Problem **1**(a):

In the matrix notation for the non-abelian gauge fields,

$$F_{\mu\nu} \;=\; \partial_\mu A_\nu \;-\; \partial_\nu A_\mu \;+\; ig[A_\mu, A_\nu], \tag{S.1}$$

hence

$$\begin{aligned}
\frac{g^2}{16\pi^2}\,\epsilon^{\alpha\beta\gamma\delta}\,\mathrm{tr}\Big(F_{\alpha\beta}F_{\gamma\delta}\Big) \;=\;& \quad\frac{g^2}{4\pi^2}\,\epsilon^{\alpha\beta\gamma\delta}\,\mathrm{tr}\Big(\partial_\alpha A_\beta\,\partial_\gamma A_\delta\Big) && \text{[two gluons]}\\
&+\;\frac{ig^3}{4\pi^2}\,\epsilon^{\alpha\beta\gamma\delta}\,\mathrm{tr}\Big([A_\alpha, A_\beta]\partial_\gamma A_\delta\Big) && \text{[three gluons]}\\
&-\;\frac{g^4}{16\pi^2}\,\epsilon^{\alpha\beta\gamma\delta}\,\mathrm{tr}\Big([A_\alpha, A_\beta]\,[A_\gamma, A_\delta]\Big) && \text{[four gluons].}
\end{aligned} \tag{S.2}$$

Thanks to the cyclic symmetry of the trace, for any matrices X, Y, and Z,

$$\mathrm{tr}([X,Y]Z) \;=\; \mathrm{tr}(X[Y,Z]). \tag{S.3}$$

Applying this rule to the 4–gluon term in the decomposition (S.2), we have

$$\text{[4–gluon anomaly]} \;\propto\; \epsilon^{\alpha\beta\gamma\delta}\,\mathrm{tr}\Big([A_\alpha, A_\beta]\,[A_\gamma, A_\delta]\Big) \;=\; \epsilon^{\alpha\beta\gamma\delta}\,\mathrm{tr}\Big(A_\alpha[A_\beta,[A_\gamma, A_\delta]]\Big). \tag{S.4}$$

In this double-commutator formula, we may use the Jacobi identity

$$[A_\beta,[A_\gamma,A_\delta]] \;+\; [A_\gamma,[A_\delta,A_\beta]] \;+\; [A_\delta,[A_\beta,A_\gamma]] \;=\; 0. \tag{S.5}$$

Since the $\epsilon^{\alpha\beta\gamma\delta}$ is symmetric with respect to *cyclic* permutations of the last three indices $\beta \to \gamma \to \delta \to \beta$, it follows that

$$\epsilon^{\alpha\beta\gamma\delta}[A_\beta,[A_\gamma,A_\delta]] \;=\; 0 \tag{S.6}$$

and hence

$$\text{[4–gluon anomaly]} \;=\; 0. \tag{S.7}$$

Quod erat demonstrandum.

Alternative proof:

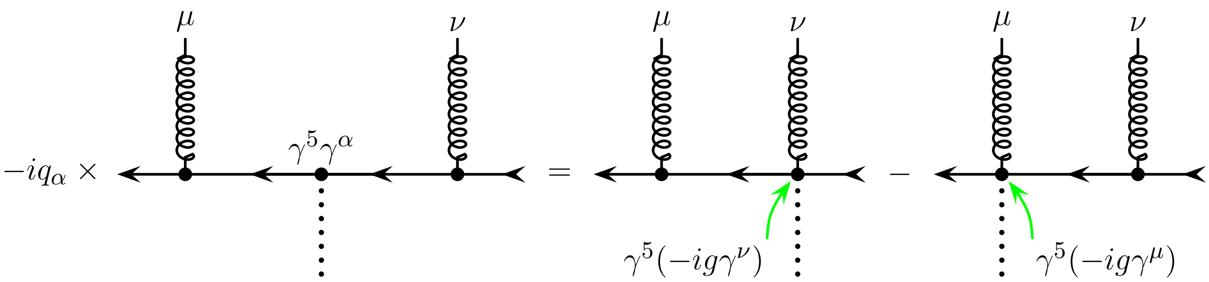
In solutions to problem 4 below, we shall see that

$$[\text{net anomaly}] \;\propto\; \epsilon^{\alpha\beta\gamma\delta}\,\mathrm{tr}\Big(F_{\alpha\beta}\,F_{\mu\nu}\Big) \;=\; 2\epsilon^{\alpha\beta\gamma\delta}\partial_\alpha\,\mathrm{tr}\left(A_\beta F_{\mu\nu}\,-\,\frac{2ig}{3}\,A_\beta A_\mu A_\nu\right). \tag{7.b}$$

By inspection, the RHS here has only 2–gluon and 3–gluon terms but no 4–gluon terms.

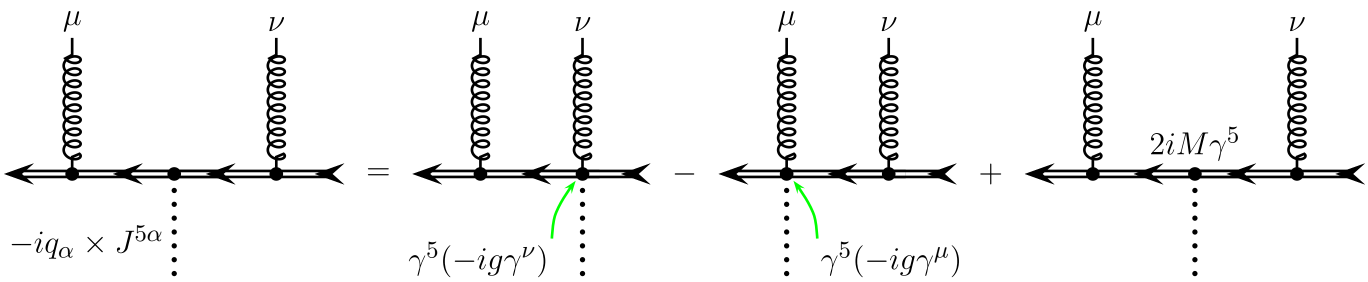
Problem **1**(b):

As explained in class, for a massless fermion



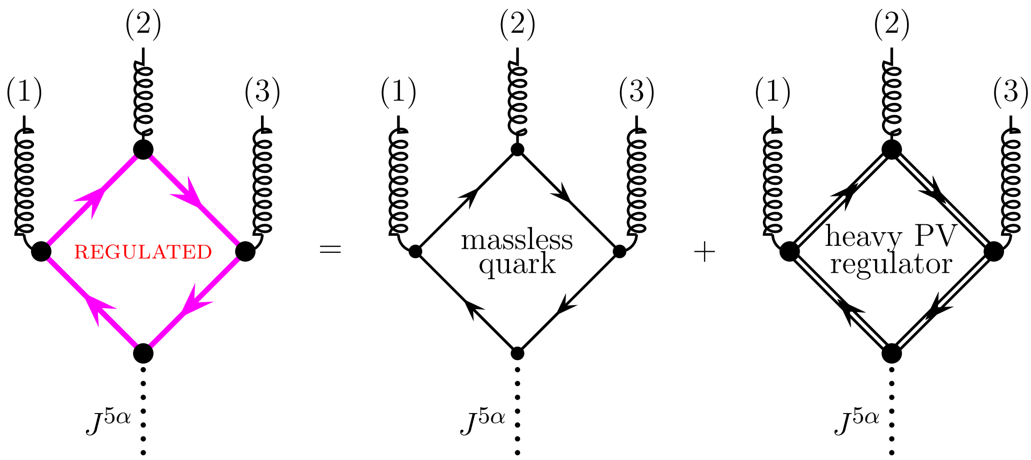
(S.8)

but for a massive fermion such as the Pauli–Villars compensator χ_{PV},



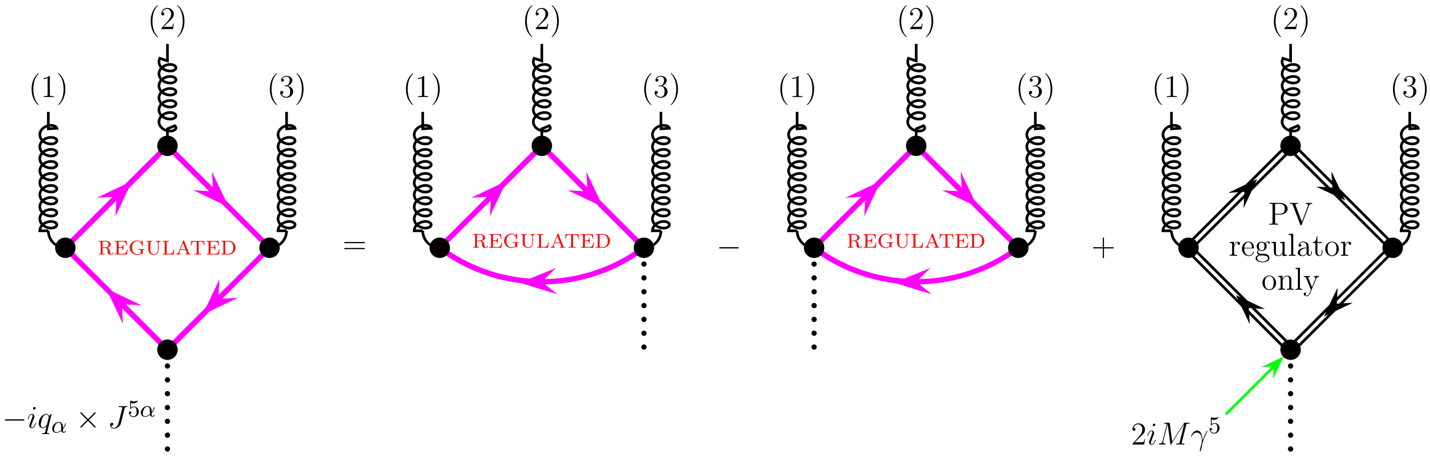
(S.9)

Now, let's apply these rules to the quadrangle anomaly diagram (3) regulated à la Pauli–Villars,



(S.10)

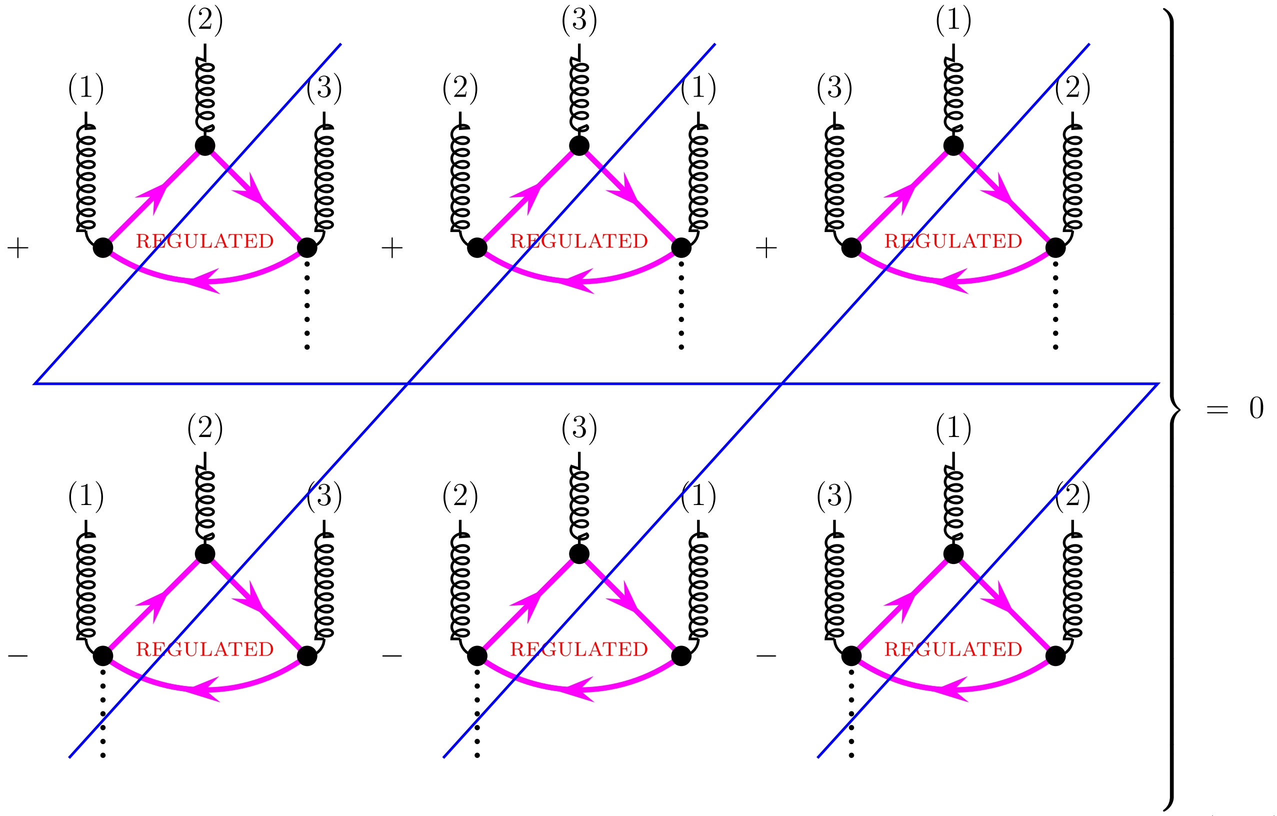
Multiplying the axial current $J^{5\alpha}(q)$ by $-iq_\alpha$, we get



(S.11)

Without the regulation, the two triangular diagrams on the right hand side would diverge as (UV cutoff scale)$^{+1}$, but subtracting similar loops of the heavy Pauli–Villars fermions makes them converge. Consequently, we may shift the momentum integration variables $p^\mu \to p^\mu + \text{const}$ separately for each diagram, and this leads to cancellation of the triangle diagrams once we sum over gluon permutations.

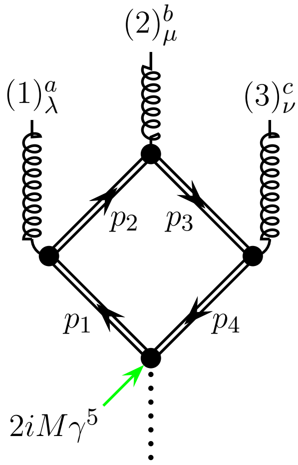
Actually, it is enough to sum over just the cyclic permutations $(1) \to (2) \to (3) \to (1)$ of the three gluons which have the same group factor $\mathrm{tr}(T_{(3)}T_{(2)}T_{(1)})$. (Note the cyclic symmetry of the trace.) Summing the triangular diagrams on the right hand side of eq. (S.11), we have diagram-by-diagram cancellation:



(S.12)

Strictly speaking, this cancellation involves separate shifting of the integration momentum for each diagram, but that's OK because the regulated diagrams are finite.

But the quadrangle diagrams involving only the massive PV regulator loops — *cf.* the last term in eq. (S.11) — do not cancel after we sum over gluon permutations, and that's what leads to the quadrangle anomaly. To see how this works, let's evaluate one such loop



$$= \; -2Mg^3\,\mathrm{Tr}(T^cT^bT^a)\times\int\!\frac{d^4p_1}{(2\pi)^4}\,\frac{\mathcal{N}}{\mathcal{D}} \tag{S.13}$$

where

$$\frac{\mathcal{N}}{\mathcal{D}} \;=\; \mathrm{tr}\left(\gamma^5\,\frac{1}{\not{p}_4-M+i0}\,\gamma^\nu\,\frac{1}{\not{p}_3-M+i0}\,\gamma^\mu\,\frac{1}{\not{p}_2-M+i0}\,\gamma^\lambda\,\frac{1}{\not{p}_1-M+i0}\right). \tag{S.14}$$

and consequently

$$\mathcal{D} \;=\; \prod_{i=1}^{4}(p_1^2-M^2+i0), \tag{S.15}$$

$$\mathcal{N} \;=\; \mathrm{tr}\Big(\gamma^5(\not{p}_4+M)\gamma^\nu(\not{p}_3+M)\gamma^\mu(\not{p}_2+M)\gamma^\lambda(\not{p}_1+M)\Big). \tag{S.16}$$

As usual, we may express the denominator in terms of the Feynman parameters x, y, z, w, thus

$$\frac{1}{\mathcal{D}} \;=\; 24\int d^4(x,y,z,w)\,\delta(x+y+z+w-1)\times\frac{1}{[\ell^2-O(k^2)-M^2+i0]^4} \tag{S.17}$$

where the details of the $O(k^2)$ expression is not important because the Pauli–Villars mass M is much bigger than all the external momenta. What's important are the relations between the loop momentum ℓ and the propagator momenta

$$\begin{aligned} p_1 &= \ell + q_1 = \ell - (y+z+w)k_1 - (z+w)k_2 - wk_3\,, \\ p_2 &= \ell + q_2 = \ell + xk_1 - (z+w)k_2 - wk_3\,, \\ p_3 &= \ell + q_3 = \ell + xk_1 + (x+y)k_2 - wk_3\,, \\ p_4 &= \ell + q_4 = \ell + xk_1 + (x+y)k_2 + (x+y+z)k_3 \end{aligned} \tag{S.18}$$

which appear in the numerator

$$\mathcal{N} = \mathrm{tr}\Big(\gamma^5(\not\ell + M + \not q_4)\gamma^\nu(\not\ell + M + \not q_3)\gamma^\mu(\not\ell + M + \not q_2)\gamma^\lambda(\not\ell + M + \not q_1)\Big). \tag{S.19}$$

The quadrangle anomaly is proportional to

$$I = M\int\frac{d^4\ell}{(2\pi)^4}\,\frac{\mathcal{N}}{\mathcal{D}} \tag{S.20}$$

or rather to its $M\to\infty$ limit, so let's expand I into powers of q_i/M. The momentum integral is UV convergent, so it's dominated by $\ell = O(M)$. By dimensional analysis, $I = O(M)$, hence

$$I(q_i) = M\times C^{(0)} + \sum_i q_i^\alpha C^{(1)}_{i,\alpha} + O\left(\frac{q^2}{M}\right), \tag{S.21}$$

and we need only the q-independent and the linear-in-q terms here. Hence, we may approximate the denominator as

$$\frac{1}{\mathcal{D}} \approx \frac{1}{[\ell^2 - M^2 + i0]^4} \tag{S.22}$$

and expand the numerator (S.19) in powers of q_i and keep only the free and linear terms,

$$\begin{aligned} \mathcal{N} &\approx \mathcal{N}_0 + \sum_{i=1}^4 q_{i,\alpha}\mathcal{N}_i^\alpha, \\ \mathcal{N}_0 &= \mathrm{tr}\Big(\gamma^5(\not\ell + M)\gamma^\nu(\not\ell + M)\gamma^\mu(\not\ell + M)\gamma^\lambda(\not\ell + M)\Big), \\ \mathcal{N}_1 &= \mathrm{tr}\Big(\gamma^5(\not\ell + M)\gamma^\nu(\not\ell + M)\gamma^\mu(\not\ell + M)\gamma^\lambda\gamma^\alpha\Big), \\ \mathcal{N}_2 &= \mathrm{tr}\Big(\gamma^5(\not\ell + M)\gamma^\nu(\not\ell + M)\gamma^\mu\gamma^\alpha\gamma^\lambda(\not\ell + M)\Big), \\ \mathcal{N}_3 &= \mathrm{tr}\Big(\gamma^5(\not\ell + M)\gamma^\nu\gamma^\alpha\gamma^\mu(\not\ell + M)\gamma^\lambda(\not\ell + M)\Big), \\ \mathcal{N}_4 &= \mathrm{tr}\Big(\gamma^5\gamma^\alpha\gamma^\nu(\not\ell + M)\gamma^\mu(\not\ell + M)\gamma^\lambda(\not\ell + M)\Big). \end{aligned} \tag{S.23}$$

Now let's evaluate these traces, starting with

$$\begin{aligned}
\mathcal{N}_1 \;&=\; M^3\times\mathrm{tr}\Big(\gamma^5\gamma^\nu\gamma^\mu\gamma^\lambda\gamma^\alpha\Big)\\
&\qquad +\; M\times\mathrm{tr}\Big(\gamma^5\big(\not\ell\gamma^\nu\not\ell\gamma^\mu+\not\ell\gamma^\nu\gamma^\mu\not\ell\,+\,\gamma^\nu\not\ell\gamma^\mu\not\ell\,\big)\gamma^\lambda\gamma^\alpha\Big)\\
&\quad\langle\!\langle\,\text{using }\;\not\ell\gamma^\nu\not\ell\gamma^\mu+\not\ell\gamma^\nu\gamma^\mu\not\ell\,+\,\gamma^\nu\not\ell\gamma^\mu\not\ell\;=\;4\ell^\nu\ell^\mu\;-\;2\ell^2\gamma^\nu\gamma^\mu\,\rangle\!\rangle\\
&=\; M(M^2-\ell^2)\times\mathrm{tr}\Big(\gamma^5\gamma^\nu\gamma^\mu\gamma^\lambda\gamma^\alpha\Big)\;+\;4M\ell^\nu\ell^\mu\times\mathrm{tr}\Big(\gamma^5\gamma^\lambda\gamma^\alpha\Big)\\
&=\; M(M^2-\ell^2)\times 4i\epsilon^{\nu\mu\lambda\alpha}\;+\;4M\ell^\nu\ell^\mu\times 0\\
&=\; 4iM(M^2-\ell^2)\epsilon^{\nu\mu\lambda\alpha}.
\end{aligned}\tag{S.24}$$

Similarly,

$$\mathcal{N}_4 \;=\; 4iM(M^2-\ell^2)\epsilon^{\alpha\nu\mu\lambda}.\tag{S.25}$$

In the remaining three traces we move the last $(\not\ell+M)$ factor forward and use

$$(\not\ell+M)\gamma^5(\not\ell+M)\;=\;(\not\ell+M)(M-\not\ell)\gamma^5\;=\;(M^2-\ell^2)\times\gamma^5.\tag{S.26}$$

Consequently,

$$\begin{aligned}
\mathcal{N}_2 \;&=\; (M^2-\ell^2)\times\mathrm{tr}\Big(\gamma^5\gamma^\nu(\not\ell+M)\gamma^\mu\gamma^\alpha\gamma^\lambda\Big)\\
&=\; 4iM(M^2-\ell^2)\epsilon^{\nu\mu\alpha\lambda},\\
\mathcal{N}_3 \;&=\; (M^2-\ell^2)\times\mathrm{tr}\Big(\gamma^5\gamma^\nu\gamma^\alpha\gamma^\mu(\not\ell+M)\gamma^\lambda\Big)\\
&=\; 4iM(M^2-\ell^2)\epsilon^{\nu\alpha\mu\lambda},\\
\mathcal{N}_0 \;&=\; (M^2-\ell^2)\times\mathrm{tr}\Big(\gamma^5\gamma^\nu(\not\ell+M)\gamma^\mu(\not\ell+M)\gamma^\lambda\Big)\\
&=\; (M^2-\ell^2)\times M\times\mathrm{tr}\Big(\gamma^5\gamma^\nu\{\not\ell,\gamma^\mu\}\gamma^\lambda\Big)\\
&=\; M(M^2-\ell^2)\times 2\ell^\mu\times\mathrm{tr}\Big(\gamma^5\gamma^\nu\gamma^\lambda\Big)\\
&=\; 0.
\end{aligned}$$

Thus, the q-independent term in the numerator vanishes while the linear-in-q terms all have the same form apart from the order of indices in the ϵ tensor. Reordering the indices and changing the sign of a_i as necessary, we arrive at

$$\mathcal{N} \;=\; 4iM(M^2-\ell^2)\epsilon^{\nu\mu\lambda\alpha}(q_1-q_2+q_3-q_4)_\alpha +\; O(M^2q^2). \tag{S.27}$$

Plugging this numerator into the momentum integral (S.20), we obtain

$$I \;=\; M\int\frac{d^4\ell}{(2\pi)^4}\,\frac{\mathcal{N}}{\mathcal{D}} \;=\; \epsilon^{\nu\mu\lambda\alpha}(q_1-q_2+q_3-q_4)_\alpha\times J +\; O\left(\frac{q^2}{M}\right) \tag{S.28}$$

where

$$\begin{aligned} J \;&=\; 4iM^2\int\frac{d^4\ell}{(2\pi)^4}\,\frac{M^2-\ell^2}{[\ell^2-M^2+i0]^4} \;=\; -4iM^2\int\frac{d^4\ell}{(2\pi)^4}\,\frac{1}{[\ell^2-M^2+i0]^3} \\ &=\; -4M^2\int\frac{d^4\ell_E}{(2\pi)^4}\,\frac{1}{[\ell_E^2+M^2]^3} \;=\; -\frac{4M^2}{16\pi^2}\int\limits_0^\infty\frac{\ell_E^2\,d\ell_E^2}{[\ell_E^2+M^2]^3} \;=\; -\frac{1}{4\pi^2}\int\limits_0^\infty\frac{x\,dx}{(x+1)^3} \\ &=\; -\frac{1}{8\pi^2}\,. \end{aligned} \tag{S.29}$$

Moreover, the gluon momenta enter eq. (S.28) in combination

$$q_1\;-\;q_2\;+\;q_3\;-\;q_4 \;=\; -k_1\;-\;k_3 \tag{S.30}$$

which does not depend on any of the Feynman parameters. Hence, integrating over those parameters is completely trivial,

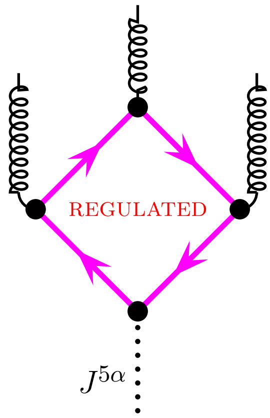
$$24\int\limits_0^1 d^4(x,y,z,w)\,\delta(x+y+z+w-1) \;=\; 1, \tag{S.31}$$

so the bottom line for the diagram (S.13) is

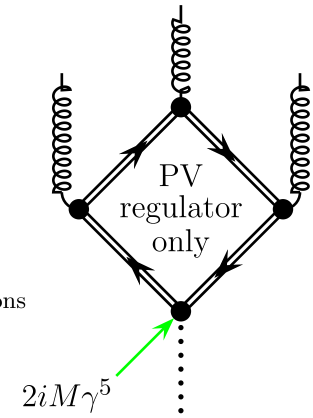
$$A(1,2,3) \;=\; -\frac{g^3}{4\pi^2}\;\mathrm{tr}(T^cT^bT^a)\,\epsilon^{\nu\mu\lambda\alpha}(k_1+k_3)_\alpha\,. \tag{S.32}$$

As we saw in eqs. (S.11) and (S.12),

$$\begin{bmatrix}\text{quadrangle}\\ \text{anomaly}\end{bmatrix} \;=\; \sum_{\substack{\text{gluon}\\ \text{permutations}}}$$



$$=\; \sum_{\substack{\text{gluon}\\ \text{permutations}}}$$



(S.33)

so all we need to do now is to sum eq. (S.32) over $3! = 6$ permutations of the three gluons. For the three cyclic permutations, the group factors is the same and the ϵ tensor is the same, hence

$$A(1,2,3) + A(2,3,1) + A(3,1,2) \;=\; -\frac{g^3}{4\pi^2}\,\mathrm{tr}(T^cT^bT^a)\,\epsilon^{\nu\mu\lambda\alpha}(2k_1+2k_2+2k_3 \;=\; 2q)_\alpha\,. \quad \text{(S.34)}$$

For the other three permutations in the opposite cyclic order, the ϵ tensor changes sign while the group factor becomes $\mathrm{tr}(T^aT^bT^c)$ instead of $\mathrm{tr}(T^cT^bT^a) = \mathrm{tr}(T^aT^cT^b)$. Thus altogether,

$$\begin{aligned}[\text{quadrangle anomaly}] \;&=\; -\frac{g^3}{4\pi^2}\,\epsilon_{\nu\mu\lambda\alpha}(2q_\alpha)\times\mathrm{tr}(T^aT^cT^b \;-\; T^aT^bT^c)\\ &=\; +\frac{g^3}{2\pi^2}\,q_\alpha\epsilon^{\alpha\lambda\mu\nu}\,\mathrm{tr}(T^a[T^b,T^c])\end{aligned} \quad \text{(S.35)}$$

or in terms of the local gluons fields A^a_λ, A^b_μ and A^c_ν,

$$\partial_\alpha J^{5\alpha}\big|_{3g} \;=\; \frac{1}{3!}\times\frac{g^3}{2\pi^2}\,i\partial_\alpha\Big(\epsilon^{\alpha\lambda\mu\nu}\,\mathrm{tr}\big(A_\lambda[A_\mu,A_\nu]\big)\Big) \;=\; \frac{ig^3}{4\pi^2}\,\epsilon^{\alpha\lambda\mu\nu}\;\mathrm{tr}\Big((\partial_\alpha A_\lambda)[A_\mu,A_\nu]\Big). \quad \text{(S.36)}$$

Comparing this formula to the second line of eq. (S.2) (part (a)), we see that the quadrangle diagrams (6) generate precisely the three–gluon part of the non-abelian anomaly (1).

Problem **3**:

In $d = 2n$ Euclidean dimension a gauge theory with massless fermions has Lagrangian

$$\mathcal{L}_E \;=\; +\tfrac{1}{4}\big(F^a_{\mu\nu}\big)^2 \;+\; \overline{\Psi}\gamma^\mu\partial_\mu\Psi \tag{S.37}$$

where $\{\gamma^\mu, \gamma^\nu\} = +2\delta^{\mu\nu}$, while the axial symmetry acts according to

$$\Psi(x) \;\to\; \exp\big(+i\theta(x)\Gamma\big)\,\Psi(x), \qquad \overline{\Psi}(x) \;\to\; \overline{\Psi}(x)\exp\big(+i\theta(x)\Gamma\big). \tag{S.38}$$

For x-independent θ this is a symmetry of the Lagrangian (S.37), while for an x–dependent $\theta(x)$

$$\Delta\mathcal{L}_E \;=\; \overline{\Psi}\gamma^\mu(i\partial_\mu\theta)\Gamma\Psi \;=\; i\partial_\mu\theta\times\overline{\Psi}\gamma^\mu\Gamma\Psi \;=\; -i\partial_\mu\theta\times J^\mu_A \tag{S.39}$$

(for the axial current $J^\mu_A = \overline{\Psi}\Gamma\gamma^\mu\Psi$), hence

$$\Delta S^{\text{classical}}_E \;=\; +i\int d^dx\,\theta(x)\times\partial_\mu J^\mu_A(x). \tag{S.40}$$

At the same time, as we saw in class for $d = 4$ — see also §22.2 of the Weinberg's book — in a non-trivial gauge field background, the axial symmetry of the fermionic path integral $\iint\mathcal{D}[\Psi(x)]\iint\mathcal{D}[\overline{\Psi}(x)]$ carries a non-trivial Jacobian $\operatorname{Det}(\exp(2i\theta(x)\Gamma)$, which is equivalent to

$$-\Delta S^{\text{quantum}}_E \;=\; \log\operatorname{Det}\big(\exp(2i\theta(x)\Gamma)\big) \;=\; 2i\operatorname{Tr}\big(\theta(x)\Gamma\big) \tag{S.41}$$

where the functional trace Tr involves both summing over Dirac and gauge indices and integration over d^dx. In terms of the ordinary matrix trace over the Dirac, color, and flavor indices,

$$\Delta S_{\text{quantum}} \;=\; -2i\int d^dx\;\operatorname{tr}\Big(\langle x|\,\Gamma\,|x\rangle_{\text{reg}}\Big)$$

where the matrix element $\langle x|\,\Gamma\,|x\rangle \;=\; \Gamma\times\delta^{(d)}(x-x)$ must be regulated to smooth out the delta-function. The $\Delta S^{\text{classical}}$ and $\Delta S^{\text{quantum}}$ should cancel each other, hence

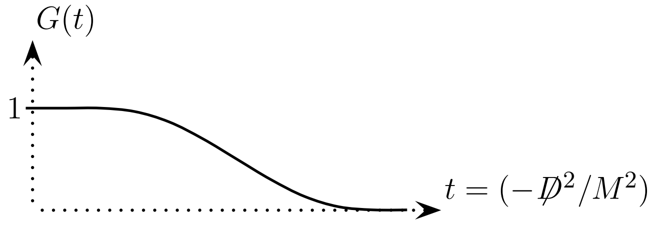
$$\partial_\mu J^\mu_A(x) \;=\; 2\operatorname{tr}\Big(\langle x|\,\Gamma\,|x\rangle_{\text{reg}}\Big) \;=\; 2\operatorname{tr}\Big(\langle x|\,\Gamma\hat{G}\,|x\rangle\Big) \tag{S.42}$$

for some regulating operator $\hat{G}$. As explained in class, this operator must commute with the Dirac operator $\not{D}$ in the background gauge field and also with the Γ matrix, hence in $d = 2n$

Euclidean dimensions

$$\hat{G} \;=\; G(-\not{D}^2/M^2) \tag{S.43}$$

for some smooth function $G(t)$ which goes to one for $t \to 0$ and to zero for $t \to \infty$,



(S.44)

Altogether,

$$\partial_\mu J^\mu_A(x) \;=\; 2\,\mathrm{tr}\Big(\langle x|\,\Gamma G(-\not{D}^2/M^2)\,|x\rangle\Big). \tag{S.45}$$

Since covariant derivatives do not commute with each other, $[D_\mu, D_\nu] = -gF_{\mu\nu}$, we have

$$\not{D}^2 \;=\; D^2 \;+\; \frac{g}{2}\,F_{\mu\nu}\sigma^{\mu\nu} \tag{S.46}$$

and consequently

$$\begin{aligned} G(-\not{D}^2/M^2) \;=\; & G(-D^2/M^2) \;-\; \frac{g}{2M^2}\,G'(-D^2/M^2) \times F_{\mu\nu}\sigma^\mu\nu \\ & +\; \frac{g^2}{8M^4}\,G''(-D^2/M^2) \times F_{\mu\nu}F_{\alpha\beta}\sigma^{\mu\nu}\sigma^{\alpha\beta} \;+\; \cdots. \end{aligned} \tag{S.47}$$

In $d = 4$, the Dirac trace $\mathrm{tr}(\gamma^5 G)$ came from the second-derivative term in this expansion because we needed four γ^μ matrices — or equivalently two $\sigma^{\mu\nu}$ matrices — to accompany the γ^5. In other even dimensions $d = 2n$, we need d γ^μ matrices or n $\sigma^{\mu\nu}$ matrices to accompany the Γ inside the Dirac trace. Specifically, in $2n$ Euclidean dimensions

$$\{\gamma^\mu, \gamma^\nu\} \;=\; 2\delta^{\mu\nu}, \quad \sigma^{\mu\nu} \;=\; \tfrac{i}{2}[\gamma^\mu, \gamma^\nu], \quad \Gamma \;=\; i^{-n}\gamma^1\gamma^2\cdots\gamma^{2n}, \tag{S.48}$$

hence

$$\mathrm{tr}\big(\Gamma\gamma^{\alpha_1}\cdots\gamma^{\alpha_k}\big) \;=\; 0 \quad \text{for } k < 2n, \qquad \mathrm{tr}\big(\Gamma\gamma^{\alpha_1}\cdots\gamma^{\alpha_{2n}}\big) \;=\; (-2i)^n\epsilon^{\alpha_1\cdots\alpha_{2n}}, \tag{S.49}$$

and

$$\mathrm{tr}(\Gamma\sigma^{\alpha_1\beta_1}\cdots\sigma^{\alpha_k\beta_k}) \;=\; 0 \quad \text{for } k<n, \qquad \mathrm{tr}(\Gamma\sigma^{\alpha_1\beta_1}\cdots\sigma^{\alpha_n\beta_n}) \;=\; +2^n\epsilon^{\alpha_1\beta_1\cdots\alpha_n\beta_n}. \tag{S.50}$$

Therefore, the Dirac trace $\mathrm{tr}(\Gamma G)$ comes from the n–th derivative term in the expansion (S.47),

$$\begin{aligned}\mathrm{tr}_{\mathrm{Dirac}}\Big(\Gamma G(-\not{D}^2/M^2)\Big) \;&=\; \frac{1}{n!}\left(\frac{-g}{2M^2}\right)^n G^{(n)}(-D^2/M^2)\times\mathrm{tr}_{\mathrm{Dirac}}\Big(\Gamma(F_{\alpha\beta}\sigma^{\alpha\beta})^n\Big)\\ &=\; \frac{1}{n!}\left(\frac{-g}{M^2}\right)^n G^{(n)}(-D^2/M^2)\times\epsilon^{\alpha_1\beta_1\cdots\alpha_n\beta_n}F_{\alpha_1\beta_1}\cdots F_{\alpha_n\beta_n}\,.\end{aligned} \tag{S.51}$$

It remains to calculate the matrix element $\langle x|\,G^{(n)}\,|x\rangle$. In the momentum basis,

$$\begin{aligned}\langle x|\,G^{(n)}(-D^2/M^2)\,|x\rangle \;&=\; \int\frac{d^dp}{(2\pi)^d}\,e^{-ipx}\,G^{(n)}(-D^2/M^2)\,e^{+ipx}\\ &=\; \int\frac{d^dp}{(2\pi)^d}\,G^{(n)}\left(\frac{(p^\mu-iD^\mu)^2_E}{M^2}\right)\end{aligned} \tag{S.52}$$

where on the last line the derivative D^μ acts on the $F(x)\cdots F(x)$ in eq. (S.51) (and also on $A^\nu(x)$ in other D^ν in the expansion of $G^{(n)}$). In the integral (S.52), the overall scale of the momentum p follows from the regulator $G(p^2/M^2)$, hence $p^\mu = O(M)$ while the derivatives D^μ are effectively O(external momenta) of the background photon or gluon fields. For the regulation purposes, we assume that M is much larger than all such external momenta, hence

$$\frac{1}{M^d}\int\frac{d^dp}{(2\pi)^d}\,G^{(n)}\left(\frac{(p^\mu-iD^\mu)^2}{M^2}\right) \;=\; \frac{1}{M^d}\int\frac{d^dp}{(2\pi)^d}\,G^{(n)}(p^2/M^2) \;+\; O\left(\frac{k^2_{\mathrm{ext}}}{M^2}\right) \tag{S.53}$$

while the leading term on the right hand side is a $O(1)$ constant. To calculate this constant, we use spherical coordinates in $2n$ Euclidean dimensions,

$$d^{2n}p \;=\; \frac{2\pi^n}{(n-1)!}\,p^{2n-1}dp \;=\; \frac{\pi^n}{(n-1)!}\,(p^2)^{n-1}dp^2. \tag{S.54}$$

Consequently,

$$\begin{aligned}
\frac{1}{M^{2n}} \int \frac{d^{2n}p}{(2\pi)^{2n}}\, G^{(n)}(p^2/M^2) &= \frac{1}{(4\pi)^n (n-1)!}\,\frac{1}{M^{2n}} \int_0^\infty dp^2\, p^{2n-2} G^{(n)}(p^2/M^2) \\
&= \frac{1}{(4\pi)^n (n-1)!} \int_0^\infty dt\, t^{n-1} G^{(n)}(t) \\
&\quad \langle\!\langle \text{ integrating } n-1 \text{ times by parts } \rangle\!\rangle \\
&= \frac{(-1)^{n-1}}{(4\pi)^n} \int_0^\infty dt\, G'(t) \\
&= \frac{(-1)^n}{(4\pi)^n}
\end{aligned}$$

regardless of the specific form of $G(t)$ functions, as long as $G(0) = 1$ and $G(\infty) = 0$, *cf.* fig. (S.44).

Therefore,

$$\langle x|\, \mathrm{tr}\Big(\Gamma G(-\not{D}^2/M^2)\Big)\, |x\rangle \;=\; \frac{1}{n!} \left(\frac{+g}{4\pi}\right)^n \epsilon^{\alpha_1\beta_1\cdots\alpha_n\beta_n} F_{\alpha_1\beta_1}(x)\cdots F_{\alpha_n\beta_n}(x), \tag{S.55}$$

and the axial anomaly follows from the trace of the RHS here over the species indices — color and flavor — of the fermionic fields,

$$\partial_\mu J^\mu_A(x) \;=\; \frac{2}{n!} \left(\frac{+g}{4\pi}\right)^n \epsilon^{\alpha_1\beta_1\cdots\alpha_n\beta_n}\, \mathrm{tr}(F_{\alpha_1\beta_1}(x)\cdots F_{\alpha_n\beta_n}(x)). \tag{S.56}$$

This completes our analysis of the axial anomaly in $d = 2n$ Euclidean dimensions. In Minkowski spacetime of $2n-1$ space dimensions plus 1 time, there is an overall minus sign hiding in the ϵ tensor, hence

$$\partial_\mu J^\mu_A(x) \;=\; -\frac{2}{n!} \left(\frac{+g}{4\pi}\right)^n \epsilon^{\alpha_1\beta_1\cdots\alpha_n\beta_n}\, \mathrm{tr}(F_{\alpha_1\beta_1}(x)\cdots F_{\alpha_n\beta_n}(x)). \tag{5}$$

Quod erat demonstrandum.

Problem **4**:

The Chern–Simons forms $\Omega_{(1)}$, $\Omega_{(3)}$, and $\Omega_{(5)}$, — or rather the vectors dual to them in $d = 2, 4, 6$ — are spelled out in eqs. (7), all we need to do is to calculate their divergences and verify eqs. (6) for the appropriate dimensions. Let's start with the $\Omega_{(1)}$ for $d = 2$:

$$\partial_\mu \Omega^\mu_{(1)} \;\equiv\; 2\epsilon^{\mu\nu}\partial_\mu \operatorname{tr}(A_\nu) \;=\; \epsilon^{\mu\nu}\operatorname{tr}(\partial_{[\mu}A_{\nu]}) \;=\; \epsilon^{\mu\nu}\operatorname{tr}(F_{\mu\nu}) \quad \text{[for the abelian } A_\nu\text{]}, \tag{S.57}$$

which is precisely the anomaly (6) in $d = 2$ dimensions.

Next, consider the $\Omega_{(3)}$ for $d = 4$. Using the Leibniz rule for the covariant derivative D_μ of products of adjoint fields,

$$\partial_\mu \operatorname{tr}(\Phi_1\Phi_2) \;=\; \operatorname{tr}\big(D_\mu(\Phi_1\Phi_2)\big) \;\operatorname{tr}\big((D_\mu\Phi_1)\Phi_2 \,+\, \Phi_1(D_\mu\Phi_2)\big) \tag{S.58}$$

we obtain

$$\begin{aligned}\partial_\mu\Omega^\mu_{(3)} \;&\equiv\; 2\epsilon^{\mu\nu\rho\sigma}\partial_\mu \operatorname{tr}\left(A_\nu F_{\rho\sigma} \,-\, \frac{2ig}{3}\,A_\nu A_\rho A_\sigma\right)\\ &=\; 2\epsilon^{\mu\nu\rho\sigma}\operatorname{tr}\Big((D_\mu A_\nu)F_{\rho\sigma} \,+\, A_\nu(D_\mu F_{\rho\sigma})\Big)\\ &\qquad -\,\frac{4ig}{3}\,\epsilon^{\mu\nu\rho\sigma}\operatorname{tr}\Big((D_\mu A_\nu)A_\rho A_\sigma \,+\, A_\nu(D_\mu A_\rho)A_\sigma \,+\, A_\nu A_\rho(D_\mu A_\sigma)\Big).\end{aligned} \tag{S.59}$$

On the second line here, the second term vanishes by the non-abelian Jacobi identity,

$$\epsilon^{\mu\nu\rho\sigma}D_\mu F_{\rho\sigma} \;=\; 0, \tag{S.60}$$

while the three terms on the last line of (S.59) are equal to each other because of the cyclic symmetry of the trace and the antisymmetry of the ϵ tensor,

$$\epsilon^{\mu\nu\rho\sigma}\operatorname{tr}\Big((D_\mu A_\nu)A_\rho A_\sigma\Big) \;=\; \epsilon^{\mu\nu\rho\sigma}\operatorname{tr}\Big(A_\nu(D_\mu A_\rho)A_\sigma\Big) \;=\; \epsilon^{\mu\nu\rho\sigma}\operatorname{tr}\Big(A_\nu A_\rho(D_\mu A_\sigma)\Big). \tag{S.61}$$

Consequently, the divergence of the $\Omega_{(3)}$ Chern–Simons form becomes

$$\begin{aligned}\partial_\mu\Omega^\mu_{(3)} \;&=\; 2\epsilon^{\mu\nu\rho\sigma}\operatorname{tr}\Big((D_\mu A_\nu)F_{\rho\sigma}\Big) \,-\, \frac{4ig}{3}\,\epsilon^{\mu\nu\rho\sigma}\operatorname{tr}\Big((D_\mu A_\nu)A_\rho A_\sigma\Big)\times 3\\ &=\; 2\epsilon^{\mu\nu\rho\sigma}\operatorname{tr}\Big((D_\mu A_\nu)\times(F_{\rho\sigma} \,-\, ig[A_\rho A_\sigma])\Big).\end{aligned} \tag{S.62}$$

Finally, $D_\mu A_\nu = \partial_\mu A_\nu] + ig[A_\mu, A_\nu]$, which after antisymmetrization in $\mu \leftrightarrow \nu$ becomes

$$D_{[\mu}A_{\nu]} \;=\; \partial_{[\mu}A_{\nu]} \;+\; 2ig[A_\mu, A_\nu] \;=\; F_{\mu\nu} \;+\; ig[A_\mu, A_\nu]. \tag{S.63}$$

Consequently,

$$\begin{aligned}\partial_\mu \Omega^\mu_{(3)} \;&=\; \epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\Big((F_{\mu\nu} \;+\; ig[A_\mu, A_\nu]) \times (F_{\rho\sigma} \;-\; ig[A_\rho A_\sigma])\Big)\\ &=\; \epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\Big(F_{\mu\nu}F_{\rho\sigma}\Big) \;+\; g^2\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\Big([A_\mu, A_\nu]\,[A_\rho A_\sigma]\Big),\end{aligned} \tag{S.64}$$

and as we saw in problem **1**(a), the second term on the last line here vanishes identically. Therefore,

$$\partial_\mu \Omega^\mu_{(3)} \;=\; \epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\Big(F_{\mu\nu}F_{\rho\sigma}\Big), \tag{S.65}$$

quod erat demonstrandum.

Finally, consider the $\Omega_{(5)}$ for $d = 6$. Again, using the Leibniz rule (S.58) we obtain

$$\begin{aligned}\partial_\mu \Omega^\mu_{(5)} \;&=\; 2\epsilon^{\mu\nu\rho\sigma\alpha\beta}\partial_\mu\,\mathrm{tr}\left(A_\nu F_{\rho\sigma}F_{\alpha\beta} \;-\; igA_\nu A_\rho A_\sigma F_{\alpha\beta} \;-\; \frac{2g^2}{5}\,A_\nu A_\rho A_\sigma A_\alpha A_\beta\right)\\ &=\; 2\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\Big((D_\mu A_\nu)F_{\rho\sigma}F_{\alpha\beta} \;+\; A_\nu(D_\mu F_{\rho\sigma})F_{\alpha\beta} \;+\; A_\nu F_{\rho\sigma}(D_\mu F_{\alpha\beta})\Big)\\ &\quad -\; 2ig\,\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\begin{pmatrix}(D_\mu A_\nu)A_\rho A_\sigma F_{\alpha\beta} \;+\; A_\nu(D_\mu A_\rho)A_\sigma F_{\alpha\beta} \;+\; A_\nu A_\rho(D_\mu A_\sigma)F_{\alpha\beta}\\ \;+\; A_\nu A_\rho A_\sigma(D_\mu F_{\alpha\beta})\end{pmatrix}\\ &\quad -\; \frac{4g^2}{5}\,\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\Big((D_\mu A_\nu)A_\rho A_\sigma A_\alpha A_\beta \;+\; 4\text{ similar terms.}\Big)\end{aligned} \tag{S.66}$$

In fact, the 5 terms on the last line here are not just similar but exactly equal to each other thanks to the cyclic symmetry of the trace and the antisymmetry of the ϵ tensor. Moreover, thanks to the non-abelian Jacobi identity (S.60) we may disregard all the terms containing $(D_\mu F_{\rho\sigma})$ or $(D_\mu F_{\alpha\beta}$, hence

$$\begin{aligned}\partial_\mu \Omega^\mu_{(5)} \;&=\; 2\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\Big((D_\mu A_\nu)F_{\rho\sigma}F_{\alpha\beta}\Big)\\ &\quad -\; 2ig\,\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\Big((D_\mu A_\nu)A_\rho A_\sigma F_{\alpha\beta} \;+\; A_\nu(D_\mu A_\rho)A_\sigma F_{\alpha\beta} \;+\; A_\nu A_\rho(D_\mu A_\sigma)F_{\alpha\beta}\Big)\\ &\quad -\; 4g^2\,\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\Big((D_\mu A_\nu)A_\rho A_\sigma A_\alpha A_\beta\Big)\end{aligned} \tag{S.67}$$

Next, let's apply eq. (S.63) — or rather $D_{[\mu}A_{\nu]} = F_{\mu\nu} + 2igA_{[\mu}A_{\nu]}$ — to every $(D_\mu A_{\ldots})$ derivative in eq. (S.67):

$$\begin{aligned}\partial_\mu\Omega^\mu_{(5)} \;&=\; \epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\Big(F_{\mu\nu}F_{\rho\sigma}F_{\alpha\beta}\Big) \;+\; 2ig\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\Big(A_\mu A_\nu F_{\rho\sigma}F_{\alpha\beta}\Big)\\ &\quad-\; ig\,\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\Big(F_{\mu\nu}A_\rho A_\sigma F_{\alpha\beta} \;+\; A_\nu F_{\mu\rho}A_\sigma F_{\alpha\beta} \;+\; A_\nu A_\rho F_{\mu\sigma}F_{\alpha\beta}\Big)\\ &\quad+\; 2g^2\,\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\Big(A_\mu A_\nu A_\rho A_\sigma F_{\alpha\beta} \;+\; A_\nu A_\mu A_\rho A_\sigma F_{\alpha\beta} \;+\; A_\nu A_\rho A_\mu A_\sigma F_{\alpha\beta}\Big)\\ &\quad-\; 2g^2\,\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\Big(F_{\mu\nu}A_\rho A_\sigma A_\alpha A_\beta\Big) \;-\; 4ig^3\,\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\Big(A_\mu A_\nu A_\rho A_\sigma A_\alpha A_\beta\Big)\end{aligned} \tag{S.68}$$

We may drastically simplify this expression using the cyclic symmetry of the trace and the antisymmetry of the ϵ tensor. In particular, the last term here vanishes identically; indeed

$$\begin{aligned}\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\Big(A_\mu A_\nu A_\rho A_\sigma A_\alpha A_\beta\Big) \;&=\; +\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\Big(A_\beta A_\mu A_\nu A_\rho A_\sigma A_\alpha\Big)\\ &\qquad \langle\!\langle\,\text{relabeling indices } \beta\to\mu\to\nu\to\rho\to\sigma\to\alpha\to\beta\,\rangle\!\rangle\\ &=\; +\epsilon^{\nu\rho\sigma\alpha\beta\mu}\,\mathrm{tr}\Big(A_\mu A_\nu A_\rho A_\sigma A_\alpha A_\beta\Big)\\ &=\; -\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\Big(A_\mu A_\nu A_\rho A_\sigma A_\alpha A_\beta\Big)\\ &\equiv\; -\text{itself}\\ &=\; 0.\end{aligned} \tag{S.69}$$

Likewise,

$$\begin{aligned}\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\Big(A_\mu F_{\nu\rho}A_\sigma F_{\alpha\beta}\Big) \;&=\; +\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\Big(A_\sigma F_{\alpha\beta}A_\mu F_{\nu\rho}\Big)\\ &=\; +\epsilon^{\sigma\alpha\beta\mu\nu\rho}\,\mathrm{tr}\Big(A_\mu F_{\nu\rho}A_\sigma F_{\alpha\beta}\Big)\\ &=\; -\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\Big(A_\mu F_{\nu\rho}A_\sigma F_{\alpha\beta}\Big)\\ &\equiv\; -\text{itself}\\ &=\; 0.\end{aligned} \tag{S.70}$$

In a similar manner most other terms on the RHS of eq. (S.68) cancel each other:

$$\begin{aligned}2ig\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\Big(A_\mu A_\nu F_{\rho\sigma}F_{\alpha\beta}\Big) \;-\; ig\,\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\Big(F_{\mu\nu}A_\rho A_\sigma F_{\alpha\beta} \;+\; A_\nu A_\rho F_{\mu\sigma}F_{\alpha\beta}\Big)&\\ =\; ig\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\Big(2A_\mu A_\nu F_{\rho\sigma}F_{\alpha\beta} \;-\; A_\rho A_\sigma F_{\alpha\beta}F_{\mu\nu} \;-\; A_\nu A_\rho F_{\mu\sigma}F_{\alpha\beta}\Big)&\\ =\; ig\;\mathrm{tr}\Big(2A_\mu A_\nu F_{\rho\sigma}F_{\alpha\beta}\Big)\times\Big(2\epsilon^{\mu\nu\rho\sigma\alpha\beta} \;-\; \epsilon^{\alpha\beta\mu\nu\rho\sigma} \;-\; \epsilon^{\rho\mu\nu\sigma\alpha\beta}\Big)&\\ =\; ig\,\mathrm{tr}(\cdots)\times\epsilon^{\mu\nu\rho\sigma\alpha\beta}\times(2\,-\,1\,-\,1\,=\,0)&\end{aligned}$$

and likewise

$$\begin{aligned}
2g^2\,\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\Big(A_\mu A_\nu A_\rho A_\sigma F_{\alpha\beta} \,+\, A_\nu A_\mu A_\rho A_\sigma F_{\alpha\beta} \,+\, A_\nu A_\rho A_\mu A_\sigma F_{\alpha\beta}\Big)& \\
-\; 2g^2\,\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\Big(F_{\mu\nu} A_\rho A_\sigma A_\alpha A_\beta\Big)& \\
=\; 2g^2\;\mathrm{tr}\Big(F_{\mu\nu} A_\rho A_\sigma A_\alpha A_\beta\Big)\,\times& \\
\times\,\Big(\epsilon^{\rho\sigma\alpha\beta\mu\nu} \,+\, \epsilon^{\sigma\rho\alpha\beta\mu\nu} \,+\, \epsilon^{\alpha\rho\sigma\beta\mu\nu} \,-\, \epsilon^{\mu\nu\rho\sigma\alpha\beta}\Big)& \\
=\; 2g^2\;\mathrm{tr}(\cdots)\times\epsilon^{\mu\nu\rho\sigma\alpha\beta}\times(1\,-\,1+\,1\,-\,1\,=\,0).&
\end{aligned} \tag{S.71}$$

Altogether, the only surviving term on the RHS of eq. (S.68) is the 6D anomaly:

$$\partial_\mu\Omega^\mu_{(5)} \;=\; \epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\Big(F_{\mu\nu}F_{\rho\sigma}F_{\alpha\beta}\Big). \tag{S.72}$$

Quod erat demonstrandum.