Problem **1**(a):

Let us evaluate the trace of the Casimir operator C_2 over an irreducible multiplet (r). On one hand,

$$\begin{aligned} \mathrm{tr}_{(r)}\left(C_2 \stackrel{\mathrm{def}}{=} \sum_a T^a T^a\right) &= \sum_a \mathrm{tr}_{(r)}\left(T^a T^a\right) = \sum_a \mathrm{tr}\left(T^a_{(r)} T^b_{(r)}\right) \\ \langle\!\langle \text{by eq. (1)} \rangle\!\rangle \quad &= \sum_a R(r) \times (\delta^{aa} = 1) = R(r) \times \dim(G) \end{aligned} \tag{S.1}$$

where $\dim(G) \stackrel{\mathrm{def}}{=} \dim(\mathrm{Adj}(G))$ is the number of the generators of the symmetry group G — which is also the dimension of the adjoint representation of G, hence the notation. On the other hand,

$$\mathrm{tr}_{(r)}(C_2) = \mathrm{tr}_{(r)}\left(C_2\big|_{(r)}\right) = \mathrm{tr}\left(C(r) \times \mathbf{1}_{(r)}\right) = C(r) \times \dim(r). \tag{S.2}$$

Together, eqs. (S.1) and (S.2) immediately imply eq. (3), *quod erat demonstrandum.*

For the special case of $G = SU(2)$, an irreducible multiplets of isospin I has $C = \mathbf{I}^2 = I(I+1)$ and dimension $2I + 1$, hence

$$R(I) = C(I) \times \frac{\dim(I)}{\dim(G)} = I(I+1) \times \frac{2I+1}{3}. \tag{S.3}$$

Problem **1**(b):

Unlike the Casimir value $C(r)$, the index $R(r)$ is well defined for any complete multiplet (r), irreducible or otherwise. For a reducible multiplet

$$(r) = \bigoplus_{i=1}^{n} (r_i) \equiv (r_1) \oplus (r_2) \oplus \cdots \oplus (r_n)$$

one has

$$\begin{aligned} \mathrm{tr}_{(r)}\left(T^a T^b\right) &= \mathrm{tr}\left(T^a T^b\Big|_{\bigoplus_{i=1}^n (r_i)}\right) = \sum_{i=1}^{n} \mathrm{tr}\left(T^a T^b\Big|_{(r_i)}\right) \\ &= \sum_{i=1}^{n} \left(R(r_i) \times \delta^{ab}\right) = \delta^{ab} \times \sum_{i=1}^{n} R(r_i) \end{aligned} \tag{S.4}$$

and thus

$$R(r) \;=\sum_{i=1}^{n} R(r_i). \tag{S.5}$$

In particular, a reducible multiplet

$$(r) \;=\; \bigoplus_{i=1}^{n} (I_i)$$

of the isospin group $SU(2)$ has index

$$R(r) \;=\sum_{i=1}^{n} \tfrac{1}{3} I(I+1)(2I+1). \tag{S.6}$$

Now consider a bigger symmetry group G which contains the 'isospin' $SU(2)$ as a subgroup. Then any complete multiplet (r) of G is automatically a complete multiplet of the $SU(2) \subset G$. However, irreducible multiplets of G usually become reducible from the $SU(2)$ point of view, $(r) = (I_1) \oplus (I_2) \oplus \cdots \oplus (I_n)$; for example, the adjoint multiplet of $SU(3)$ decomposes into $(0) \oplus (\frac{1}{2}) \oplus (\frac{1}{2}) \oplus (1)$ of the $SU(2) \subset SU(3)$. Let T^1, T^2, and T^3 be generators of the $SU(2)$ subgroup of G. Then according to eq. (S.6),

$$\text{for } a,b=1,2,3, \quad \operatorname{tr}_{(r)}\left(T^a T^b\right) \;=\; \delta^{ab} \times \sum_{i=1}^{n} \tfrac{1}{3} I(I+1)(2I+1). \tag{S.7}$$

Now, let us suppose that the Lie group G is *simple*, that is, all its generators are related to each other by the symmetry G itself. In this case, for any complete multiplet (r) of G

$$\operatorname{tr}_{(r)}\left(T^a T^b\right) \;=\; R(r) \times \delta^{ab}, \quad \text{same } R(r)\ \forall a,b=1,\ldots,\dim(G). \tag{S.8}$$

Combining this formula with eq. (S.7) we immediately obtain

$$R(r) \;=\sum_{i=1}^{n} \tfrac{1}{3} I(I+1)(2I+1), \tag{4}$$

quod erat demonstrandum.

Caveat: We have silently assumed that $T^{1,2,3}$ have the same normalization as generators of G as the have as generators of the $SU(2) \subset G$. This assumption is correct for the $SU(2) \subset SU(N)$ discussed in parts (c) and (d) of this problem, but it would fail for a different (*i.e.*, inequivalent) $SU(2)$ subgroup. In general, properly normalized $SU(2) \subset G$ generators $I^{1,2,3}$ are related to the properly normalized generators of G as

$$I^a = T^{(a)} \times \sqrt{k} \tag{S.9}$$

where $T^{(1)}$, $T^{(2)}$, and $T^{(3)}$ are 3 generators of G which happen to satisfy $[T^{(a)}, T^{(b)}] = i\epsilon^{abc}T^{(c)}/\sqrt{k}$. The k here is always a positive integer; it's called *the level of embedding of the $SU(2)$ into G*. For example, consider the $SU(2)$ subgroup of $SU(3)$ which acts on the fundamental triplet as a real $SO(3)$ rotation. This subgroup is generated by the

$$I^1 = \sqrt{4} \times T^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & +i \\ 0 & -i & 0 \end{pmatrix}, \quad I^2 = \sqrt{4} \times T^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ +i & 0 & 0 \end{pmatrix}, \quad I^3 = \sqrt{4} \times T^2 = \begin{pmatrix} 0 & +i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{S.10}$$

(note $T^a = \frac{1}{2}\lambda^a$), so its embedding level is $k = 4$.

When you decompose a multiplet (r) of G into irreducible multiplets of an $SU(2)$ subgroup, you should take into account the level at which this $SU(2)$ is embedded into G. As written, eq. (4) works only for the $k = 1$ subgroups; for other embedding levels,

$$R(r) = \frac{1}{k} \sum_{i=1}^{n} \tfrac{1}{3} I(I+1)(2I+1). \tag{S.11}$$

Note that the decomposition of the G multiplet (r) into $SU(2)$ multiplets depends on the $SU(2)$ embedding into G. For example, under the $k = 1$ subgroup $SU(2) \subset SU(3)$

$$\text{triplet} = (\tfrac{1}{2}) \oplus (0), \qquad \text{octet} = (1) \oplus (\tfrac{1}{2}) \oplus (\tfrac{1}{2}) \oplus (0), \tag{S.12}$$

while under the $k = 4$ subgroup (S.10)

$$\text{triplet} = (1), \qquad \text{octet} = (1) \oplus (2). \tag{S.13}$$

In both cases, eq. (S.11) produces the same index R for each $SU(3)$ multiplet, *e.g.* $R(\text{triplet}) = \frac{1}{2}$, $R(\text{octet}) = 3$, but only if you remember the $1/k$ factor in front of the sum.

Problem $\mathbf{1}$(c):

From the $SU(2) \subset SU(N)$ point of view, the fundamental representation $\mathbf{N}$ of the $SU(N)$ decomposes into one doublet plus $(N-2)$ singlets,

$$\mathbf{N} \;=\; \mathbf{2} \;+\; (N-2)\times\mathbf{1} \;\equiv\; (I=\tfrac{1}{2}) \;+\; (N-2)\times(I=0), \tag{S.14}$$

hence according to eq. (4),

$$R(\mathbf{N}) \;=\; R(I=\tfrac{1}{2}) \;+\; (N-2)\times R(I=0) \;=\; \tfrac{1}{2} \;+\; (N-2)\times 0 \;=\; \tfrac{1}{2}$$

and consequently

$$C(\mathbf{N}) \;=\; R(\mathbf{N})\times\frac{\dim(G)}{\dim(\mathbf{N})} \;=\; \frac{1}{2}\times\frac{N^2-1}{N} \;=\; \frac{N^2-1}{2N} \tag{4}$$

Now consider the adjoint representation of the $SU(N)$. Let us form a tensor product of the fundamental representation $\mathbf{N}$ and the conjugate (anti-fundamental) representation $\overline{\mathbf{N}}$. Given the transformation laws

$$\begin{aligned} \Psi &\to U\Psi, \qquad i.e.\ \ \Psi'_j \;=\; U_j^{\,k}\Psi_k\,, \\ \overline{\Psi} &\to \overline{\Psi}U^\dagger, \qquad i.e.\ \ \overline{\Psi}'^{\ell} \;=\; \overline{\Psi}^m\,U^{*\ell}_{\ m}\,, \end{aligned}$$

it follows that the tensor product is a hermitian $N\times N$ matrix Φ_j^k which transforms as

$$\Phi' \;=\; U\Phi U^\dagger \qquad i.e.\ \ \Phi'^{\ell}_j \;=\; U_j^{\,k}\Phi_k^m U^{*\ell}_{\ m}\,. \tag{5}$$

This matrix is a reducible multiplet Adj $+\,\mathbf{1}$ of the $SU(N)$: The trace $\mathrm{tr}(\Phi)$ is an invariant singlet, while the traceless part $\Phi_i^{\,j} - \delta_i^j \times \mathrm{tr}(\Phi)/N$ forms the adjoint multiplet (*cf.* homework set 3 back in September). In other words,

$$\mathbf{N}\otimes\overline{\mathbf{N}} \;=\; \mathrm{Adj}\oplus\mathbf{1} \tag{S.15}.$$

In $SU(2)$ $\overline{\mathbf{2}} = \mathbf{2}$, so from the $SU(2)\subset SU(N)$ point of view, both the fundamental and the anti-fundamental multiplets of the $SU(N)$ decompose into similar sets of one doublet and $N-2$

singlets. Therefore,

$$\begin{aligned}
[\mathrm{Adj}\ +\ \mathbf{1}]_{SU(N)} &= \left[\mathbf{N}\otimes\overline{\mathbf{N}}\right]_{SU(N)} \\
&= [\mathbf{2}\ +\ (N-2)\times\mathbf{1}]_{SU(2)}\ \otimes\ [\mathbf{2}\ +\ (N-2)\times\mathbf{1}]_{SU(2)} \\
&= \left[(\mathbf{2}\otimes\mathbf{2})\ +\ 2(N-2)\times(\mathbf{2}\otimes\mathbf{1})\ +\ (N-2)^2\times(\mathbf{1}\otimes\mathbf{1})\right]_{SU(2)} \\
&= \left[\mathbf{3}\ +\ \mathbf{1}\ +\ 2(N-2)\times\mathbf{2}\ +\ (N-2)^2\times\mathbf{1}\right]_{SU(2)}\ , \\
\textit{i.e.}\quad [\mathrm{Adj}]_{SU(N)} &= \left[\mathbf{3}\ +\ 2(N-2)\times\mathbf{2}\ +\ (N-2)^2\times\mathbf{1}\right]_{SU(2)}\ ,
\end{aligned} \tag{S.16}$$

and consequently

$$\begin{aligned}
R(\mathrm{Adj}) &= R_{SU(2)}(\mathbf{3})\ +\ 2(N-2)\times R_{SU(2)}(\mathbf{2})\ +\ (N-2)^2\times R_{SU(2)}(\mathbf{1}) \\
&= 2\ +\ 2(N-2)\times\tfrac{1}{2}\ +\ (N-2)^2\times 0\ =\ N.
\end{aligned} \tag{S.17}$$

Finally,

$$C(G)\ \stackrel{\mathrm{def}}{=}\ C(\mathrm{Adj}(G))\ =\ R(\mathrm{Adj})\times\frac{\dim(G)}{\dim(G)}\ =\ R(\mathrm{Adj})\ = N. \tag{S.18}$$

Problem **1**(d):

Consider the two-index symmetric tensor $S_{(ij)}$ representation of the $SU(N)$ symmetry group. Denote the index $i=\alpha$ if $i=1,2$ or $i=\mu$ if $i=3,4,\ldots,N$ and likewise $j=\beta$ if $j=1,2$ and $j=\nu$ if $j=3,4,\ldots,N$. Thus, the complete set of independent $S_{(ij)}$ decomposes into $S_{(\alpha\beta)}$, $S_{\alpha,\mu}\equiv S_{\mu,\alpha}$ and $S_{(\mu\nu)}$. The $SU(2)\subset SU(N)$ acts on indices $\alpha,\beta=1,2$ and ignores indices $\mu,\nu=3,4,\ldots,N$, so from the $SU(2)$ point of view, $S_{(\alpha\beta)}$ is a triplet, $S_{\alpha,\mu}$ are $N-2$ separate doublets, and $S_{(\mu\nu)}$ are $(N-2)(N-1)/2$ singlets. Consequently,

$$\begin{aligned}
R(S) &= R_{SU(2)}(\mathbf{3})\ +\ (N-2)\times R_{SU(2)}(\mathbf{2})\ +\ \tfrac{1}{2}(N-1)(N-2)\times R_{SU(2)}(\mathbf{1}) \\
&= 2\ +\ (N-2)\times\tfrac{1}{2}\ +\ \tfrac{1}{2}(N-1)(N-2)\times 0\ =\ \tfrac{1}{2}(N+2),
\end{aligned} \tag{S.19}$$

and hence

$$C(S) =\ R(S)\times\frac{\dim(G)}{\dim(S)}\ =\ \frac{N+2}{2}\times\frac{N^2-1}{\frac{1}{2}N(N+1)}\ =\ \frac{N^2+N-2}{N}\,. \tag{S.20}$$

Similarly, the two-index anti-symmetric tensor $A_{[ij]}$ decomposes into $A_{[\alpha\beta]}$, $A_{\alpha,\mu}$, and $A_{[\mu\nu]}$. In $SU(2)$, the $A_{[\alpha\beta]}$ is equivalent to the trivial singlet $A\times\epsilon_{[\alpha\beta]}$, the $A_{\alpha,\mu}$ are $N-2$ doublets, and

$A_{[\mu\nu]}$ are $(N-2)(N-3)/2$ singlets. Altogether

$$(A) \;=\; (N-2)\times\mathbf{2} \;+\; \text{singlets},$$

therefore

$$R(A) \;=\; (N-2)\times\tfrac{1}{2} \;+\; 0 \;=\; \tfrac{1}{2}(N-2) \tag{S.21}$$

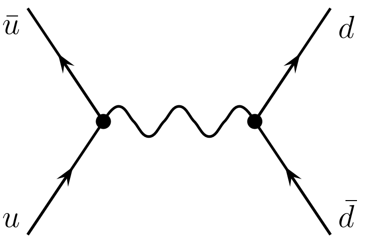
and

$$C(A) = \; R(A)\times\frac{\dim(G)}{\dim(A)} \;=\; \frac{N-2}{2}\times\frac{N^2-1}{\frac{1}{2}N(N-1)} \;=\; \frac{N^2-N-2}{N}\,. \tag{S.22}$$

<u>Problem **2**</u>:

At the tree level of QCD,

$$i\mathcal{M}(u\bar{u}\to d\bar{d}) \;=$$



(S.23)

$$= \;\frac{ig^2}{s}\times\bar{v}(\bar{u})\gamma^\mu u(u)\,(T^a)^i{}_j\times\bar{u}(d)\gamma_\mu v(d)\,(T^a)^k{}_\ell$$

where $s=E^2_{\text{c.m.}}$, the quarks and the antiquarks have color indices i,j,k,ℓ, and the virtual gluon has gauge index a in the adjoint representation; the summation over a is implicit. Except for the gauge indices, the $u\bar{u}\to d\bar{d}$ process in QCD is completely analogous to the $e^-e^+\to\mu^-\mu^+$ pair production in QED. In particular, summing / averaging $|\mathcal{M}|^2$ over the fermion's spins yields

$$\begin{aligned}\tfrac{1}{4}\sum_{\text{all spins}}\left|\bar{v}(\bar{u})\gamma^\mu u(u)\,\bar{u}(d)\gamma_\mu v(d)\right|^2 \;&\approx\; \tfrac{1}{4}\,\mathrm{tr}(\not{p}_{\bar{u}}\gamma^\mu\,\not{p}_u\gamma^\nu)\times\mathrm{tr}(\not{p}_{\bar{d}}\gamma_\mu\,\not{p}_d\gamma_\nu)\\ &=\; 2(t^2+u^2) \;=\; s^2(1+\cos^2\theta_{\text{c.m.}})\end{aligned} \tag{S.24}$$

where the approximation is neglecting the quark masses m_u and m_d.

The new part of this exercise is summing / averaging over the color indices. By hermiticity of the Lie Algebra matrices T^a, we have

$$\left((T^a)^i_{\ j}\,(T^a)^k_{\ \ell}\right)^* \;=\; (T^a)^j_{\ i}\,(T^a)^\ell_{\ k} \;=\; (T^b)^j_{\ i}\,(T^b)^\ell_{\ k} \tag{S.25}$$

— note the implicit summation over a or b — and hence

$$\begin{aligned}\sum_{i,j,k,\ell}\left|(T^a)^i_{\ j}\,(T^a)^k_{\ \ell}\right|^2 &= \sum_{i,j,k,\ell}(T^a)^i_{\ j}\,(T^a)^k_{\ \ell}\times(T^b)^j_{\ i}\,(T^b)^\ell_{\ k}\\ &= \sum_{ij}(T^a)^i_{\ j}\,(T^b)^j_{\ i}\times\sum_{k,\ell}(T^a)^k_{\ \ell}\,(T^b)^\ell_{\ k}\\ &= \operatorname{tr}(T^aT^b)\times\operatorname{tr}(T^aT^b)\end{aligned} \tag{S.26}$$

For the moment, let us consider 'quarks' belonging to some generic multiplet (r) of some generic gauge group G. In such a generic case, $\operatorname{tr}(T^aT^b) = R(r)\times\delta^{ab}$ where $R(r)$ is the index of the quark multiplet, *cf.* problem 1, and hence

$$\sum_{a,b}\operatorname{tr}(T^aT^b)\times\operatorname{tr}(T^aT^b) \;=\; R^2(r)\times\sum_{a,b}\delta^{ab}\delta^{ab} \;=\; R^2(r)\times\dim(G). \tag{S.27}$$

Substituting this formula into eq. (S.26) then gives

$$\sum_{i,j,k,\ell}\left|\sum_a(T^a)^i_{\ j}\,(T^a)^k_{\ \ell}\right|^2 \;=\; R^2(r)\times\dim(G),$$

or, for the average over the initial 'colors' i and j,

$$\frac{1}{\dim^2(r)}\sum_{i,j}\sum_{k,\ell}\left|\sum_a(T^a)^i_{\ j}\,(T^a)^k_{\ \ell}\right|^2 \;=\; \frac{R^2(r)\,\dim(G)}{\dim^2(r)} \;=\; \frac{C^2(r)}{\dim(G)}\,. \tag{S.28}$$

Specializing to the 'quarks' in the fundamental representation of an $SU(N)$ gauge group, we have $R(r)=\frac{1}{2}$, $\dim(r)=N$ and $\dim(G)=N^2-1$, hence eq. (S.28) evaluates to $(N^2-1)/(4N^2)$; for the actual QCD $N=3$ and the color sum / average (S.28) gives 2/9.

Altogether, $|\mathcal{M}|^2$ summed / averaged over both spins and colors of all the fermions is

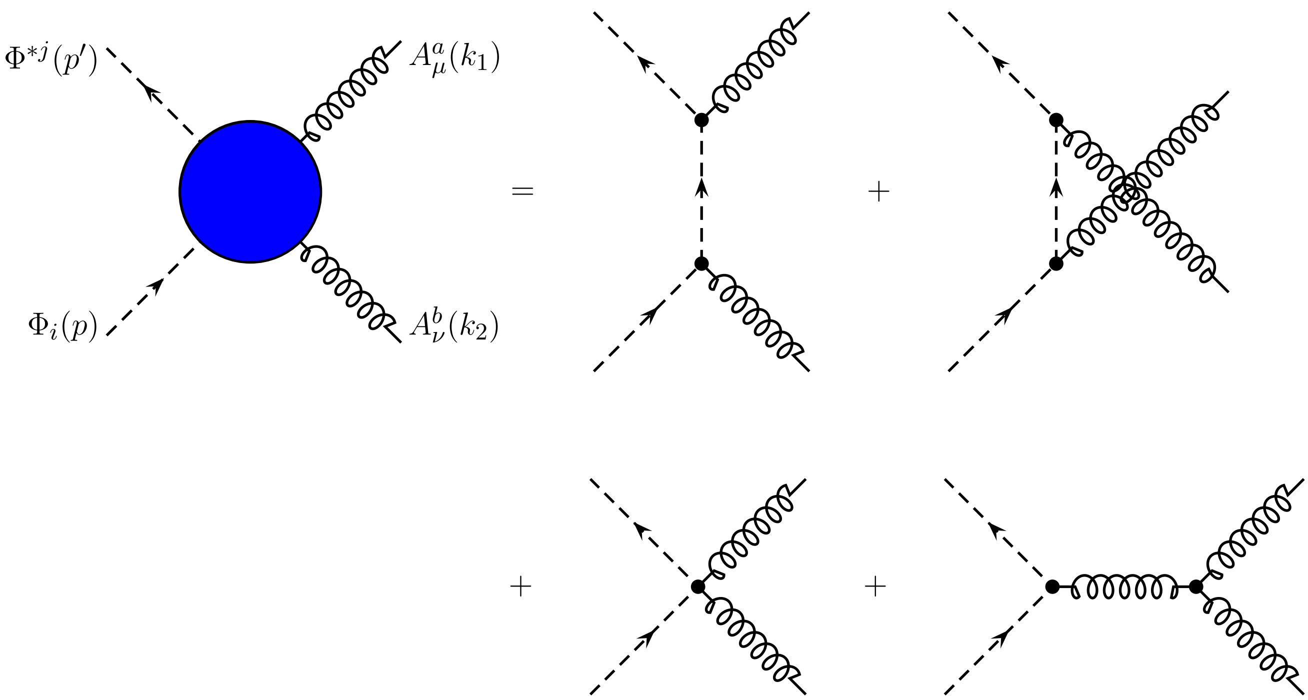
$$\frac{2}{9} \times g^4 (1 + \cos^2 \theta_{\rm c.m.}) \tag{S.29}$$

and hence the total cross section

$$\sigma(u\bar{u} \to d\bar{d}) \;=\; \frac{8\pi\alpha_{\rm QCD}^2}{27 E_{\rm c.m.}^2}\,. \tag{S.30}$$

Problem **3**(a):

In problem 2(b) of the previous homework set you should have calculated the tree-level amplitude for the $\Phi\Phi^* \to gg$ annihilation process. Copying from the the solutions, there are 4 tree diagrams



(S22.36)

and the net tree-level amplitude is $\mathcal{M} = \mathcal{M}^{\mu\nu} \times e^*_{1\mu} e^*_{2\nu}$ where

$$\begin{aligned}
\mathcal{M} &= \mathcal{M}^{\mu\nu} \times e^*_{1\mu} e^*_{2\nu}\,, \\
\mathcal{M}^{\mu\nu} &= \frac{g^2}{(p-k_1)^2 - m^2}\,(k_2 - 2p')^\nu (2p - k_1)^\mu \left(T^b T^a\right)^j_{\ i} \\
&+ \frac{g^2}{(p-k_2)^2 - m^2}\,(k_1 - 2p')^\mu (2p - k_2)^\nu \left(T^a T^b\right)^j_{\ i} \\
&- g^2 g^{\mu\nu} \left\{T^a, T^b\right\}^j_{\ i} \\
&- \frac{ig^2}{(k_1+k_2)^2}\,(p-p')_\lambda \left(T^c\right)^j_{\ i} \\
&\quad \times f^{abc}\Big(g^{\mu\nu}(k_1-k_2)^\lambda + g^{\nu\lambda}(2k_2+k_1)^\mu + g^{\lambda\mu}(-2k_1-k_2)^\nu\Big).
\end{aligned} \tag{S22.37}$$

For the transverse gauge bosons $(e^*_1 k_1) = (e^*_2 k_2) = 0$, this formula simplifies to

$$\begin{aligned}
\mathcal{M} \equiv \mathcal{M}^{\mu\nu} e^*_{1\mu} e^*_{2\nu} = &- \frac{4g^2}{t - m^2}\,(e^*_1 p)(e^*_2 p') \times \left(T^b T^a\right)^j_{\ i} \\
&- \frac{4g^2}{u - m^2}\,(e^*_1 p')(e^*_2 p) \times \left(T^a T^b\right)^j_{\ i} \\
&- g^2 (e^*_1 e^*_2) \times \left\{T^a, T^b\right\}^j_{\ i} \\
&- \frac{ig^2}{s}\,\Big[(u-t)(e^*_1 e^*_2) + 2(e^*_1 k_2)\big(e^*_2 (p-p')\big) - 2(e^*_2 k_1)\big(e^*_1 (p-p')\big)\Big] \\
&\quad \times f^{abc} \left(T^c\right)^j_{\ i}
\end{aligned} \tag{S.31}$$

where and s, t and u are Mandelstam's kinematic variables and

$$(u - t) = (p - p')_\lambda (k_1 - k_2)^\lambda.$$

Clearly,

$$\begin{aligned}
(T^a T^b) &= \tfrac{1}{2}\{T^a, T^b\} + \tfrac{1}{2}[T^a, T^b]\,, \\
(T^b T^a) &= \tfrac{1}{2}\{T^a, T^b\} - \tfrac{1}{2}[T^a, T^b]\,, \\
i f^{abc} T^c &= [T^a, T^b]\,,
\end{aligned} \tag{S.32}$$

so indeed, every term in eq. (S.31) can be written in the form (8). Specifically,

$$\begin{aligned} F &= -\frac{2g^2(e_1^*p)(e_2^*p')}{t-m^2} - \frac{2g^2(e_1^*p')(e_2^*p)}{u-m^2} - g^2(e_1^*e_2^*), \\ iG &= +\frac{2g^2(e_1^*p)(e_2^*p')}{t-m^2} - \frac{2g^2(e_1^*p')(e_2^*p)}{u-m^2} \\ &\quad - \frac{g^2}{s}\left[(u-t)(e_1^*e_2^*) + 2(e_1^*k_2)\big(e_2^*(p-p')\big) - 2(e_2^*k_1)\big(e_1^*(p-p')\big)\right]. \end{aligned} \tag{S.33}$$

Problem **3**(b):

First, let us average over the scalar particles' color indices $i, j = 1, 2, \ldots, \dim(r)$. For fixed gauge bosons a and b, let

$$M = F\{T^a_{(r)}, T^b_{(r)}\} + iG\,[T^a_{(r)}, T^b_{(r)}] \tag{S.34}$$

be a matrix (in the representation (r) of the gauge group) whose elements $M^j_{\ i}$ are annihilation amplitudes (6) for the scalar particles Φ_i and Φ^{*j} of specific colors i, j. Then averaging over those colors gives

$$\frac{1}{\dim^2(r)}\sum_{i,j}\left|M^j_{\ i}\right|^2 = \frac{1}{\dim^2(r)}\sum_{i,j} M^j_{\ i}(M^\dagger)^i_{\ j} = \frac{1}{\dim^2(r)}\,\mathrm{tr}(MM^\dagger). \tag{S.35}$$

For the specific form (S.34) of the matrix M, we write

$$\begin{aligned} M &= (F+iG)\,T^a_{(r)}T^b_{(r)} + (F-iG)\,T^b_{(r)}T^a_{(r)}, \\ M^\dagger &= (F+iG)^*\,T^b_{(r)}T^a_{(r)} + (F-iG)^*\,T^a_{(r)}T^b_{(r)}, \end{aligned} \tag{S.36}$$

and therefore

$$\begin{aligned} \mathrm{tr}(MM^\dagger) &= |F+iG|^2\ \mathrm{tr}_{(r)}(T^aT^bT^bT^a) + (F-iG)(F+iG)^*\ \mathrm{tr}_{(r)}(T^bT^aT^bT^a) \\ &\quad + (F+iG)(F-iG)^*\ \mathrm{tr}_{(r)}(T^aT^bT^aT^b) + |F-iG|^2\ \mathrm{tr}_{(r)}(T^bT^aT^aT^b) \\ &= 2(|F|^2+|G|^2)\ \mathrm{tr}_{(r)}(T^aT^aT^bT^b) + 2(|F|^2-|G|^2)\ \mathrm{tr}_{(r)}(T^aT^bT^aT^b) \\ &= 4|F|^2\ \mathrm{tr}_{(r)}(T^aT^aT^bT^b) + 2(|G|^2-|F|^2)\,\mathrm{tr}_{(r)}(T^a[T^a,T^b]T^b). \end{aligned} \tag{S.37}$$

where the second equality follows from the cyclic symmetry of the traces.

Our next step is to sum over the color indices a and b of the gauge bosons. In the context of eq. (S.37), we have

$$\sum_{a,b} \mathrm{tr}_{(r)}(T^aT^aT^bT^b) \;=\; \mathrm{tr}_{(r)}\left(\Big(\sum_a T^aT^a\Big)\Big(\sum_b T^bT^b\Big)\right) \;=\; \mathrm{tr}_{(r)}\,(C_2\,C_2) \;=\; C^2(r)\times\dim(r) \tag{S.38}$$

and

$$\begin{aligned}\sum_{a,b} \mathrm{tr}_{(r)}(T^a[T^a,T^b]T^b) &= \sum_{a,b}\sum_c if^{abc}\,\mathrm{tr}_{(r)}(T^aT^cT^b)\\ &= \tfrac{1}{2}\sum_{a,b,c} if^{abc}\,\mathrm{tr}_{(r)}(T^aT^cT^b - T^aT^bT^c)\\ &= \tfrac{1}{2}\sum_{a,b,c} if^{abc}\sum_d if^{cbd}\,\mathrm{tr}_{(r)}(T^aT^d)\\ &= \tfrac{1}{2}\sum_{a,b,c,d} (if^{abc})(if^{dbc})\times R(r)\delta^{ad}\\ &= \tfrac{1}{2}R(r)\sum_a\sum_{b,c}\Big(if^{abc} = (T^a_{(\mathrm{adj})})^{bc}\Big)\,\Big(if^{acb} = (T^a_{(\mathrm{adj})})^{cb}\Big)\\ &= \tfrac{1}{2}R(r)\sum_a \mathrm{tr}\Big(T^a_{(\mathrm{adj})}T^a_{(\mathrm{adj})}\Big)\\ &= \tfrac{1}{2}R(r)\times C(G)\,\dim(G)\\ &= \tfrac{1}{2}C(G)C(r)\,\dim(r).\end{aligned} \tag{S.39}$$

Therefore,

$$\sum_{a,b} \mathrm{tr}(MM^\dagger) \;=\; C(r)\,\dim(r)\times\big[4C(r)|F|^2 \;+\; C(G)\big(|G|^2-|F|^2\big)\big] \tag{S.40}$$

and hence in light of eq. (S.35),

$$\frac{1}{\dim^2(r)}\sum_{ij}\sum_{ab}|\mathcal{M}|^2 \;=\; \frac{C(r)}{\dim(r)}\times\big(4C(r)\,|F|^2 \,+\, C(\mathrm{adj})\,(|G|^2-|F|^2)\big)\,. \tag{9}$$

Eq. (10) follows from this as a special case.

Problem **3**(c):

Let us take a closer look at eqs. (S.33). In the center of mass frame, $\mathbf{p}' = -\mathbf{p}$, $\mathbf{k}_2 = -\mathbf{k}_1$, and the vector bosons' polarizations $e^\mu_{1,2}$ are purely spatial and transverse, $e^0_{1,2} = 0$ and $\mathbf{k}_{1,2} \cdot \mathbf{e}_{1,2} = 0$. Consequently, eqs. (S.33) simplify to

$$\begin{aligned} F &= 2g^2(\mathbf{e}_1^*\mathbf{p})(\mathbf{e}_2^*\mathbf{p}) \left(\frac{1}{t-m^2} + \frac{1}{u-m^2}\right) + g^2(\mathbf{e}_1^*\mathbf{e}_2^*), \\ G &= 2ig^2(\mathbf{e}_1^*\mathbf{p})(\mathbf{e}_2^*\mathbf{p}) \left(\frac{1}{t-m^2} - \frac{1}{u-m^2}\right) - ig^2\,\frac{u-t}{s}\,(\mathbf{e}_1^*\mathbf{e}_2^*). \end{aligned} \tag{S.41}$$

Furthermore, in the center of mass frame $E = E' = \omega_1 = \omega_2$, $|\mathbf{k}| = \omega = E$, $|\mathbf{p}| = \beta E$,

$$s = 4E^2, \qquad t - m^2 = -2E^2(1 - \beta\cos\theta), \qquad u - m^2 = -2E^2(1 + \beta\cos\theta),$$

hence

$$\begin{aligned} \frac{u-t}{s} &= \beta\cos\theta, \\ \frac{1}{t-m^2} + \frac{1}{u-m^2} &= \frac{-1}{m^2 + \mathbf{p}^2\sin^2\theta}, \\ \frac{1}{t-m^2} - \frac{1}{u-m^2} &= \frac{-\beta\cos\theta}{m^2 + \mathbf{p}^2\sin^2\theta}, \end{aligned}$$

and therefore

$$\begin{aligned} F &= g^2\left((\mathbf{e}_1^*\mathbf{e}_2^*) - \frac{2(\mathbf{e}_1^*\mathbf{p})(\mathbf{e}_2^*\mathbf{p})}{m^2 + \mathbf{p}^2\sin^2\theta}\right), \\ G &= -ig^2\left((\mathbf{e}_1^*\mathbf{e}_2^*) + \frac{2(\mathbf{e}_1^*\mathbf{p})(\mathbf{e}_2^*\mathbf{p})}{m^2 + \mathbf{p}^2\sin^2\theta}\right) \times \beta\cos\theta. \end{aligned} \tag{S.42}$$

Now consider the gluons' polarization vectors. For the problem at hand it is easier to use linear polarizations for which the $\mathbf{e}_1$ and $\mathbf{e}_2$ are real unit vectors. Specifically, for each gluon there is a choice of two transverse $\mathbf{e}$, one parallel to the $(\mathbf{p}, \mathbf{k})$ plane and one perpendicular to it. In the coordinate system where

$$\mathbf{p} = \beta E(0,0,1) \qquad \text{and} \qquad \mathbf{k} = E(\sin\theta, 0, \cos\theta), \tag{S.43}$$

the two polarization vectors are

$$\mathbf{e}_\parallel = (-\cos\theta, 0, +\sin\theta) \quad \text{and} \quad \mathbf{e}_\perp = (0,1,0). \tag{S.44}$$

For these vectors

$$(\mathbf{p}\mathbf{e}_{\|}) \;=\; \beta E \sin\theta, \qquad (\mathbf{p}\mathbf{e}_{\perp}) \;=\; 0, \tag{S.45}$$

so according to eqs. (S.42),

$$\text{for } \mathbf{e}_1 = \mathbf{e}_2 = \mathbf{e}_{\perp}\,, \quad F \;=\; g^2 \quad \text{and} \quad G \;=\; -ig^2\beta\cos\theta, \tag{S.46}$$

$$\text{for } \mathbf{e}_1 = \mathbf{e}_2 = \mathbf{e}_{\|}\,, \quad F \;=\; g^2(1-2A) \quad \text{and} \quad G \;=\; -ig^2(1+2A)\beta\cos\theta \tag{S.47}$$

where

$$A \;=\; \frac{\mathbf{p}^2\sin^2\theta}{m^2+\mathbf{p}^2\sin^2\theta}\,, \tag{S.48}$$

and finally

$$\text{for } \mathbf{e}_1 = \mathbf{e}_{\perp},\, \mathbf{e}_2 = \mathbf{e}_{\|} \text{ or } \textit{vice verse}, \quad F \;=\; G \;=0. \tag{S.49}$$

<u>Problem **3**(d)</u>:

According to eq. (S.49), the two gauge bosons produced in the $\Phi\Phi^*$ annihilation must have similar polarizations: either both are polarized $\|$ to the $(\mathbf{p},\mathbf{k})$ plane of scattering or both are polarized $\perp$ to the plane. Consequently, there are only two polarized partial cross sections to consider, namely

$$\begin{aligned}
\left(\frac{d\sigma(\perp)}{d\Omega}\right)_{\text{c.m.}} &= \frac{g^4}{64\pi^2E^2_{\text{c.m.}}\beta}\,\frac{C(r)}{\dim(r)}\Big(4C(r)\,-\,(1-\beta^2\cos^2\theta)C(G)\Big),\\
\left(\frac{d\sigma(\|)}{d\Omega}\right)_{\text{c.m.}} &= \frac{g^4}{64\pi^2E^2_{\text{c.m.}}\beta}\,\frac{C(r)}{\dim(r)}\Big(4C(r)(1-2A)^2\,-\,C(G)\big((1-2A)^2-\beta^2(1+2A)^2\cos^2\theta\big)\Big).
\end{aligned} \tag{S.50}$$

Note that the angular dependence of the $\|$ polarized partial cross section is more complicated than it looks because A is θ–dependent according to eq. (S.48).

In the limit of non-relativistic scalar particles, $\beta \ll 1$ leads to $A \ll 1$ and hence to the expected isotropy and polarization independence of the annihilation cross-section,

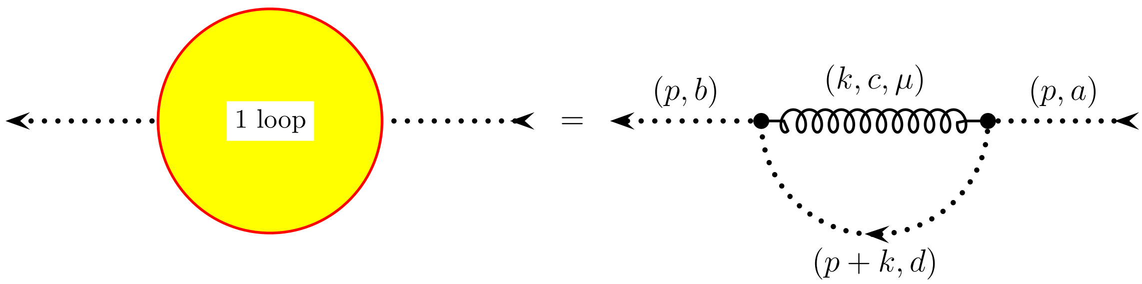
$$\left(\frac{d\sigma(\|)}{d\Omega}\right)_{\text{c.m.}} \approx \left(\frac{d\sigma(\perp)}{d\Omega}\right)_{\text{c.m.}} \approx \frac{g^4}{256\pi^2m^2\beta}\,\frac{C(r)\big(4C(r)-C(G)\big)}{\dim(r)}\,. \tag{S.51}$$

In the opposite limit of ultra-relativistic scalars, $\beta \approx 1$ leads to $A \approx 1$ (except for $\theta \approx 0$) and therefore

$$\begin{aligned}
\left(\frac{d\sigma(\perp)}{d\Omega}\right)_{\text{c.m.}} &\approx \frac{g^4}{16\pi^2 E_{\text{c.m.}}^2}\,\frac{C^2(r)}{\dim(r)}\,,\\
\left(\frac{d\sigma(\|)}{d\Omega}\right)_{\text{c.m.}} &\approx \frac{g^4}{16\pi^2 E_{\text{c.m.}}^2}\,\frac{C(r)}{\dim(r)}\left[C(r)\,+\,C(G)\,\frac{9\cos^2\theta-1}{4}\right].
\end{aligned} \tag{S.52}$$

Problem **4**(a–c):

At the one-loop level, the $\delta_2^{(\text{gh})}$ cancels the divergence of a single diagram



which evaluates (in the Feynman gauge) to

$$-i\Sigma^{ba}_{1\,\text{loop}}(p) \;=\; \int\!\frac{d^4k}{(2\pi)^4}\,\frac{i}{(p+k)^2+i0}\times\frac{-i}{k^2+i0}\times -gf^{cad}(p+k)_\mu\times -gf^{cdb}p^\mu. \tag{S.53}$$

In particular, the group factor here is

$$\sum_{c,d} f^{cad}f^{cdb} \;=\; \sum_{c,d}\bigl(-iT^a_{\text{adj}}\bigr)^{dc}\bigl(-iT^b_{\text{adj}}\bigr)^{cd} \;=\; -\operatorname{tr}_{\text{adj}}\bigl(T^aT^b\bigr) \;=\; -R(\text{adj})\times\delta^{ab}. \tag{S.54}$$

Using $R(\text{adj}) = C(\text{adj}) \equiv G(G)$ and taking care of all the signs and $\pm i$ factors, we arrive at

$$\Sigma^{ba}_{1\,\text{loop}}(p) \;=\; -ig^2C(G)\delta^{ba}\times p^\mu\times\int\!\frac{d^4k}{(2\pi)^4}\,\frac{(p+k)_\mu}{(k^2+i0)\times((p+k)^2+i0)}\,. \tag{S.55}$$

Note: the p^μ factor from the outgoing ghost vertex can be pulled outside the integral, which reduces its UV divergence form quadratic to linear. Moreover, by Lorentz symmetry the linear divergence cancels out, and the remaining integral becomes $p_\mu \times O(\log\Lambda)$. Consequently, the whole amplitude has form

$$\Sigma^{ba}_{1\,\text{loop}}(p) \;=\; \delta^{ba}\times p^2\times \Pi_{1\,\text{loop}}(p) \tag{S.56}$$

and we do not need the ghost-mass counterterm. Also, the logarithmic divergence of $\Pi(p)$ may be canceled by the $\delta_2^{(\text{gh})}$ counterterm as

$$\Pi(p) \;=\; P_{\text{loop}}(p) \;-\; \delta_2^{(\text{gh})}. \tag{S.57}$$

Indeed, introducing the Feynman parameter x into the momentum integral (S.55), we have

$$\begin{aligned}\int\frac{d^4k}{(2\pi)^4}\,\frac{(p+k)_\mu}{(k^2+i0)\times((p+k)^2+i0)} \;&=\; \int_0^1\!dx\int\frac{d^4\ell}{(2\pi)^4}\frac{\ell_\mu+(1-x)p_\mu}{[\ell^2+x(1-x)p^2+i0]^2}\\ &=\; \int_0^1\!dx\,(1-x)p_\mu\times\int\frac{d^4\ell}{(2\pi)^4}\frac{1}{[\ell^2+x(1-x)p^2+i0]^2}\end{aligned} \tag{S.58}$$

where the second equality follows from the $\ell\to-\ell$ symmetry of the integral. Using dimensional regularization for the UV divergence of the remaining integral, we have

$$\int\frac{d^4\ell}{(2\pi)^4}\frac{1}{[\ell^2+x(1-x)p^2+i0]^2} \;=\; \frac{i}{16\pi^2}\left(\frac{1}{\epsilon}\,+\,\log\frac{\mu^2}{-x(1-x)p^2}\,+\,\text{finite constant}\right), \tag{S.59}$$

hence

$$\int_0^1\!dx\,(1-x)p_\mu\times[\cdots] \;=\; \frac{ip_\mu}{32\pi^2}\left(\frac{1}{\epsilon}\,+\,\log\frac{\mu^2}{-p^2}\,+\,\text{finite constant}\right) \tag{S.60}$$

and consequently

$$\Sigma^{ba}_{1\,\text{loop}}(p) \;=\; +\frac{g^2C(G)}{32\pi^2}\times\delta^{ba}p^2\times\left(\frac{1}{\epsilon}\,+\,\log\frac{\mu^2}{-p^2}\,+\,\text{finite constant}\right). \tag{S.61}$$

To cancel the UV divergence here, we need

$$\delta_2^{(\text{gh})}[1\,\text{loop}] \;=\; +\frac{g^2C(G)}{32\pi^2}\times\frac{1}{\epsilon}\,. \tag{S.62}$$

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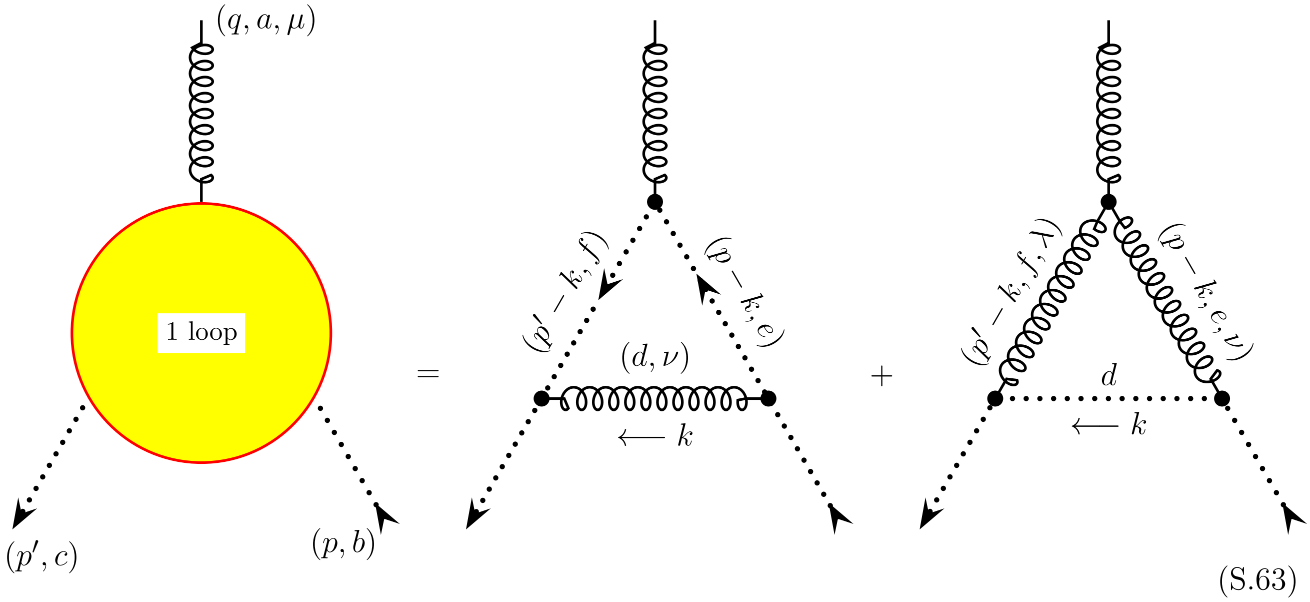
Now consider the $\delta_2^{(\mathrm{gh})}$ counterterm. At the one-loop level, it cancels the UV divergence of two diagrams



(S.63)

The first diagram here evaluates to

$$-igG^{\mu,abc}(1^{\mathrm{st}}) = \int\frac{d^4k}{(2\pi)^4}\,\frac{i}{(p-k)^2+i0}\times\frac{i}{(p'-k)^2+i0}\times\frac{-i}{k^2+i0}\times \\ \times -gf^{dbe}(p-k)^\nu\times -gf^{aef}(p'-k)^\mu\times -gf^{dfc}p'_\nu\,. \tag{S.64}$$

In particular, the group factor on the second line here amounts to

$$X_1^{abc} \equiv \sum_{d,e,f} f^{dbe}f^{aef}f^{dfc} = -\sum_{d,e,f} f^{dcf}f^{afe}f^{deb} = -\sum_{d,e,f}\left(-iT^d_{\mathrm{adj}}\right)^{cf}\left(-iT^a_{\mathrm{adj}}\right)^{fe}\left(-iT^d_{\mathrm{adj}}\right)^{eb} \\ = -i\sum_d\left(T^d_{\mathrm{adj}}T^a_{\mathrm{adj}}T^d_{\mathrm{adj}}\right)^{cb}. \tag{S.65}$$

The simplest way to take the last sum here is to use the abstract generators $\hat{T}^a$ of the Lie algebra instead of the specific matrices representing them in the adjoint multiplet:

$$
\begin{aligned}
\sum_d \hat{T}^d\hat{T}^a\hat{T}^d &= \sum_d \hat{T}^d\hat{T}^d\hat{T}^a + \sum_d \hat{T}^d\big[\hat{T}^a,\hat{T}^d\big] = \Big(\sum_d \hat{T}^d\hat{T}^d\Big)\times\hat{T}^a + \sum_{d,h}\hat{T}^d\times if^{adh}\hat{T}^h \\
&= \hat{C}_2\times\hat{T}^a + \frac{i}{2}\sum_{d,h} f^{adh}\times\big[\hat{T}^d,\hat{T}^h\big] = \hat{C}_2\times\hat{T}^a - \frac{1}{2}\sum_{d,h,j} f^{adh}f^{dhj}\hat{T}^j \\
&= \hat{C}_2\times\hat{T}^a - \frac{C(G)}{2}\times\hat{T}^a
\end{aligned}
\tag{S.66}
$$

where the last equality follows from eq. (S.54), $\sum_{dh} f^{adh}f^{dhj} = C(G)\times\delta^{aj}$. Consequently, in the adjoint representation of the Lie algebra

$$
\sum_d T^d_{\rm adj}T^a_{\rm adj}T^d_{\rm adj} = C(\text{adj})\times T^a_{\rm adj} - \frac{C(G)}{2}\times T^a_{\rm adj} = \frac{C(G)}{2}\times T^a_{\rm adj} \tag{S.67}
$$

and hence

$$
X_1^{abc} = -i\frac{C(G)}{2}\times\big(T^a_{\rm adj}\big)^{cb} = +\frac{C(G)}{2}\times f^{acb} = -\frac{C(G)}{2}\times f^{abc}. \tag{S.68}
$$

Plugging this group factor into the loop amplitude (S.64) and pulling all the constant factors outside the integral, we obtain

$$
-igG^{\mu,abc}(1^{\rm st}) = -\frac{g^3C(G)}{2}\times f^{abc}p'_\nu\times H_1^{\nu\mu} \tag{S.69}
$$

where

$$
H_1^{\mu\nu} = -i\int\!\frac{d^4k}{(2\pi)^4}\,\frac{(p-k)^\nu(p'-k)^\mu}{[(p-k)^2+i0]\times[(p'-k)^2+i0]\times[k^2+i0]}\,. \tag{S.70}
$$

Note that thanks to the k–independent factor p'_ν of the left vertex which we pulled out from the momentum integral, the remaining integral $H^{\mu\nu}$ is only logarithmically divergent. Consequently, the infinite part of $H^{\mu\nu}$ depends only on the leading terms of the numerator and denominator (as polynomials in k), thus

$$\begin{aligned}\left[H_1^{\mu\nu}\right]_\infty &= -i\int\frac{d^4k}{(2\pi)^4}\,\frac{k^\nu k^\mu+\cdots}{(k^2+i0)^3+\cdots} = -\int\frac{d^4k_E}{(2\pi)^4}\,\frac{k_E^\mu k_E^\nu+\cdots}{(k_E^2)^3+\cdots}\\ &= +\frac{g^{\mu\nu}}{4}\times\int\frac{d^4k_E}{(2\pi)^4}\,\frac{k_E^2+\cdots}{(k_E^2)^3+\cdots} = +\frac{g^{\mu\nu}}{4}\times\frac{1}{16\pi^2}\times\frac{1}{\epsilon}\,.\end{aligned} \tag{S.71}$$

Altogether, the divergent part of the first diagram amounts to

$$\left[-igG^{\mu,abc}(1^{\rm st})\right]_\infty = -gf^{abc}p'^\mu\times\frac{g^2C(G)}{128\pi^2}\times\frac{1}{\epsilon}\,. \tag{S.72}$$

Note that its dependence on the colors of the 3 external particle, on the index μ of the gluon, and on the ghosts' momenta have just the right form to be cancelled by the $\delta_1^{(\rm gh)}$ counterterm vertex,

$$-igG^{\mu,abc}(\text{counterterm}) = -gf^{abc}p'^\mu\times\delta_1^{(\rm gh)}\,. \tag{S.73}$$

In particular, to cancel just the first diagram we need

$$\delta_1^{(\rm gh)}(1^{\rm st}) = -\frac{g^2C(G)}{128\pi^2}\times\frac{1}{\epsilon}\,. \tag{S.74}$$

Now consider the second diagram (S.63), which evaluates to

$$\begin{aligned}-igG^{\mu,abc}(2^{\rm nd}) = \int\frac{d^4k}{(2\pi)^4}&\,\frac{-i}{(p-k)^2+i0}\times\frac{-i}{(p'-k)^2+i0}\times\frac{i}{k^2+i0}\times\\ &\times -gf^{ebd}k_\nu\times -gf^{fdc}p'_\lambda\times\\ &\times -gf^{aef}\begin{bmatrix} g^{\mu\lambda}(q-(k-p'))^\nu\\ +g^{\lambda\nu}((k-p')-(p-k))^\mu\\ +g^{\mu\nu}((p-k)-q)^\lambda\end{bmatrix}.\end{aligned} \tag{S.75}$$

Despite other complications, the group factor here is the same as in the first diagram,

$$X_2^{abc} = \sum_{d,e,f} f^{ebd}f^{fdc}f^{aef} = +\sum_{d,e,f} f^{dbe}f^{aef}f^{dfc} = X_1^{abc} = -\frac{C(G)}{2}\times f^{abc} \tag{S.76}$$

Pulling this group factor — as well as other k–independent factors outside of the integral (S.75), we obtain

$$-igG^{\mu,abc}(2^{\rm nd})\;=\;-\frac{g^3C(G)}{2}\times f^{abc}p'_\lambda\times H_2^{\lambda\mu} \tag{S.77}$$

where

$$H_2^{\lambda\mu}\;=\;+i\int\!\frac{d^4k}{(2\pi)^4}\,\frac{k_\nu\times\left[g^{\lambda\mu}(q+p'-k)^\nu\,+\,g^{\lambda\nu}(2k-p-p')^\mu\,+\,g^{\mu\nu}(p-q-k)^\lambda\right]}{[(p-k)^2+i0]\times[(p'-k)^2+i0]\times[k^2+i0]}\,. \tag{S.78}$$

Again, thanks to the k–independent factor p'_λ of the left vertex which we pulled out from the momentum integral, the remaining integral $H_2^{\lambda\mu}$ is only logarithmically divergent. Although the numerator in the integral (S.78) is much messier than in the integral (S.70) for the first diagram, its leading term for $k\to\infty$ is fairly simple

$$\text{(numerator)}\;=\;g^{\lambda\mu}\times(-k^2)\,+\,k^\lambda\times(2k)^\mu\,+\,k^\mu\times(-k)^\lambda\,+\,\cdots\;=\;-g^{\lambda\mu}k^2\,+\,k^\lambda k^\mu\,+\,\cdots \tag{S.79}$$

and that's all we need to get the infinite part of the integral. Specifically,

$$\begin{aligned}\left[H_2^{\lambda\mu}\right]_\infty\;&=\;+i\int\!\frac{d^4k}{(2\pi)^4}\,\frac{-g^{\mu\lambda}k^2\,+\,k^\lambda k^\mu\,+\,\cdots}{(k^2+i0)^3\,+\,\cdots}\\ &=\;+\int\!\frac{d^4k_E}{(2\pi)^4}\,\frac{+g^{\lambda\mu}_{\rm Mink}\times k_E^2\,+\,k_E^\lambda k_E^\mu\,+\,\cdots}{(k_E^2)^3\,+\,\cdots}\\ &=\;+\left(g^{\lambda\mu}_{\rm Mink}\,-\,\frac14\,g^{\lambda\mu}_{\rm Mink}\right)\times\int\!\frac{d^4k_E}{(2\pi)^4}\,\frac{k_E^2+\cdots}{k_E^6+\cdots}\\ &=\;\frac34\,g^{\lambda\mu}\times\frac{1}{16\pi^2}\times\frac{1}{\epsilon}\end{aligned} \tag{S.80}$$

and hence the infinite part of the second diagram's amplitude

$$\left[-igG^{\mu,abc}(2^{\rm nd})\right]_\infty\;=\;-gf^{abc}p'^\mu\times\frac{3g^2C(G)}{128\pi^2}\times\frac{1}{\epsilon}\,. \tag{S.81}$$

Again, this divergence has exactly the right form to be canceled by the $\delta_1^{\rm (gh)}$ counterterm vertex. This time, to cancel just the divergence of the second diagram we need

$$\delta_1^{\rm (gh)}(2^{\rm nd})\;=\;-\frac{3g^2C(G)}{128\pi^2}\times\frac{1}{\epsilon}\,. \tag{S.82}$$

Finally, combining the two diagrams' contributions, we get the net one-loop counterterm coefficient

$$\delta_1^{(\mathrm{gh})}[1\,\mathrm{loop}] \;=\; -\frac{g^2C(G)}{128\pi^2}\times\frac{1}{\epsilon}\;-\;\frac{3g^2C(G)}{128\pi^2}\times\frac{1}{\epsilon}\;=\;-\frac{g^2C(G)}{32\pi^2}\times\frac{1}{\epsilon}\,. \tag{S.83}$$

Comparing to the $\delta_2^{(\mathrm{gh})}$ for the ghosts' wave function renormalization

$$\delta_2^{(\mathrm{gh})}[1\,\mathrm{loop}] \;=\; +\frac{g^2C(G)}{32\pi^2}\times\frac{1}{\epsilon}\,, \tag{S.62}$$

we immediately obtain the difference

$$\delta_1^{(\mathrm{gh})}[1\,\mathrm{loop}] \;-\; \delta_2^{(\mathrm{gh})}[1\,\mathrm{loop}] \;=\; -\frac{g^2C(G)}{16\pi^2}\times\frac{1}{\epsilon}\,. \tag{S.84}$$

As promised, this difference agrees with the $\delta_1-\delta_2$ difference for the quarks we had calculated in class

$$\delta_1^{(\mathrm{q})}[1\,\mathrm{loop}] \;-\; \delta_2^{(\mathrm{q})}[1\,\mathrm{loop}] \;=\; -\frac{g^2C(G)}{16\pi^2}\times\frac{1}{\epsilon}\,. \tag{S.85}$$