A_{\rm tree} = 1$ and hence

$$A_{\rm loops}(p^2)\ =\ -\frac{d\Sigma_{\rm loops}}{dp^2}\,. \tag{S.80}$$

Now consider the renormalization condition for the one-photon vertex in scalar QED. Altogether, including the tree vertex, the loops, and the $\delta_1^{(1\gamma)}$ counterterm,

$$G^\mu_{\rm net}(p',p)\ =\ -iq(p'+p)^\mu\ +\ G^\mu_{\rm loops}(p',p)\ -\ q(p'+p)^\mu\times\delta_1^{(1\gamma)}, \tag{S.81}$$

hence for $p'=p$

$$A_{\rm net}(p^2)\ =\ 1\ +\ A_{\rm loops}(p^2)\ +\ \delta_1^{(1\gamma)}. \tag{S.82}$$

For the on-shell momenta $G^\mu_{\rm net}$ measures the physical net electric charge of the scalar particle, thus $Q_{\rm phys}\ =\ qA_{\rm net}(p^2=m^2)$ and the renormalization condition

$$A_{\rm net}(p^2=m^2)\ =\ A_{\rm tree}\ =\ 1 \tag{7.a}$$

means that there are no quantum corrections to the particle charge. In terms of the $\delta_1^{(1\gamma)}$ counterterm, this condition implies

$$\delta_1^{(1\gamma)}\ =\ -A_{\rm loops}(p^2=m^2). \tag{S.83}$$

At the same time, the renormalization condition for the δ_2 counterterm is

$$\left.\frac{d\Sigma_{\rm net}}{dp^2}\right|_{p^2=m^2}\ =\ 0\quad\text{for}\quad \Sigma_{\rm net}(p^2)\ =\ \Sigma_{\rm loops}(p^2)\ +\ \delta_m\ -\ \delta^2\times p^2, \tag{S.84}$$

hence

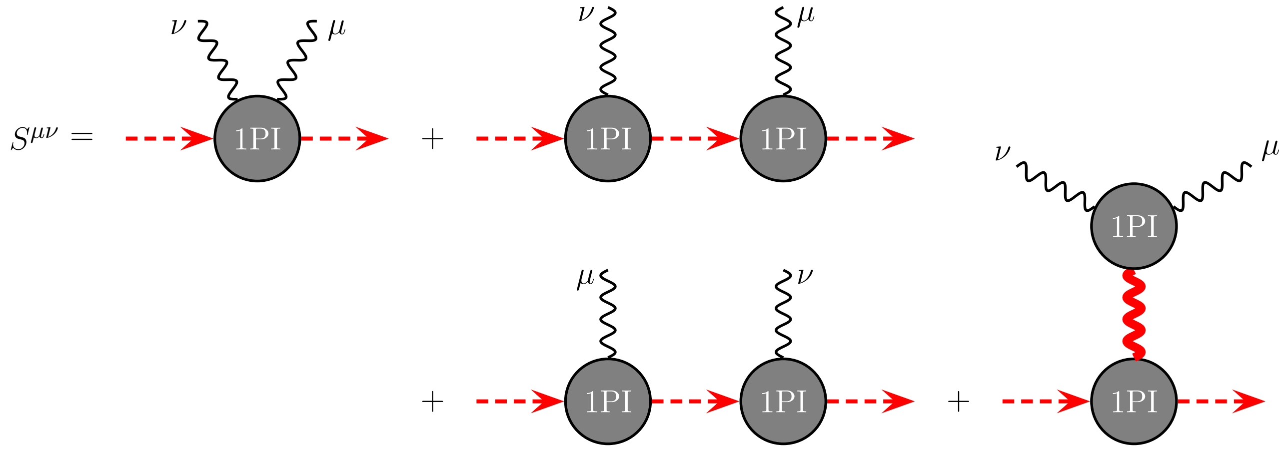
$$\delta_2\ =\ +\left.\frac{d\Sigma_{\rm loops}}{dp^2}\right|_{p^2=m^2} \tag{S.85}$$

In light of eqs. (S.83) and (S.85), the Ward identity (S.80) immediately implies

$$\delta_1^{(1\gamma)}\ =\ -A_{\rm loops}(p^2=m^2)\ =\ +\left.\frac{d\Sigma_{\rm loops}}{dp^2}\right|_{p^2=m^2}\ =\ \delta_2\,. \tag{S.86}$$

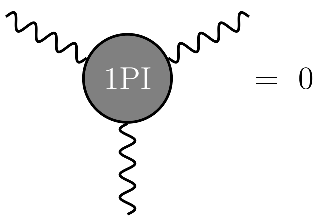
⋆ ⋆ ⋆

To prove the relation (5) for the two-photon counterterm $\delta_1^{(2\gamma)}$ we need the two-photon amplitude $G^{\mu\nu}(k_1, k_2; p', p)$. Again, this is an un-amputated amplitude comprised of an amputated core and two dressed scalar propagators. However, in this case the amputated core is not necessarily one-particle irreducible: Instead, it may comprise two 1PI sub-diagrams connected by a dressed propagator. Altogether, we have four distinct diagram topologies



(S.87)

Fortunately, the last topology here does not contribute because its top 1PI sub-amplitude vanishes by the charge-conjugation symmetry,



(S.88)

Spelling out the other three topologies in terms of the dressed *scalar* propagators and the 1PI one-photon and two-photon amplitudes, we have

$$\begin{aligned} S^{\mu\nu}(p', p; k_1, k_2) &= S(p') \times G^{\mu\nu}(p', p; k_1, k_2) \times S(p) \\ &+ S(p') \times G^{\nu}(p', p + k_1; k_2) \times S(p + k_1) \times G^{\mu}(p + k_1, p; k_1) \times S(p) \\ &+ S(p') \times G^{\mu}(p', p' - k_1; k_1) \times S(p' - k_1) \times G^{\nu}(p' - k_1, p; k_2) \times S(p). \end{aligned} \tag{S.89}$$

Let us multiply both sides of eq. (S.89) by the $k_{1\mu}$. On the left had side, we use the WT identity (S.70) and obtain

$$\begin{aligned} k_{1\mu} \times S^{\mu\nu}(p', p; k_1, k_2) &= -qS^{\nu}(p', p+k_1; k_2) \,+\, qS^{\nu}(p'-k_1, p; k_2) \\ &= -qS(p') \times G^{\nu}(p', p+k_1; k_2) \times S(p+k_1) \\ &\quad +\, qS(p'-k_1) \times G^{\nu}(p'-k_1, p; k_2) \times S(p). \end{aligned} \tag{S.90}$$

On the right hand side of eq. (S.89), we use the WT identity (S.69) (in the form of eq. (S.74)) for the last two terms, hence

$$\begin{aligned} k_{1\mu} \times S^{\mu\nu}(p', p; k_1, k_2) &= S(p') \times k_{1\mu} \times G^{\mu\nu}(p', p; k_1, k_2) \times S(p) \\ &\quad +\, S(p') \times G^{\nu}(p', p+k_1; k_2) \times \big[-qS(p+k_1) \,+\, qS(p)\big] \\ &\quad +\, \big[-qS(p') \,+\, qS(p'-k_1)\big] \times G^{\nu}(p'-k_1, p; k_2) \times S(p). \end{aligned} \tag{S.91}$$

Equating the last two formulæ, we obtain

$$\begin{aligned} S(p') \times k_{1\mu} \times G^{\mu\nu}(p', p; k_1, k_2) \times S(p) &= -qS(p') \times G^{\nu}(p', p+k_1; k_2) \times S(p) \\ &\quad +\, S(p') \times G^{\nu}(p'-k_1, p; k_2) \times S(p) \end{aligned} \tag{S.92}$$

— other terms cancel out — and consequently

$$k_{1\mu} \times G^{\mu\nu}(p', p; k_1, k_2) \;=\; -qG^{\nu}(p', p+k_1; k_2) \,+\, qG^{\nu}(p'-k_1, p; k_2). \tag{S.93}$$

Or rather

$$k_{1\mu}\times G^{\mu\nu}_{\text{tree+loops}}(p', p; k_1, k_2) \;=\; -qG^{\nu}_{\text{tree+loops}}(p', p+k_1; k_2) + qG^{\nu}_{\text{tree+loops}}(p'-k_1, p; k_2) \tag{S.94}$$

because we started with un-amputated WT identities for the bare amplitudes without the counterterms. Moreover, the tree-level vertices $G^{\mu\nu}_{\text{tree}} = 2iq^2 g^{\mu\nu}$ and $G^{\nu}_{\text{tree}} = -iq(p'+p)^{\nu}$ satisfy eq. (S.94) all by themselves, so the 1PI loop amplitudes satisfy their own Ward identity

$$k_{1\mu} \times G^{\mu\nu}_{\text{loops}}(p', p; k_1, k_2) \;=\; -qG^{\nu}_{\text{loops}}(p', p+k_1; k_2) \,+\, qG^{\nu}_{\text{loops}}(p'-k_1, p; k_2). \tag{S.95}$$

It is this identity which assures counterterm the equality $\delta_1^{(2\gamma)} = \delta_1^{(1\gamma)} = \delta_2$.

To relate the Ward identity (S.95) to the counterterms, we need to take both photon's momenta to zero in order to use the renormalization conditions (6) and (7). Let's take the $k_2 \to 0$ limit first, and only then take $k_1 \to 0$. For $k_2 = 0$ and hence $p' = p + k_1$, eq. (S.95) becomes

$$\begin{aligned} k_{1\mu} \times G^{\mu\nu}_{\text{loops}}(p', p; k_1, 0) &= -qG^{\nu}_{\text{loops}}(p', p'; 0) + qG^{\nu}_{\text{loops}}(p, p; 0) \\ \langle\!\langle \text{ by the first eq. (6) } \rangle\!\rangle &= +iq^2(2p')^\nu \times A_{\text{loops}}(p'^2) - iq^2(2p)^\nu \times A(p^2) \\ \langle\!\langle \text{ by eq. (S.80) } \rangle\!\rangle &= -iq^2(2p')^\nu \times \frac{d\Sigma_{\text{loops}}(p'^2)}{dp'^2} + iq^2(2p)^\nu \times \frac{d\Sigma_{\text{loops}}(p^2)}{dp^2}. \end{aligned} \tag{S.96}$$

Now let's take the $k_1 \to 0$ limit and expand the bottom line of eq. (S.96) in powers of $p' - p = k_1$:

$$\begin{aligned} -iq^2(2p')^\nu \times \frac{d\Sigma_{\text{loops}}(p'^2)}{dp'^2} &+ iq^2(2p)^\nu \times \frac{d\Sigma_{\text{loops}}(p^2)}{dp^2} \\ &= -2iq^2(p' - p)_\mu \times \frac{\partial}{\partial p_\mu}\left[p^\nu \times \frac{d\Sigma_{\text{loops}}(p^2)}{dp^2}\right] + O(|p' - p|^2). \end{aligned} \tag{S.97}$$

Consequently, for $k_1 = k_2 = 0$,

$$\begin{aligned} G^{\mu\nu}_{\text{loops}}(p' = p; k_1 = k_2 = 0) &= -2iq^2 \times \frac{\partial}{\partial p_\mu}\left[p^\nu \times \frac{d\Sigma_{\text{loops}}(p^2)}{dp^2}\right] \\ &= -2iq^2 \times g^{\mu\nu} \times \frac{d\Sigma_{\text{loops}}(p^2)}{dp^2} - 4iq^2 \times p^\mu p^\nu \times \frac{d^2\Sigma(p^2)}{(dp^2)^2}. \end{aligned} \tag{S.98}$$

or in terms of the second eq. (6),

$$B_{\text{loops}}(p^2) = -2\frac{d\Sigma_{\text{loops}}(p^2)}{dp^2}, \qquad C_{\text{loops}}(p^2) = -4\frac{d^2\Sigma_{\text{loops}}(p^2)}{(dp^2)^2}. \tag{S.99}$$

Now let's relate this formula to the counterterms. The net 1PI two-scalar two-photon amplitude comprises the tree-level seagull vertex, the loop corrections, and the $\delta_1^{(2\gamma)}$ counterterm, thus

$$G^{\mu\nu}_{\text{net}} = 2iq^2 g^{\mu\nu} + G^{\mu\nu}_{\text{loops}}(p', p; k_1, k_2) + 2iq^2 g^{\mu\nu} \times \delta_1^{(2\gamma)}. \tag{S.100}$$

In the $k_1 = k_2 = 0$ limit, this becomes

$$B_{\text{net}}(p^2) = 2_{\text{tree}} + 2B_{\text{loops}}(p^2) + 2\delta_1^{(2\gamma)}, \qquad C_{\text{net}}(p^2) = C_{\text{loops}}(p^2). \tag{S.101}$$

Physically, the seagull vertex dominates the Thompson scattering of a low-energy photon off a scalar particle, $\mathcal{M} \approx q^2(\epsilon\cdot\epsilon')B_{\rm net}(p^2=m^2)$. Thus, $q^2\times B_{\rm net}$ for the on-shell $p^2=m^2$ probes the electric charge[2] of the scalar particle, so to avoid the quantum corrections to the net electric charge, we need the renormalization condition

$$B_{\rm net}(p^2=m^2)\;=\;B_{\rm tree}\;=\;2. \tag{7b}$$

In terms of the counterterm $\delta_1^{(2\gamma)}$, this condition means

$$\delta_1^{(2\gamma)}\;=\;-B_{\rm loops}(p^2=m^2). \tag{S.102}$$

In light of this condition — and also the condition (S.85) for the δ_2 — the Ward identity (S.99) immediately leads to

$$\delta_1^{(2\gamma)}\;=\;-B_{\rm loops}(p^2=m^2)\;=\;+2\,\frac{d\Sigma_{\rm loops}(p^2)}{dp^2}\bigg|_{p^2=m^2}\;=\;\delta_2\,. \tag{S.103}$$

And this (almost) completes our proof of eqs. (5).

However, we still have a loophole to close. At higher loop orders, the loop amplitudes $\Sigma_{\rm loops}(p^2)$, $G^\mu_{\rm loops}(k;p',p)$, and $G^{\mu\nu}_{\rm loops}(k_1,k_2;p',p)$ involved in the renormalization conditions for the δ_2, $\delta_1^{(1\gamma)}$, and $\delta_1^{(2\gamma)}$ counterterms include both pure loops of the bare theory, but also loops containing some counterterm vertices of lower order in α. On the other hand, in the previous homework we have derived the Ward–Takahashi identities (S.69) and (S.70) for the bare theory, without the counterterms. It's fairly easy to extend that derivation to the renormalized theory with all the counterterms, but that requires gauge invariance of the counterterm Lagrangian. In other words, we must assume eqs. (5) in order to prove the WT identities we have used here to prove eqs. (5).

In order to break this logical circle, we expand the counterterms in powers of the couplings $\alpha=e^2/4\pi$ and λ, and prove eqs. (5) by induction in the loop order. As the base of the induction, we notice that at the one-loop level of the perturbative expansion, the loop amplitudes are pure loops and do not contain the counterterm vertices. Consequently, at this level, the WT identities (S.69) and (S.70) must hold true, and hence our proof of eqs. (5) is valid at the one-loop level.

Now, suppose we have proved eqs. (5) to the n–loop level. At the $n+1$ loop order, the loop diagrams may contain counterterm vertices of orders n or less; by induction assumption, these vertices are gauge invariant, and hence the $n+1$ loop amplitudes do satisfy the WT identities. Consequently, our proof of eqs. (5) is valid to the $n+1$ loop order, and this gives us the induction step.

Therefore, by induction, eqs. (5) hold true to all orders of the perturbation theory. $\mathcal{Q}.\mathcal{E}.\mathcal{D}.$.