

Problem 1(a):

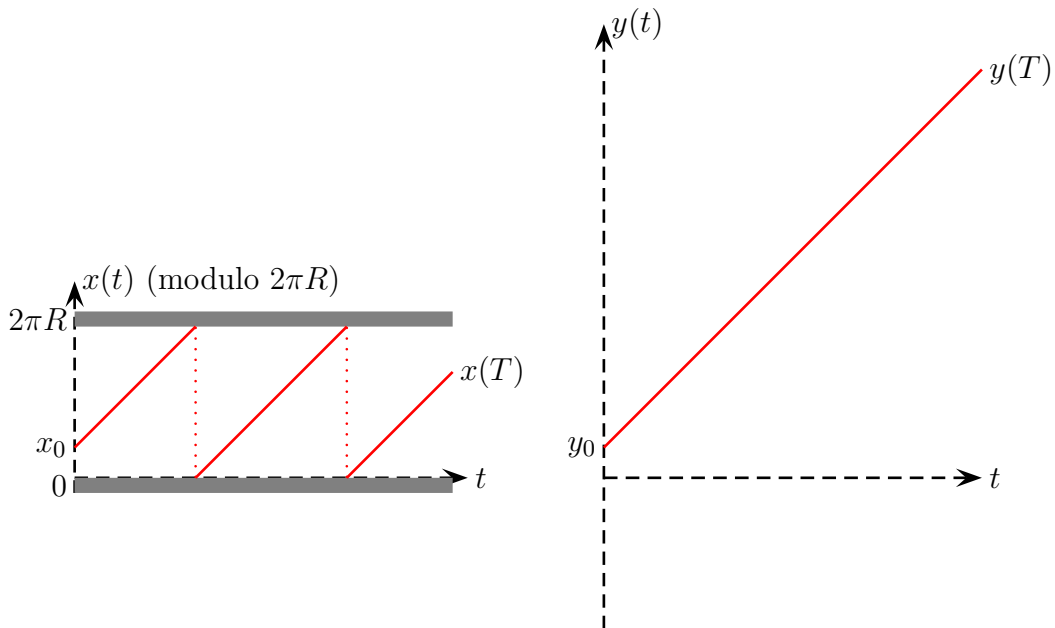
The difference between a circle and a straight line is that on a circle the path of a particle going from point  $x_0$  to point  $x'$  does not need to be 'straight' but may wrap around the whole circle one or more times. Indeed, let us compare a particle moving on a circle according to  $x(t)$  (modulo  $2\pi R$ ) with a particle moving on an infinite line according to  $y(t)$ . If the two particles have exactly the same velocities at all times,

$$\frac{dx}{dt} \equiv \frac{dy}{dt} \tag{S.1}$$

and similar initial positions  $x_0 = y_0$  (according to some coordinate systems) at time  $t = 0$ , then after time  $T$  one generally has

$$y(T) = x(T) + 2\pi R \times n \tag{S.2}$$

for some integer  $n = 0, \pm 1, \pm 2, \pm 3, \dots$  because the  $x(t)$  path may wrap around the circle  $n$  times while the  $y(t)$  path may not wrap. For example, the two paths depicted below have same (constant) velocities and begin at  $y_0 = x_0$  but end at  $y(T) = x(T) + 2\pi R \times 2$ :



It is easy to see that the paths  $x(t)$  (modulo  $2\pi R$ ) and  $y(t)$  (modulo nothing) are in one-to-one correspondence with each other, provided we restrict the initial point  $y_0$  of the particle on

the infinite line to a particular interval of length  $L = 2\pi R$ , say  $0 \leq y_0 < 2\pi R$ . Consequently, in the path integral for the particle on the circle

$$\int_{x(t=0)=x_0 \pmod L}^{x(t=T)=x' \pmod L} \mathcal{D}'[x(t) \pmod L] = \sum_{n=-\infty}^{+\infty} \int_{y(t=0)=x_0}^{y(t=T)=x'+nL} \mathcal{D}'[y(t)]. \quad (\text{S.3})$$

Furthermore, in the absence of potential energy, the circle path  $x(t) \pmod L$  and the corresponding  $\infty$  line path  $y(t)$  have equal actions

$$S[x(t) \pmod L] = S[y(t)] = \int_0^T dt \left[ \frac{M}{2} \dot{x}^2 = \frac{M}{2} \dot{y}^2 \right], \quad (\text{S.4})$$

and therefore

$$\begin{aligned} U_{\text{circle}}(x'; x_0) &= \int_{x(t=0)=x_0 \pmod L}^{x(t=T)=x' \pmod L} \mathcal{D}'[x(t) \pmod L] e^{iS[x(t) \pmod L]/\hbar} \\ &= \sum_{n=-\infty}^{+\infty} \int_{y(t=0)=x_0}^{y(t=T)=x'+nL} \mathcal{D}'[y(t)] e^{iS[y(t)]/\hbar} \\ &= \sum_{n=-\infty}^{+\infty} U_{\infty \text{ line}}(y' = x' + nL; y_0 = x_0). \end{aligned} \quad (1)$$

*Q.E.D.*

Problem 1(b):

For a free particle living on an infinite line the evolution kernel is given by

$$U_{\infty \text{ line}}(y'; y_0) = \sqrt{\frac{M}{2\pi i \hbar T}} \times \exp\left(\frac{i}{\hbar} S_{\text{classical}} = \frac{i}{\hbar} \frac{M(x' - x_0)^2}{2T}\right), \quad (3)$$

hence according to eq. (1), a particle on a circle has kernel

$$U_{\text{circle}}(x'; x_0) = \sqrt{\frac{M}{2\pi i \hbar T}} \times \sum_{n=-\infty}^{+\infty} \exp\left(\frac{iM}{2\hbar T} (x' - x_0 + nL)^2\right). \quad (\text{S.5})$$

To evaluate this sum, we use Poisson re-summation formula (2), which gives

$$\sum_{n=-\infty}^{+\infty} \exp\left(\frac{iM}{2\hbar T}(x' - x_0 + nL)^2\right) = \sum_{\ell=-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\nu \exp\left(\frac{iM}{2\hbar T}(x' - x_0 + \nu L)^2\right) \times e^{2\pi i \ell \nu}. \quad (\text{S.6})$$

Rearranging the exponential, we have

$$\frac{iM}{2\hbar T}(x' - x_0 + \nu L)^2 + 2\pi i \ell \nu = \frac{iML^2}{2\hbar T} \left( \nu + \frac{x' - x_0}{L} + \frac{2\pi \ell \hbar T}{ML^2} \right) - 2\pi i \ell \frac{x' - x_0}{L} - \frac{i\hbar T(2\pi \ell)^2}{ML^2}, \quad (\text{S.7})$$

and therefore

$$\int_{-\infty}^{+\infty} d\nu \exp\left(\frac{iM}{2\hbar T}(x' - x_0 + \nu L)^2\right) \times e^{2\pi i \ell \nu} = \sqrt{\frac{2\pi i \hbar T}{ML^2}} \times \exp\left(-2\pi i \ell \frac{x' - x_0}{L} - \frac{(2\pi \ell)^2 i \hbar T}{ML^2}\right). \quad (\text{S.8})$$

Consequently,

$$\begin{aligned} U_{\text{circle}}(x'; x_0) &= \sqrt{\frac{M}{2\pi i \hbar T}} \times \sqrt{\frac{2\pi i \hbar T}{ML^2}} \times \sum_{\ell=-\infty}^{+\infty} \exp\left(-2\pi i \ell \frac{x' - x_0}{L} - \frac{(2\pi \ell)^2 i \hbar T}{ML^2}\right) \\ &= \frac{1}{L} \sum_{\ell=-\infty}^{+\infty} e^{ip(x' - x_0)/\hbar} \times e^{-iTE/\hbar} \end{aligned} \quad (\text{S.9})$$

where

$$p = -\frac{2\pi \hbar \ell}{L} = -\frac{\hbar \ell}{R} \quad \text{and} \quad E = \frac{p^2}{2M}. \quad (\text{S.10})$$

Problem 1(c): This is obvious from eqs. (S.9) and (S.10).

Problem 3 — the textbook problem **9.2**, part (c):

In class, we have learned the path-integral formula for the “partition function” of a quantum particle,

$$Z(T) \equiv \exp(-iT\hat{H}) = \int_{\mathbf{x}(T)=\mathbf{x}(0)} \mathcal{D}[\mathbf{x}(t)] \exp(iS[\mathbf{x}(t)]). \quad (\text{S.11})$$

In statistical mechanics, the partition function at temperature  $\mathcal{T}$  is defined as

$$Z(\mathcal{T}) \equiv \text{Tr} \exp(-\beta\hat{H}), \quad \beta = \frac{1}{\mathcal{T}}. \quad (\text{S.12})$$

The two partition functions are related by analytic continuation of the real time  $T$  to the imaginary time  $-i\beta = -i\mathcal{T}^{-1}$ , thus

$$Z_{\text{SM}}(\mathcal{T}) = Z_{\text{QM}}(T = -i\mathcal{T}^{-1}). \quad (\text{S.13})$$

In terms of the path integrals, this relation corresponds to going from the Minkowski path integral to the Euclidean path integral

$$Z(\mathcal{T}) = \int_{\mathbf{x}(\beta)=\mathbf{x}(0)} \mathcal{D}[\mathbf{x}(t_E)] \exp(-S_E[\mathbf{x}(t_E)]) \quad (\text{S.14})$$

where the Euclidean action is

$$S_E[\mathbf{x}(t_E)] = \int_0^\beta dt_E \left( \frac{m}{2} \left( \frac{d\mathbf{x}}{dt_E} \right)^2 + V(\mathbf{x}) \right). \quad (\text{S.15})$$

Note the boundary conditions for the Euclidean path integral (S.14): after the Euclidean time interval  $\beta = 1/\mathcal{T}$ , the particle must come back to its starting point. In other words, *at finite temperatures, the motion in Euclidean time is periodic with period  $\beta = 1/\mathcal{T}$ .*

Generalizing eq. (S.14) from particle mechanics to field theory is quite straightforward. For a real scalar field  $\phi(x)$  with a Euclidean Lagrangian

$$\mathcal{L}_E = \frac{1}{2}(\partial\phi)^2 + V(\phi) \quad (\text{S.16})$$

we have finite-temperature Partition function

$$Z(\beta) = \int_{\phi(\mathbf{x},\beta)=\phi(\mathbf{x},0)} \mathcal{D}[\phi(\mathbf{x}, x_4)] \exp \left[ - \int d^3\mathbf{x} \int_0^\beta dx_4 \left( \frac{1}{2}(\partial\phi)^2 + V(\phi) \right) \right]. \quad (\text{S.17})$$

Again, the finite temperature translates into the geometry of the Euclidean 4D spacetime: The Euclidean time  $x_4 = it$  is of finite extent  $\beta = 1/\mathcal{T}$  and the scalar field is subject to the periodic boundary condition; the other 3 dimensions  $x_1, x_2, x_3$  are infinite as usual.

For the free scalar field, the Euclidean action is a quadratic functional

$$S_E[\phi(x_E)] = \frac{1}{2} \int d^4x_E \phi(m^2 - \partial_E^2)\phi, \quad (\text{S.18})$$

so the path integral (S.17) is a Gaussian integral that can be formally evaluated as the determinant

$$Z(\beta) = \text{const} \times \left( \text{Det}[m^2 - \partial_E^2] \right)^{-1/2} \quad (\text{S.19})$$

Or in terms of the Helmholtz's free energy,

$$\mathcal{F} \equiv -\mathcal{T} \log Z = \text{const} + \frac{\mathcal{T}}{2} \log \text{Det}[m^2 - \partial_E^2] = \text{const} + \frac{\mathcal{T}}{2} \text{Tr} \log[m^2 - \partial_E^2]. \quad (\text{S.20})$$

To actually evaluate the trace here, we diagonalize the  $m^2 - \partial_E^2$  operator via Fourier transform to the momentum space. However, because of the periodicity of the Euclidean time coordinate,

the Euclidean “energies”  $k_4$  have discrete rather than continuous spectrum,

$$k_4 = \frac{2\pi}{\beta} \times \text{integer}. \quad (\text{S.21})$$

Consequently, the trace evaluates to

$$\text{Tr} \log[m^2 - \partial_E^2] = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_{k_4=(2\pi\mathcal{T}n)} \log(m^2 + \mathbf{k}^2 + k_4^2) \quad (\text{S.22})$$

and the free energy of the quantum field becomes

$$\mathcal{F} = \text{const} + \frac{\mathcal{T}}{2} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_{k_4=(2\pi\mathcal{T}n)} \log(m^2 + k_E^2) \quad (\text{S.23})$$

Now let’s use the Poisson’s resummation formula (2) to re-arrange the sum over discrete  $k_4$  momenta as

$$\sum_{k_4} F(k_4) = \beta \sum_{\ell=-\infty}^{+\infty} \int \frac{dk_4}{2\pi} F(k_4) e^{i\beta\ell k_4}. \quad (\text{S.24})$$

Consequently, the free energy (S.23) becomes

$$\mathcal{F} = \text{const} + \frac{1}{2} \sum_{\ell=-\infty}^{+\infty} \int \frac{d^4k_E}{(2\pi)^4} e^{i\beta\ell k_4} \log(k_E^2 + m^2). \quad (\text{S.25})$$

In the zero-temperature limit  $\beta \rightarrow \infty$ , the sum  $\sum_{\ell}$  reduces to the  $\ell = 0$  term while all the other terms are suppressed by the rapidly changing phase  $e^{i\beta\ell k_4}$ . In the general spirit of subtracting the zero-point energy contribution, we should therefore get rid of the  $\ell = 0$  term. Since all the other terms come in symmetric pairs  $\pm\ell \neq 0$ , we arrive at

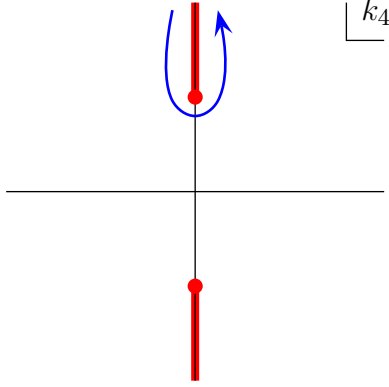
$$\mathcal{F} = \sum_{\ell=1}^{\infty} \int \frac{d^4k_E}{(2\pi)^4} e^{i\beta\ell k_4} \log(k_E^2 + m^2). \quad (\text{S.26})$$

Formula (S.26) has a nice 4D form, but for the purpose of comparison with the ordinary statistical mechanics, let us integrate over the  $k_4$  before we integrate over the 3-momentum  $\mathbf{k}$ .

For fixed  $\mathbf{k}$  and  $\ell$ , we need to calculate

$$I = \int \frac{dk_4}{2\pi} e^{i\beta\ell k_4} \log(k_4^2 + E^2) \quad (\text{S.27})$$

where  $E^2 = m^2 + \mathbf{k}^2$ , and the best way to evaluate this integral is to deform the integration contour in the complex  $k_4$  plane away from the real axis. In terms of  $k_4$ , the logarithm  $\log(k_4^2 + E^2)$  has one branch cut running from  $+iE$  to  $+i\infty$  and another cut running from  $-iE$  to  $-i\infty$ . Lets move the integration contour up (towards  $+i\infty$ ) until it wraps around the upper branch cut:



In other words,  $k_4 = iE(1+x+i\epsilon)$  on its way down from  $x = +\infty$  to  $x = 0$  and  $k_4 = iE(1+x-i\epsilon)$  on its way up from  $x = 0$  back to  $k = +\infty$ . Consequently, the integral becomes

$$\begin{aligned} I &= \frac{iE}{2\pi} \int_0^{+\infty} dx e^{-\beta\ell E(1+x)} \times [\log(E^2(-2x - x^2 + i\epsilon)) - \log(E^2(-2x - x^2 - i\epsilon)) = 2\pi i] \\ &= -E \int_0^{+\infty} dx e^{-\beta\ell E(1+x)} = -\frac{e^{-\beta\ell E}}{\beta\ell}. \end{aligned} \quad (\text{S.28})$$

and hence

$$\mathcal{F} = \sum_{\ell=1}^{\infty} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{-e^{-\beta\ell E(k)}}{\beta\ell}. \quad (\text{S.29})$$

At this point, it's convenient to sum over  $\ell$  before integrating over the 3D momentum  $\mathbf{k}$ , thus

$$-\frac{1}{\beta} \sum_{\ell=1}^{\infty} \frac{(e^{-\beta E})^\ell}{\ell} = \mathcal{T} \log(1 - e^{-\beta E}), \quad (\text{S.30})$$

and therefore

$$\mathcal{F}(\mathcal{T}, m) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \mathcal{T} \log \left( 1 - e^{-\beta E_{\mathbf{k}}} \right). \quad (\text{S.31})$$

Finally, let us compare our result (S.31) for the free energy of the free scalar quantum field with the conventional statistical mechanics of identical spinless relativistic bosons. In the SM of identical bosons,

$$\mathcal{F}(\mathcal{T}, m) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \mathcal{F}_{\text{oscillator}}^{\text{harmonic}}(\mathcal{T}, E_{\mathbf{k}}) \quad (\text{S.32})$$

where each oscillator mode contributes

$$\mathcal{F}_{\text{oscillator}}^{\text{harmonic}}(\mathcal{T}, E_{\mathbf{k}}) = -\mathcal{T} \log Z_{\text{oscillator}}^{\text{harmonic}} = \mathcal{T} \log(2 \sinh(E\beta/2)) = \frac{1}{2}E + \mathcal{T} \log \left( 1 - e^{-\beta E} \right). \quad (\text{S.33})$$

Subtracting the zero-point energy  $\frac{1}{2}E$  and substituting into eq. (S.32), we arrive at precisely eq. (S.31). Thus, functional quantization of the field theory correctly reproduces the free energy of the field's quanta.

Problem 11.1(a):

The easiest way to compute the correlation function of  $\exp(+i\Phi(x_1))$  and  $\exp(-i\Phi(x_2))$  is in terms of functional integrals:

$$\begin{aligned} \left\langle \mathbf{T} e^{+i\hat{\Phi}(x_1)} e^{-i\hat{\Phi}(x_2)} \right\rangle &= \frac{\iint \mathcal{D}[\Phi(x)] e^{iS[\Phi(x)]} e^{+i\Phi(x_1)} e^{-i\Phi(x_2)}}{\iint \mathcal{D}[\Phi(x)] e^{iS[\Phi(x)]}} \\ &= \frac{\iint \mathcal{D}[\Phi(x)] \exp \left( i \int (\mathcal{L} + J\Phi) d^d x \right)}{\iint \mathcal{D}[\Phi(x)] \exp \left( i \int \mathcal{L} d^d x \right)} \equiv \frac{Z[J]}{Z[0]} \end{aligned} \quad (\text{S.34})$$

where

$$J(x) = \delta^{(d)}(x - x_1) - \delta^{(d)}(x - x_2). \quad (\text{S.35})$$



Moreover, for a *free* scalar field  $\Phi(x)$

$$Z[J] = Z[0] \times \exp\left(-\frac{1}{2} \int d^d x \int d^d y J(x) G^F(x-y) J(y)\right). \quad (\text{S.36})$$

For the source as in eq. (S.35), the double integral inside the exponential is simply  $2G^F(0) - 2G_F(x_1 - x_2)$ , hence

$$\left\langle \mathbf{T} e^{+i\hat{\Phi}(x_1)} e^{-i\hat{\Phi}(x_2)} \right\rangle = \frac{Z[J]}{Z[0]} \exp\left(G^F(x_1 - x_2) - G^F(0)\right). \quad (\text{S.37})$$

Problem 11.1(b):

Under the axionic symmetry  $\Phi(x) \mapsto \Phi(x) - \alpha$ , the derivatives  $\partial_\mu \Phi$ ,  $\partial_\mu \partial_\nu \Phi$ , *etc.* are invariant but the field  $\Phi$  is not. Consequently, the most general effective Lagrangian for the quantum  $\Phi(x)$  field must be a function of its derivatives only,

$$\mathcal{L} = \frac{\rho}{2} (\partial\Phi)^2 + \frac{A}{2} (\partial^2\Phi)^2 + \frac{B}{4} (\partial\Phi)^4 + \dots \quad (\text{S.38})$$

Now consider the renormalizability. In  $d$  spacetime dimensions, the scalar field  $\Phi$  has dimensionality  $\frac{d}{2} - 1$ , hence a Lagrangian term involving  $n$  fields and  $m$  derivatives has dimensionality

$$\Delta = \frac{nd}{2} + (m - n). \quad (\text{S.39})$$

Renormalizability requires  $\Delta \leq d$  and hence

$$\frac{d}{2}(n - 2) + (m - n) \leq 0. \quad (\text{S.40})$$

On the other hand, axionic symmetry requires  $m \geq n$  (no field without a derivative) while any term with  $n < 2$  can be disregarded as a total derivative. All these conditions leave just one possibility (assuming  $d > 0$ ), namely  $m = n = 2$  and hence

$$\mathcal{L}_{\text{renorm.}} = \frac{\rho}{2} (\partial\Phi)^2 + \text{nothing else.} \quad (\text{S.41})$$

In other words, in the infrared regime where the non-renormalizable interactions become irrelevant,  $\Phi(x)$  is a free massless field.

Problem 11.1(c):

Suppose a theory with a global phase symmetry  $U(1)$  appears to have a symmetry-breaking VEV of a complex field

$$S(x) = A(x) \times e^{i\Phi(x)}, \quad \langle A \rangle > 0. \quad (\text{S.42})$$

The radial field  $A$  is massive, so its fluctuations decouple from the low-energy effective theory. All we have at low energies is the VEV  $\langle A \rangle > 0$  and the massless Goldstone field  $\Phi(x)$  — and we saw in part (b) that  $\Phi(x)$  is effectively a free field with Lagrangian (S.41). It's non-canonically normalized, so

$$\langle \mathbf{T}\Phi(x)\Phi(y) \rangle = \frac{1}{\rho} \times G_0(x-y) \quad (\text{S.43})$$

where  $G_0(x-y)$  is the usual Feynman propagator for  $m^2 = 0$ . Consequently, according to eq. (S.37),

$$\begin{aligned} \langle \mathbf{T}S(x)S^*(y) \rangle &= \langle A \rangle^2 \times \left\langle \mathbf{T}e^{+i\hat{\Phi}(x_1)}e^{-i\hat{\Phi}(x_2)} \right\rangle \\ &= \langle A \rangle^2 \times \exp\left(\frac{G_0(x-y) - G_0(0)}{\rho}\right) \\ &\equiv C^2 \times \exp(G_0(x-y)/\rho) \end{aligned} \quad (\text{S.44})$$

where  $C = \langle A \rangle \times e^{-G_0(0)/2\rho}$  absorbs the UV corrections to the bare  $A$  parameter.

Similarly, in the Euclidean  $d$ -dimensional space of the statistical mechanics, the correlation function becomes

$$\langle S(x)S^*(y) \rangle = C^2 \times \exp(G_0^E(x-y)/\rho) \quad (\text{S.45})$$

where

$$G_0^E(x-y) = \int \frac{d^d p_E}{(2\pi)^E} \frac{e^{ip(x-y)}}{p_E^2} \quad (\text{S.46})$$

is the Euclidean propagator of a massless scalar. By the  $SO(d)$  symmetry, it depends only on

the Euclidean distance  $r = |x - y|$ , and for  $m^2 = 0$  this dependence is a pure power law:

$$G_0^E(x) = \frac{\Gamma(\frac{d}{2} - 1)}{4\pi^{d/2}} r^{2-d}. \quad (\text{S.47})$$

Specifically, in  $d = 3$  Euclidean dimensions  $D_0^E = 1/4\pi r$  and hence

$$\langle S(x)S^*(y) \rangle_{d=3} = C^2 \times \exp\left(\frac{1}{4\pi\rho} \times \frac{1}{r}\right). \quad (\text{S.48})$$

Likewise, in  $d = 4$  dimensions,

$$\langle S(x)S^*(y) \rangle_{d=4} = C^2 \times \exp\left(\frac{1}{2\pi^2\rho} \times \frac{1}{r^2}\right). \quad (\text{S.49})$$

In both cases, we see extremely strong self-correlations of the  $S(x)$  field at very short distances. On the other hand, in the long-distance limit the correlated expectation values (S.48) and (S.49) remain finite. Indeed, in any dimension greater than two

$$\langle S(x)S^*(y) \rangle_{d>2} \xrightarrow[r \rightarrow \infty]{} C^2 > 0. \quad (\text{S.50})$$

Physically, such asymptotic behavior is characteristic of non-trivial vacuum expectation values: By cluster expansion,

$$\langle S(x)S^*(y) \rangle_{d>2} \xrightarrow[r \rightarrow \infty]{} \langle S(x) \rangle \times \langle S^*(y) \rangle \equiv |\langle S \rangle|^2, \quad (\text{S.51})$$

thus the physical meaning of the limit (S.50) is

$$|\langle S(x) \rangle| = C > 0. \quad (\text{S.52})$$

In other words, *the  $U(1)$  symmetry of the complex  $S(x)$  field is spontaneously broken.*

On the other hand, in one Euclidean dimension,  $G_0^E = -\frac{1}{2}r$  and hence

$$\langle S(x)S^*(y) \rangle_{d=1} = C^2 \times \exp\left(-\frac{r}{2\rho}\right). \quad (\text{S.53})$$

This correlated expectation value remains finite at short distance and decreases exponentially at large distances. Indeed, for any dimension  $d$  less than two  $G_0^E(x-y) \rightarrow -\infty$  for  $r = |x-y| \rightarrow \infty$  and hence

$$\langle S(x)S^*(y) \rangle_{d<2} \xrightarrow{r \rightarrow \infty} 0. \quad (\text{S.54})$$

In terms of the cluster expansion (S.51), this means  $\langle S \rangle = 0$  — *despite the classical formula  $|S(x)| \equiv A > 0$ , the quantum theory has a zero VEV in  $d < 2$  and the  $U(1)$  phase symmetry remains unbroken.*

In the borderline case of exactly two dimensions,  $D_0^E = \frac{-1}{2\pi} \log r + \text{const}$  and hence

$$\langle S(x)S^*(y) \rangle_{d=2} \propto r^{-1/2\pi\rho}. \quad (\text{S.55})$$

Such scaling behavior correspond to the absence of dimensionful parameters in the theory — the spin wave modulus  $\rho$  is dimensionless for  $d = 2$ . Among other things, the scaling behavior (S.55) provides for vanishing of the correlated expectation value in the infinite distance limit. Thus, similarly to the  $d < 2$  case, we again have  $\langle S \rangle = 0$  and the unbroken  $O(2)$  symmetry.

The bottom line of this exercise is to illustrate the general rule: In  $d > 2$  spacetime dimensions, exact symmetries of the action may become spontaneously broken by vacuum expectation values such as  $\langle S \rangle$ . *But in  $d \leq 2$  dimensions, quantum effects destroy any VEV that would break a continuous symmetry.* However, the discrete symmetries may be spontaneously broken in two dimensions or in fractional dimensions  $d > 1$ .