Dirac Trace Techniques

Consider a QED amplitude involving one incoming electron with momentum p and spin s, one outgoing electron with momentum p' and spin s', and some photons. There may be several Feynman diagrams contributing to this amplitude, but they all have the same external legs and the corresponding factors $u(p,s)$ and $\bar{u}(p',s')$. Consequently, we may write the amplitude as

$$\left\langle e^{-\prime}, \ldots \right| \mathcal{M} \left| e^{-}, \ldots \right\rangle \;=\; \bar{u}(p',s')\Gamma u(p,s) \tag{1}$$

where Γ is comprises all the other factors of the QED Feynman rules; for the moment, we don't want to be specific, so Γ is just some kind of a 4×4 matrix.

In many experiments, the initial electrons come in un-polarized beam, 50% having spin $s=+\frac{1}{2}$ and 50% having $s=-\frac{1}{2}$. At the same time, the detector of the final electrons measures their momenta p' but is blind to their spins s'. The cross-section $\overline{\sigma}$ measured by such an experiment would be the average of the polarized cross-sections $\sigma(s,s')$ with respect to initial spins s and the sum over the final spins s', thus

$$\overline{\sigma} \;=\; \frac{1}{2}\sum_{s}\sum_{s'}\sigma(s,s'). \tag{2}$$

Similar averaging / summing rules apply to the un-polarized partial cross-sections,

$$\overline{\frac{d\sigma}{d\Omega}} \;=\; \frac{1}{2}\sum_{s}\sum_{s'}\frac{d\sigma(s,s')}{d\Omega}\,, \tag{3}$$

etc., *etc.* Since all total or partial cross-sections are proportional to mod-squares $|\mathcal{M}|^2$ of amplitudes $\mathcal{M}$, we need to know how to calculate

$$\overline{|\mathcal{M}|^2} \;\stackrel{\text{def}}{=}\; \frac{1}{2}\sum_{s}\sum_{s'}\bigl|\mathcal{M}(s,s')\bigr|^2 \tag{4}$$

for amplitudes such as (1).

To do such a calculation efficiently, we need to recall two things about Dirac spinors. First,

$$\textbf{If} \quad \mathcal{M} \;=\; \bar{u}(p',s')\Gamma u(p,s) \qquad \textbf{then} \quad \mathcal{M}^* \;=\; \bar{u}(p,s)\overline{\Gamma}u(p',s') \tag{5}$$

where $\overline{\Gamma} = \gamma^0\Gamma^\dagger\gamma^0$ is the Dirac conjugate of the matrix Γ; for a product $\gamma_\lambda\cdots\gamma_\nu$ of Dirac matrices, $\overline{\gamma_\lambda\cdots\gamma_\nu} = \gamma_\nu\cdots\gamma_\lambda$. Second,

$$\sum_s u_\alpha(p,s)\times\bar{u}_\beta(p,s) \;=\; (\not{p}+m)_{\alpha\beta} \tag{6}$$

and likewise

$$\sum_{s'} u_\gamma(p',s')\times\bar{u}_\delta(p',s') \;=\; (\not{p}'+m)_{\gamma\delta}\,. \tag{7}$$

Combining these two facts, we obtain

$$\begin{aligned}
\sum_{s,s'}|\mathcal{M}=\bar{u}(p',s')\Gamma u(p,s)|^2 \;&=\; \sum_{s,s'}\bar{u}(p',s')\Gamma u(p,s)\times\bar{u}(p,s)\overline{\Gamma}u(p',s') \\
&=\; \sum_{s,s'}\left(\sum_{\delta,\alpha}\bar{u}_\delta(p',s')\,\Gamma_{\delta\alpha}\,u_\alpha(p,s)\times\sum_{\beta,\gamma}\bar{u}_\beta(p,s)\,\overline{\Gamma}_{\beta\gamma}\,u_\gamma(p',s')\right) \\
&=\; \sum_{\alpha,\beta,\gamma,\delta}\Gamma_{\delta\alpha}\,\overline{\Gamma}_{\beta\gamma}\times\left(\sum_s u_\alpha(p,s)\,\bar{u}_\beta(p,s)\right) \\
&\qquad\qquad\times\left(\sum_{s'}u_\gamma(p',s')\,\bar{u}_\delta(p',s')\right) \\
&=\; \sum_{\alpha,\beta,\gamma,\delta}\Gamma_{\delta\alpha}\,\overline{\Gamma}_{\beta\gamma}\times(\not{p}+m)_{\alpha\beta}\times(\not{p}'+m)_{\gamma\delta} \\
&=\; \sum_\gamma\Big(\text{matrix product } (\not{p}'+m)\,\Gamma\,(\not{p}+m)\,\overline{\Gamma}\Big)_{\gamma\gamma} \\
&=\; \operatorname{tr}\Big((\not{p}'+m)\,\Gamma\,(\not{p}+m)\,\overline{\Gamma}\Big)
\end{aligned} \tag{8}$$

and hence

$$\big\langle e^{-\prime},\ldots\big|\,\mathcal{M}\,\big|e^-,\ldots\big\rangle \;=\; \bar{u}(p',s')\Gamma u(p,s) \quad\Longrightarrow\quad \frac{1}{2}\sum_{s,s'}|\mathcal{M}|^2 \;=\; \frac{1}{2}\operatorname{tr}\Big((\not{p}'+m)\,\Gamma\,(\not{p}+m)\,\overline{\Gamma}\Big). \tag{9}$$

A similar trace formula exists for un-polarized scattering of positrons. In this case, the amplitude is

$$\left\langle e^{+\prime},\ldots\right|\mathcal{M}\left|e^{+},\ldots\right\rangle \;=\; \bar{v}(p,s)\Gamma v(p',s') \tag{10}$$

(note that $v(p',s')$ belongs to the outgoing positron while $\bar{v}(p,s)$ belongs to the incoming s^+), and we need to average $|\mathcal{M}|^2$ over s and sum over s'. Using

$$\sum_s v_\alpha(p,s)\times\bar{v}_\beta(p,s) \;=\; (\not{p}-m)_{\alpha\beta} \tag{11}$$

and working through algebra similar to eq. (8), we arrive at

$$\left\langle e^{+\prime},\ldots\right|\mathcal{M}\left|e^{+},\ldots\right\rangle \;=\; \bar{v}(p,s)\Gamma v(p',s') \quad\Longrightarrow\quad \frac{1}{2}\sum_{s,s'}|\mathcal{M}|^2 \;=\; \frac{1}{2}\,\mathrm{tr}\Big((\not{p}-m)\,\Gamma\,(\not{p}'-m)\,\overline{\Gamma}\Big). \tag{12}$$

Now suppose an electron with momentum p_1 and spin s_1 and a positron with momentum p_2 and spin s_2 come in and annihilate each other. In this case, the amplitude has form

$$\left\langle\ldots\right|\mathcal{M}\left|e_1^-,e_2^+,\ldots\right\rangle \;=\; \bar{v}(p_2,s_2)\Gamma u(p_1,s_1) \tag{13}$$

for some Γ, and if both electron and positron beams are un-polarized, we need to average the $|\mathcal{M}|^2$ over both spins s_1 and s_2. Again, there is a trace formula for such averaging, namely

$$\left\langle\ldots\right|\mathcal{M}\left|e_1^-,e_2^+,\ldots\right\rangle \;=\; \bar{v}(p_2,s_2)\Gamma u(p_1,s_1) \quad\Longrightarrow\quad \frac{1}{4}\sum_{s_1,s_2}|\mathcal{M}|^2 \;=\; \frac{1}{4}\,\mathrm{tr}\Big((\not{p}_2-m)\,\Gamma\,(\not{p}_1+m)\,\overline{\Gamma}\Big). \tag{14}$$

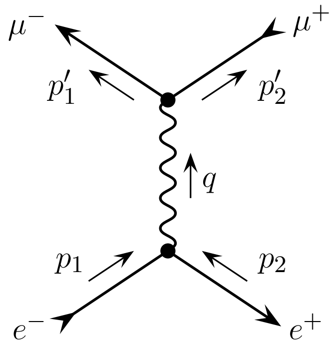
Finally, for a process in which an electron-positron pair is created, the amplitude has form

$$\left\langle e_1^{-\prime},e_2^{+\prime},\ldots\right|\mathcal{M}\left|\ldots\right\rangle \;=\; \bar{u}(p_1',s_1')\Gamma v(p_2',s_2'), \tag{15}$$

and if we do not detect the spins of the outgoing electron and positron but only their momenta, then we should sum the $|\mathcal{M}|^2$ over both spins s_1 and s_2. Again, there is a trace formula for this sum, namely

$$\left\langle e_1^{-\prime}, e_2^{+\prime}, \ldots \middle| \mathcal{M} \middle| \ldots \right\rangle \;=\; \bar{u}(p_1', s_1')\Gamma v(p_2', s_2') \quad\Longrightarrow\quad \sum_{s_1,s_2} |\mathcal{M}|^2 \;=\; \operatorname{tr}\Big((\not{p}_1' + m)\,\Gamma\,(\not{p}_2' - m)\,\overline{\Gamma} \Big). \tag{16}$$

Processes involving 4 or more un-polarized fermions also have trace formulae for the spin sums. For example, consider an $e^- + e^+$ collision in which a muon pair $\mu^- + \mu^+$ is created. There is one tree diagram for this process,



(17)

and it evaluates to

$$\begin{aligned} i \left\langle \mu^-, \mu^+ \middle| \mathcal{M} \middle| e^-, e^+ \right\rangle \;&=\; \frac{-ig^{\lambda\nu}}{q^2} \times \bar{u}(\mu^-)(ie\gamma_\lambda)v(\mu^+) \times \bar{v}(e^+)(ie\gamma_\nu)u(e^-) \\ &=\; \frac{ie^2}{s} \times \bar{u}(\mu^-)\gamma^\nu v(\mu^+) \times \bar{v}(e^+)\gamma_\nu u(e^-) \end{aligned} \tag{18}$$

where

$$s \;=\; q^2 \;=\; (p_1 + p_2)^2 \;=\; (p_1' + p_2')^2 \;=\; E^2_{\text{c.m.}} \tag{19}$$

is the square of the total energy in the center-of-mass frame.

The amplitude (18) depends on the spins of all 4 particles involved. To get the partial cross-section for an experiment using un-polarized beams of initial electrons and positrons and a spin-blind muon detector, we need to average the $|\mathcal{M}|^2$ over both initial spins s_1, s_2 and sum it over both final spins s_1', s_2', thus

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{c.m.}} \;=\; \frac{\overline{|\mathcal{M}|^2}}{64\pi^2 s} \quad \text{where} \quad \overline{|\mathcal{M}|^2} \;\equiv\; \frac{1}{4} \sum_{s_1,s_2,s_1',s_2'} |\mathcal{M}|^2. \tag{20}$$

For the amplitude (18) at hand,

$$\begin{aligned}
\mathcal{M}\times\mathcal{M}^* &= \frac{e^4}{s^2}\times\Big(\bar u(\mu^-)\gamma^\nu v(\mu^+)\times\bar v(e^+)\gamma_\nu u(e^-)\Big)\times\Big(\bar v(\mu^+)\gamma^\lambda u(\mu^-)\times\bar u(e^-)\gamma_\lambda v(e^+)\Big)\\
&\qquad\text{— note sums over two separate Lorentz indices } \nu \text{ and } \lambda \text{ —}\\
&= \frac{e^4}{s^2}\times\Big(\bar u(\mu^-)\gamma^\nu v(\mu^+)\times\bar v(\mu^+)\gamma^\lambda u(\mu^-)\Big)\times\Big(\bar v(e^+)\gamma_\nu u(e^-)\times\bar u(e^-)\gamma_\lambda v(e^+)\Big)
\end{aligned}\tag{21}$$

and consequently

$$\begin{aligned}
\overline{|\mathcal{M}|^2} &= \frac{e^4}{4s^2}\times\left(\sum_{s'_1,s'_2}\bar u(\mu^-)\gamma^\nu v(\mu^+)\times\bar v(\mu^+)\gamma^\lambda u(\mu^-)\right)\times\\
&\qquad\times\left(\sum_{s_1,s_2}\bar v(e^+)\gamma_\nu u(e^-)\times\bar u(e^-)\gamma_\lambda v(e^+)\right)\\
&= \frac{e^4}{4s^2}\times\mathrm{tr}\Big((\not p'_1+M_\mu)\,\gamma^\nu\,(\not p'_2-M_\mu)\,\gamma^\lambda\Big)\times\mathrm{tr}\Big((\not p_2-m_e)\,\gamma_\nu\,(\not p_1+m_e)\,\gamma_\lambda\Big).
\end{aligned}\tag{22}$$

Calculating Dirac Traces

Thus far, we have learned how to express un-polarized cross-sections in terms of Dirac traces (*i.e.*, traces of products of the Dirac γ^λ matrices). In this section, we shall learn how to calculate such traces.

Dirac traces do not depend on the specific form of the $\gamma^0,\gamma^1,\gamma^2,\gamma^4$ matrices but are completely determined by the Clifford algebra

$$\{\gamma^\mu,\gamma^\nu\}\ \equiv\ \gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu\ =\ 2g^{\mu\nu}.\tag{23}$$

To see how this works, please recall the key property of the trace of any matrix product:

$\star$ $\mathrm{tr}(AB)=\mathrm{tr}(BA)$ for any two matrices A and B. Proof:

$$\mathrm{tr}(AB)\ =\ \sum_\alpha(AB)_{\alpha\alpha}\ =\ \sum_\alpha\sum_\beta A_{\alpha\beta}B_{\beta\alpha}\ =\ \sum_\beta(BA)_{\beta\beta}\ =\ \mathrm{tr}(BA).\tag{24}$$

This symmetry has two important corollaries:

- All commutators have zero traces, $\mathrm{tr}([A,B])=0$ for any A and B.
- Traces of products of several matrices have *cyclic symmetry*

$$\mathrm{tr}(ABC\cdots YZ) \;=\; \mathrm{tr}(BC\cdots YZA) \;=\; \mathrm{tr}(C\cdots YZAB) \;=\; \cdots \;=\; \mathrm{tr}(ZABC\cdots Y). \tag{25}$$

Using these properties it is easy to show that

$$\mathrm{tr}(\gamma^\mu\gamma^\nu) \;=\; 4g^{\mu\nu} \quad\Longrightarrow\quad \mathrm{tr}(\not{a}\,\not{b}) \;=\; 4(ab) \;\equiv\; 4a_\mu b^\mu. \tag{26}$$

Indeed,

$$\mathrm{tr}(\gamma^\mu\gamma^\nu) \;=\; \mathrm{tr}(\gamma^\nu\gamma^\mu) \;=\; \mathrm{tr}\left(\tfrac{1}{2}\{\gamma^\mu,\gamma^\nu\}\right) \;=\; \mathrm{tr}(g^{\mu\nu}) \;=\; g^{\mu\nu}\times\mathrm{tr}(1) \;=\; g^{\mu\nu}\times 4, \tag{27}$$

where the last equality follows from Dirac matrices being 4×4 and hence

$$\mathrm{tr}(1) \;=\; 4. \tag{28}$$

Next, all products of any *odd* numbers of the γ^μ matrices have zero traces,

$$\mathrm{tr}(\gamma^\mu) \;=\; 0, \quad \mathrm{tr}(\gamma^\lambda\gamma^\mu\gamma^\nu) \;=\; 0, \quad \mathrm{tr}(\gamma^\lambda\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) \;=\; 0,\ \textit{etc.}, \tag{29}$$

and hence

$$\mathrm{tr}(\not{a}) \;=\; 0, \quad \mathrm{tr}(\not{a}\,\not{b}\,\not{c}) \;=\; 0, \quad \mathrm{tr}(\not{a}\,\not{b}\,\not{c}\,\not{d}\,\not{e}) \;=\; 0,\ \textit{etc.} \tag{30}$$

To see how this works, we can use the γ^5 matrix which anticommutes with all the γ^μ and hence with any product Γ of an odd number of the Dirac γ's, $\gamma^5\Gamma=-\Gamma\gamma^5$. Consequently,

$$\Gamma \;=\; \gamma^5\gamma^5\Gamma \;=\; -\gamma^5\Gamma\gamma^5 \;=\; -\tfrac{1}{2}[\gamma^5\Gamma,\gamma^5] \tag{31}$$

and hence

$$\mathrm{tr}(\Gamma) \;=\; -\tfrac{1}{2}\,\mathrm{tr}([\gamma^5\Gamma,\gamma^5]) \;=\; 0. \tag{32}$$

Products of even numbers $n=2m$ of γ matrices have non-trivial traces, and we may calculate them recursively in n. We already know the traces for $n=0$ and $n=2$, so consider a product $\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu$ of $n=4$ matrices. Thanks to the cyclic symmetry of the trace,

$$\mathrm{tr}(\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu) \;=\; \mathrm{tr}(\gamma^\lambda\gamma^\mu\gamma^\nu\gamma^\kappa) \;=\; \mathrm{tr}\left(\tfrac{1}{2}\{\gamma^\kappa,\gamma^\lambda\gamma^\mu\gamma^\nu\}\right) \tag{33}$$

where the anticommutator follows from the Clifford algebra (23) and the Leibnitz rule,

$$\begin{aligned}\{\gamma^\kappa,\gamma^\lambda\gamma^\mu\gamma^\nu\} \;&=\; \{\gamma^\kappa,\gamma^\lambda\}\gamma^\mu\gamma^\nu \;-\; \gamma^\lambda\{\gamma^\kappa,\gamma^\mu\}\gamma^\nu \;+\; \gamma^\lambda\gamma^\mu\{\gamma^\kappa,\gamma^\nu\}\\ &=\; 2g^{\kappa\lambda}\times\gamma^\mu\gamma^\nu \;-\; 2g^{\kappa\mu}\times\gamma^\lambda\gamma^\nu \;+\; 2g^{\kappa\nu}\times\gamma^\lambda\gamma^\mu.\end{aligned} \tag{34}$$

Therefore,

$$\begin{aligned}\mathrm{tr}(\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu) \;&=\; g^{\kappa\lambda}\times\mathrm{tr}(\gamma^\mu\gamma^\nu) \;-\; g^{\kappa\mu}\times\mathrm{tr}(\gamma^\lambda\gamma^\nu) \;+\; g^{\kappa\nu}\times\mathrm{tr}(\gamma^\lambda\gamma^\mu)\\ &=\; 4g^{\kappa\lambda}g^{\mu\nu} \;-\; 4g^{\kappa\mu}g^{\lambda\nu} \;+\; 4g^{\kappa\nu}g^{\lambda\mu}\end{aligned} \tag{35}$$

and hence

$$\mathrm{tr}(\not{a}\,\not{b}\,\not{c}\,\not{d}) \;=\; 4(ab)(cd) \;-\; 4(ac)(bd) \;+\; 4(ad)(bc). \tag{36}$$

Note that in eq. (35) we have expressed the trace of a 4–γ product to traces of 2–γ products. Similar recursive formulae exist for all even numbers of γ matrices,

$$\begin{aligned}\mathrm{tr}(\gamma^{\nu_1}\gamma^{\nu_2}\cdots\gamma^{\nu_n}) \;&=\; \mathrm{tr}\left(\tfrac{1}{2}\{\gamma^{\nu_1},\gamma^{\nu_2}\cdots\gamma^{\nu_n}\}\right)\\ &\qquad\text{for even } n \text{ only}\\ &=\; \sum_{k=2}^{n}(-1)^k g^{\nu_1\nu_k}\times\mathrm{tr}\left(\gamma^{\nu_2}\cdots\cancel{\gamma^{\nu_k}}\cdots\gamma^{\nu_n}\right).\end{aligned} \tag{37}$$

For example, for $n=6$

$$\begin{aligned}\mathrm{tr}(\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) \;&=\; g^{\kappa\lambda}\times\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) \;-\; g^{\kappa\mu}\times\mathrm{tr}(\gamma^\lambda\gamma^\nu\gamma^\rho\gamma^\sigma) \;+\; g^{\kappa\nu}\times\mathrm{tr}(\gamma^\lambda\gamma^\mu\gamma^\rho\gamma^\sigma)\\ &\qquad -\; g^{\kappa\rho}\times\mathrm{tr}(\gamma^\lambda\gamma^\mu\gamma^\nu\gamma^\sigma) \;+\; g^{\kappa\sigma}\times\mathrm{tr}(\gamma^\lambda\gamma^\mu\gamma^\nu\gamma^\rho)\\ &=\; 4g^{\kappa\lambda}\times\left(g^{\mu\nu}g^{\rho\sigma} \;-\; g^{\mu\rho}g^{\nu\sigma} \;+\; g^{\mu\sigma}g^{\nu\rho}\right)\\ &\quad -\; 4g^{\kappa\mu}\times\left(g^{\lambda\nu}g^{\rho\sigma} \;-\; g^{\lambda\rho}g^{\nu\sigma} \;+\; g^{\lambda\sigma}g^{\nu\rho}\right)\\ &\quad +\; 4g^{\kappa\nu}\times\left(g^{\lambda\mu}g^{\rho\sigma} \;-\; g^{\lambda\rho}g^{\mu\sigma} \;+\; g^{\lambda\sigma}g^{\mu\rho}\right)\\ &\quad -\; 4g^{\kappa\rho}\times\left(g^{\lambda\mu}g^{\nu\sigma} \;-\; g^{\lambda\nu}g^{\mu\sigma} \;+\; g^{\lambda\sigma}g^{\mu\nu}\right)\\ &\quad +\; 4g^{\kappa\sigma}\times\left(g^{\lambda\mu}g^{\nu\rho} \;-\; g^{\lambda\nu}g^{\mu\rho} \;+\; g^{\lambda\rho}g^{\mu\nu}\right).\end{aligned} \tag{38}$$

For products of more γ matrices, the recursive formulae (37) for traces produce even more terms (105 terms for $n - 8$, 945 terms for $n = 10$, *etc.*, *etc.*), so it helps to reduce n whenever possible. For example, if the matrix product inside the trace contains two $\not{a}$ matrices (for the same 4–vector a^μ) next to each other, you can simplify the product using $\not{a}\,\not{a} = a^2$, thus

$$\operatorname{tr}(\not{a}\,\not{a}\,\not{b}\cdots\not{c}) \;=\; a^2 \times \operatorname{tr}(\not{b}\cdots\not{c}). \tag{39}$$

Also, when a product contains γ^α and γ_α with the same Lorentz index α which should be summed over, we may simplify the trace using

$$\gamma^\alpha\gamma_\alpha \;=\; 4, \quad \gamma^\alpha\not{a}\gamma_\alpha \;=\; -2\,\not{a}, \quad \gamma^\alpha\not{a}\,\not{b}\gamma_\alpha \;=\; +4(ab), \quad \gamma^\alpha\not{a}\,\not{b}\,\not{c}\gamma_\alpha \;=\; -2\,\not{c}\,\not{b}\,\not{a}, \tag{40}$$

etc., *cf.* homework set 8, problem 1(a).

In the electroweak theory, one often needs to calculate traces of products containing the γ^5 matrix. If the γ^5 appears more than once, we may simplify the product using $\gamma^5\gamma^5 = 1$ and $\gamma^5\gamma^\nu = -\gamma^\nu\gamma^5$. For example,

$$\begin{aligned}
\operatorname{tr}\Big(\gamma^\mu(1-\gamma^5)\,\not{p}\,\gamma^\nu(1-\gamma^5)\,\not{q}\Big) \;&=\; \operatorname{tr}\Big(\gamma^\mu(1-\gamma^5)\,\not{p}\,(1+\gamma^5)\gamma^\nu\,\not{q}\Big)\\
&=\; \operatorname{tr}\Big(\gamma^\mu(1-\gamma^5)(1-\gamma^5)\,\not{p}\,\gamma^\nu\,\not{q}\Big)\\
&=\; 2\operatorname{tr}\Big(\gamma^\mu(1-\gamma^5)\,\not{p}\,\gamma^\nu\,\not{q}\Big)\\
&\qquad \text{because } (1-\gamma^5)^2 \;=\; 1-2\gamma^5+\gamma^5\gamma^5 \;=\; 2(1-\gamma^5)\\
&=\; 2\operatorname{tr}\Big((1+\gamma^5)\gamma^\mu\,\not{p}\,\gamma^\nu\,\not{q}\Big)\\
&=\; 2\operatorname{tr}\Big(\gamma^\mu\,\not{p}\,\gamma^\nu\,\not{q}\Big) \;+\; 2\operatorname{tr}\Big(\gamma^5\gamma^\mu\,\not{p}\,\gamma^\nu\,\not{q}\Big).
\end{aligned} \tag{41}$$

When the γ^5 appears just one time, we may use $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ to show that

$$\operatorname{tr}(\gamma^5) \;=\; 0, \quad \operatorname{tr}(\gamma^5\gamma^\nu) \;=\; 0, \quad \operatorname{tr}(\gamma^5\gamma^\mu\gamma^\nu) \;=\; 0, \quad \operatorname{tr}(\gamma^5\gamma^\lambda\gamma^\mu\gamma^\nu) \;=\; 0, \tag{42}$$

while

$$\operatorname{tr}(\gamma^5\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu) \;=\; -4i\epsilon^{\kappa\lambda\mu\nu}. \tag{43}$$

For more γ^ν matrices accompanying the γ^5 we have

$$\operatorname{tr}(\gamma^5\gamma^{\nu_1}\cdots\gamma^{\nu_n}) \;=\; 0 \quad \forall \text{ odd } n, \tag{44}$$

while for even $n = 6, 8, \ldots$ there are recursive formulae based on the identity

$$\gamma^5\gamma^\lambda\gamma^\mu\gamma^\nu \;=\; g^{\lambda\mu}\times\gamma^5\gamma^\nu -\; g^{\lambda\nu}\times\gamma^5\gamma^\mu +\; g^{\mu\nu}\times\gamma^5\gamma^\lambda -\; i\epsilon^{\lambda\mu\nu\rho}\times\gamma_\rho\,. \tag{45}$$

Muon Pair Production

As an example of trace technology, let us calculate the traces (22) for the muon pair production. Let's start with the trace due to summing over muons' spins,

$$\begin{aligned}\mathrm{tr}\Big((\not p'_1 + M_\mu)\gamma^\lambda(\not p'_2 - M_\mu)\gamma^\nu\Big) \;&=\; \mathrm{tr}(\not p'_1\gamma^\lambda\not p'_2\gamma^\nu)\\ &\quad+\; M_\mu\times\mathrm{tr}(\gamma^\lambda\not p'_2\gamma^\nu) \;-\; M_\mu\times\mathrm{tr}(\not p'_1\gamma^\lambda\gamma^\nu)\\ &\quad-\; M_\mu^2\times\mathrm{tr}(\gamma^\lambda\gamma^\nu).\end{aligned} \tag{46}$$

On the second line here, we have three γ matrices inside each trace, so those traces vanish. On the third line, $\mathrm{tr}(\gamma^\lambda\gamma^\nu) = 4g^{\lambda\nu}$. Finally, the trace on the first line follows from eq. (35),

$$\begin{aligned}\mathrm{tr}(\not p'_1\gamma^\lambda\not p'_2\gamma^\nu) \;&=\; p'_{a\alpha}p'_{2\beta}\times\mathrm{tr}(\gamma^\alpha\gamma^\lambda\gamma^\beta\gamma^\nu)\\ &=\; p'_{a\alpha}p'_{2\beta}\times 4\Big(g^{\alpha\lambda}\times g^{\beta\nu} \;-\; g^{\alpha\beta}\times g^{\lambda\nu} \;+\; g^{\alpha\nu}\times g^{\lambda\beta}\Big)\\ &=\; 4p_1'^\lambda\times p_2'^\nu \;-\; 4(p'_1p'_2)\times g^{\lambda\nu} \;+\; 4p_1'^\nu\times p_2'^\lambda\,.\end{aligned} \tag{47}$$

Altogether,

$$\begin{aligned}\mathrm{tr}\Big((\not p'_1 + M_\mu)\gamma^\lambda(\not p'_2 - M_\mu)\gamma^\nu\Big) \;&=\; 4p_1'^\lambda p_2'^\nu \;+\; 4p_1'^\nu p_2'^\lambda \;-\; 4(M_\mu^2 + (p'_1p'_2))\times g^{\lambda\nu}\\ &=\; 4p_1'^\lambda p_2'^\nu \;+\; 4p_1'^\nu p_2'^\lambda \;-\; 2s\times g^{\lambda\nu}\end{aligned} \tag{48}$$

where on the last line I have used

$$s \;\equiv\; (p'_1 + p'_2)^2 \;=\; p_1'^2 \;+\; p_2'^2 \;+\; 2(p'_1p'_2) \;=\; 2M_\mu^2 \;+\; 2(p'_1p'_2). \tag{49}$$

Similarly, for the second trace (22) due to averaging over electron's and positron's spins, we have

$$\begin{aligned}
\mathrm{tr}\Big((\not{p}_2 - m_e)\,\gamma_\nu\,(\not{p}_1 + m_e)\,\gamma_\lambda\Big) \;&=\; \mathrm{tr}(\not{p}_2\gamma_\nu\not{p}_1\gamma_\lambda) \\
&\quad +\; m_e \times \mathrm{tr}(\not{p}_2\gamma_\nu\gamma_\lambda) \;-\; m_e \times \mathrm{tr}(\gamma_\nu\not{p}_1\gamma_\lambda) \\
&\quad -\; m_e^2 \times \mathrm{tr}(\gamma_\nu\gamma_\lambda) \\
&=\; 4p_{2\nu} \times p_{1\lambda} \;-\; 4(p_2p_1) \times g_{\nu\lambda} +\; 4p_{2\lambda} \times p_{1\nu} \\
&\quad + m_e \times 0 \;-\; m_e \times 0 \;-\; m_e^2 \times 4g_{\nu\lambda} \\
&=\; 4p_{2\nu}p_{1\lambda} \;+\; 4p_{2\lambda}p_{1\nu} \;-\; 4((p_2p_1) + m_e^2) \times g_{\lambda\nu} \\
&=\; 4p_{2\nu}p_{1\lambda} \;+\; 4p_{2\lambda}p_{1\nu} \;-\; 2s \times g_{\lambda\nu}\,,
\end{aligned} \tag{50}$$

where on the last line I have used

$$s \;\equiv\; (p_1+p_2)^2 \;=\; p_1^2 \;+\; p_2^2 \;+\; 2(p_2p_1) \;=\; 2m_e^2 \;+\; 2(p_2p_1). \tag{51}$$

It remains to multiply the two traces and sum over the Lorentz indices λ and ν:

$$\begin{aligned}
&\mathrm{tr}\Big((\not{p}'_1 + M_\mu)\gamma^\nu(\not{p}'_2 - M_\mu)\gamma^\lambda\Big) \;\times \mathrm{tr}\Big((\not{p}_2 - m_e)\,\gamma_\nu\,(\not{p}_1 + m_e)\,\gamma_\lambda\Big) \;= \\
&\qquad=\; \Big(4p_1'^\nu p_2'^\lambda \;+\; 4p_1'^\lambda p_2'^\nu \;-\; 2s \times g^{\nu\lambda}\Big) \times \Big(4p_{2\nu}p_{1\nu} \;+\; 4p_{2\lambda}p_{1\nu} \;-\; 2s \times g_{\nu\lambda}\Big) \\
&\qquad=\; 16(p_1'^\nu p_2'^\lambda \;+\; p_1'^\lambda p_2'^\nu) \times (p_{2\nu}p_{1\lambda} \;+\; p_{2\lambda}p_{1\nu}) \\
&\qquad\quad -\, 8s\, g^{\nu\lambda} \times (p_{2\nu}p_{1\lambda} \;+\; p_{2\lambda}p_{1\nu}) \\
&\qquad\quad -\, 8s\, g_{\nu\lambda} \times (p_1'^\lambda p_2'^\nu \;+\; p_1'^\nu p_2'^\lambda) \\
&\qquad\quad +\, 4s^2 \times g^{\lambda\nu}g_{\lambda\nu} \\
&\qquad=\; 16 \times 2 \times \Big((p'_1p_1)(p'_2p_2) \;+\; (p'_2p_1)(p'_1p_2)\Big) \\
&\qquad\quad -\, 8s \times 2(p_1p_2) \;-\; 8s \times 2(p'_1p'_2) \;+\; 4s^2 \times 4 \\
&\qquad=\; 32(p'_1p_1)(p'_2p_2) \;+\; 32(p'_2p_1)(p'_1p_2) \;+\; 16s \times (M_\mu^2 + m_e^2)
\end{aligned} \tag{52}$$

where the last line follows from eqs. (49) and (51). Hence, in eq. (22) we have

$$\overline{|\mathcal{M}|^2} \;\equiv\; \frac{1}{4}\sum_{\text{all spins}} |\mathcal{M}|^2 \;=\; \frac{4e^4}{s^2} \times \Big(2(p'_1p_1)(p'_2p_2) \;+\; 2(p'_2p_1)(p'_1p_2) \;+\; s(M_\mu^2 + m_e^2)\Big). \tag{53}$$

Finally, let's work out the kinematics of pair production. In the center-of-mass frame, $p_{1,2}^{\nu} = (E, \pm\mathbf{p})$ and $p_{1,2}^{\prime\nu} = (E, \mathbf{p}')$, same $E = \frac{1}{2}E_{\rm cm}$ but $\mathbf{p}' \neq \mathbf{p}$. Therefore,

$$\begin{aligned}
(p_1'p_1) \;=\; (p_2'p_2) &\;=\; E^2 \;-\; \mathbf{p}'\cdot\mathbf{p}, \\
(p_2'p_1) \;=\; (p_1'p_2) &\;=\; E^2 \;+\; \mathbf{p}'\cdot\mathbf{p}, \\
s &\;=\; 4E^2, \\
2(p_1'p_1)(p_2'p_2) \;+\; 2(p_2'p_1)(p_1'p_2) &\;=\; 2(E^2 - \mathbf{p}'\cdot\mathbf{p})^2 \;+\; 2(E^2 + \mathbf{p}'\cdot\mathbf{p})^2 \\
&\;=\; 4E^4 \;+\; 4(\mathbf{p}'\cdot\mathbf{p})^2 \\
&\;=\; 4E^4 \;+\; 4\mathbf{p}'^2\,\mathbf{p}^2 \times \cos^2\theta,
\end{aligned} \tag{54}$$

and consequently

$$\overline{|\mathcal{M}|^2} \;=\; e^4\left(1 \;+\; \frac{\mathbf{p}'^2\,\mathbf{p}^2}{E^4} \times \cos^2\theta \;+\; \frac{M_\mu^2 + m_e^2}{E^2}\right). \tag{55}$$

where $\mathbf{p}'^2 = E^2 - M_\mu^2$ and $\mathbf{p}^2 = E^2 - m_e^2$.

We may simplify this expression a bit using the experimental fact that the muon is much heavier than the electron, $M_\mu \approx 207 m_e$, so we need ultra-relativistic $e^\mp$ to produce $\mu^\mp$, $E > M_\mu \gg m_e$. This allows us to neglect the m_e^2 term in eq. (55) and let $\mathbf{p}^2 = E^2$, thus

$$\overline{|\mathcal{M}|^2} \;=\; e^4\left(\left(1 \;+\; \frac{M_\mu^2}{E^2}\right) \;+\; \left(1 \;-\; \frac{M_\mu^2}{E^2}\right) \times \cos^2\theta\right), \tag{56}$$

and consequently the partial cross-section is

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm c.m.} \;=\; \frac{\alpha^2}{4s} \times \left(\left(1 \;+\; \frac{M_\mu^2}{E^2}\right) \;+\; \left(1 \;-\; \frac{M_\mu^2}{E^2}\right) \times \cos^2\theta\right) \times \sqrt{1 \;-\; \frac{M_\mu^2}{E^2}} \tag{57}$$

where the root comes from the phase-space factor $|\mathbf{p}'|/|\mathbf{p}|$ for inelastic processes.

Looking at the angular dependence of this partial cross-section, we see that just above the energy threshold, for $E = M_\mu + \text{small}$, the muons are produced isotropically in all directions.

On the other hand, for very high energies $E \gg M_\mu$ when all 4 particles are ultra-relativistic,

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{c.m.}} \propto\ 1\,+\,\cos^2\theta. \tag{58}$$

In the homework set 11 you will see that the polarized cross sections depend on the angle as $(1\pm\cos\theta)^2$ where the sign $\pm$ depends on the helicities of initial and final particles; for the un-polarized particles, we average / sum over helicities, and that produces the averaged angular distribution $1+\cos^2$.

Finally, the total cross-section of muon pair production follows from eq. (57) using

$$\int\!d^2\Omega\ =\ 4\pi,\qquad \int\!d^2\Omega\,\cos^2\theta\ =\ \frac{4\pi}{3}\,, \tag{59}$$

hence

$$\sigma^{\text{tot}}(e^-e^+\to\mu^-\mu^+)\ =\ \frac{4\pi}{3}\,\frac{\alpha^2}{s}\times\left(1\,+\,\frac{M_\mu^2}{2E^2}\right)\sqrt{1\,-\,\frac{M_\mu^2}{E^2}}\,. \tag{60}$$

Well above the threshold

$$\sigma^{\text{tot}}\ \approx\ \sigma_0^{\text{tot}}\ \equiv\ \frac{4\pi}{3}\,\frac{\alpha^2}{s}\,. \tag{61}$$

At the threshold the cross-section is zero, but it rises very rapidly with energy and for $E=1.25M_\mu$, σ^{tot} reaches about 80% of σ_0^{tot}.