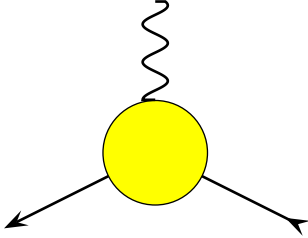
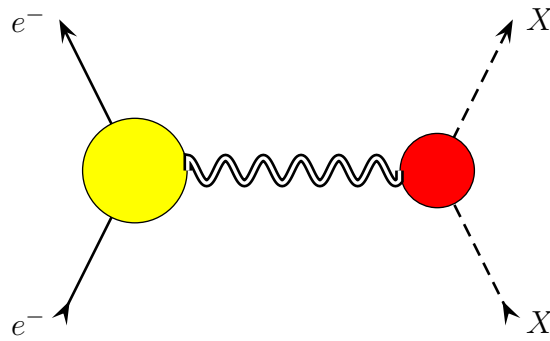


# QED Vertex Correction

In these notes I shall calculate the one-loop correction to the 1PI electron-electron-photon vertex in QED,

$$ie\Gamma^\mu(p', p) = \text{Diagram} \tag{1}$$


We are interested in this vertex in the context of elastic Coulomb scattering,

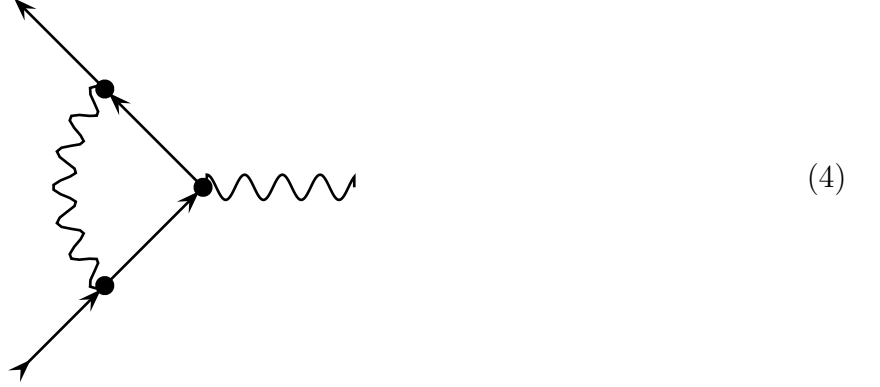
$$\text{Diagram} \tag{2}$$


so we take the incoming and the outgoing electrons to be on-shell,  $p^2 = p'^2 = m^2$ , but the photon is off-shell,  $q^2 \neq 0$ . Moreover, we put the vertex in the context of the complete electron line — including the external line factors, thus  $\bar{u}(p') \times ie\Gamma^\mu \times u(p)$ . As discussed in class, this simplifies the Lorentz and Dirac structure of the vertex and allows us to write it as

$$\Gamma^\mu(p', p) = F_{\text{el}}(q^2) \times \frac{(p' + p)^\mu}{2m} + F_{\text{mag}}(q^2) \times \frac{i\sigma^{\mu\nu}q_\nu}{2m} = F_1(q^2) \times \gamma^\mu + F_2(q^2) \times \frac{i\sigma^{\mu\nu}q_\nu}{2m}. \tag{3}$$

## Working Through the Algebra

At the one-loop level of QED, the 1PI vertex correction comes from a single Feynman diagram



Using the Feynman gauge for the internal photon's propagators, this diagram evaluates to

$$\begin{aligned}
 ie\Gamma_{1\text{loop}}^\mu(p', p) &= \int_{\text{reg}} \frac{d^4k}{(2\pi)^4} \frac{-ig^{\nu\lambda}}{k^2 + i0} \times ie\gamma_\nu \times \frac{i}{\not{p}' + \not{k} - m + i0} \times ie\gamma^\mu \times \frac{i}{\not{p} + \not{k} - m + i0} \times ie\gamma_\lambda \\
 &= e^3 \int_{\text{reg}} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + i0} \times \gamma^\nu \times \frac{\not{p}' + \not{k} + m}{(p' + k)^2 - m^2 + i0} \times \gamma^\mu \times \frac{\not{p} + \not{k} + m}{(p + k)^2 - m^2 + i0} \times \gamma_\nu \\
 &= e^3 \int_{\text{reg}} \frac{d^4k}{(2\pi)^4} \frac{\mathcal{N}^\mu}{\mathcal{D}}
 \end{aligned} \tag{5}$$

where

$$\mathcal{N}^\mu = \gamma^\nu (\not{k} + \not{p}' + m) \gamma^\mu (\not{k} + \not{p} + m) \gamma_\nu \tag{6}$$

and

$$\mathcal{D} = [k^2 + i0] \times [(p + k)^2 - m^2 + i0] \times [(p' + k)^2 - m^2 + i0]. \tag{7}$$

The purpose of this section of the notes is to simplify these numerator and denominator. Using the Feynman parameter trick, we may combine the 3 denominator factors as

$$\frac{1}{\mathcal{D}} = \int_0^1 \int_0^1 \int_0^1 dx dy dz \delta(x + y + z - 1) \frac{2}{\left[ x((p + k)^2 - m^2) + y((p' + k)^2 - m^2) + z(k^2) + i0 \right]^3}. \tag{8}$$

Inside the big square brackets here we have

$$\begin{aligned}
[\dots] &= x \times ((p+k)^2 - m^2) + y \times ((p'+k)^2 - m^2) + z \times k^2 \\
&= k^2 \times (x+y+z=1) + 2k_\mu(xp+yp')^\mu + x(p^2 - m^2) + y(p'^2 - m^2) \\
&= (k+xp+yp')^2 - \Delta
\end{aligned} \tag{9}$$

where

$$\begin{aligned}
\Delta &= (xp+yp')^2 - xp^2 - yp'^2 + (x+y)m^2 \\
&= xy \times (2p \cdot p' = p^2 + p'^2 - (p'-p)^2) - x(1-x) \times p^2 - y(1-y) \times p'^2 + (x+y) \times m^2 \\
&= -xy \times q^2 - x(1-x-y) \times p^2 - y(1-x-y) \times p'^2 + (x+y) \times m^2 \\
&= -xy \times q^2 - xz \times p^2 - yz \times p'^2 + (1-z) \times m^2
\end{aligned} \tag{10}$$

For the on-shell electron momenta,  $p^2 = p'^2 = m^2$ , we may further simplify

$$(1-z) \times m^2 - xz \times p^2 - yz \times p'^2 = m^2 \times \left( (1-z) - (x+y)z = (1-z)^2 \right) \tag{11}$$

which gives

$$\Delta = (1-z)^2 \times m^2 - xy \times q^2. \tag{12}$$

Let us also define the shifted loop momentum

$$\ell = k + xp + yp', \tag{13}$$

then we can rewrite the denominator as

$$\frac{1}{\mathcal{D}} = \int_0^1 \int \int dx dy dz \delta(x+y+z-1) \frac{2}{[\ell^2 - \Delta + i0]^3}. \tag{14}$$

As usual, we plug this denominator into the loop integral (5), then change the order of integration —  $\int$  over the loop momentum before  $\int$  over the Feynman parameters, — and then shift

the momentum integration variable from  $k$  to  $\ell$ , thus

$$\Gamma_{1\text{loop}}^\mu(p', p) = -2ie^2 \iiint_0^1 dx dy dz \delta(x + y + z - 1) \int_{\text{reg}} \frac{d^4\ell}{(2\pi)^4} \frac{\mathcal{N}^\mu}{[\ell^2 - \Delta + i0]^3}. \quad (15)$$

But to make full use of the momentum shift, we need to re-express the numerator  $\mathcal{N}^\mu$  in terms of the shifted momentum  $\ell$ . It would also help to simplify the numerator (6) *in the context of this monstrous integral*.

The first step towards simplifying the  $\mathcal{N}^\mu$  is obvious: Let us get rid of the  $\gamma^\nu$  and  $\gamma_\nu$  factors using the  $\gamma$  matrix algebra, *eg.*,  $\gamma^\nu \not{a} \gamma_\nu = -2 \not{a}$ , *etc.*. However, in order to allow for the dimensional regularization, we need to re-work the algebra for an arbitrary spacetime dimension  $D$  where  $\gamma^\nu \gamma_\nu = D \neq 4$ . Consequently,

$$\begin{aligned} \gamma^\nu \not{a} \gamma_\nu &= -2 \not{a} + (4 - D) \not{a}, \\ \gamma^\nu \not{a} \not{b} \gamma_\nu &= 4(ab) - (4 - D) \not{a} \not{b}, \\ \gamma^\nu \not{a} \not{b} \not{c} \gamma_\nu &= -2 \not{c} \not{b} \not{a} + (4 - D) \not{a} \not{b} \not{c}, \end{aligned} \quad (16)$$

and therefore

$$\begin{aligned} \mathcal{N}^\mu &\stackrel{\text{def}}{=} \gamma^\nu (\not{k} + \not{p}' + m) \gamma^\mu (\not{k} + \not{p} + m) \gamma_\nu \\ &= -2m^2 \gamma^\mu + 4m(p' + p + 2k)^\mu - 2(\not{p}' + \not{k}) \gamma^\mu (\not{p}' + \not{k}) \\ &\quad + (4 - D)(\not{p}' + \not{k} - m) \gamma^\mu (\not{p}' + \not{k} - m). \end{aligned} \quad (17)$$

The second step is to re-express this numerator in terms of the loop momentum  $\ell$  rather than  $k$  using eq. (13). Expanding the result in powers of  $\ell$ , we get quadratic, linear and  $\ell$ -independent terms, but the linear terms do not contribute to the  $\int d^D\ell$  integral because they are odd with respect to  $\ell \rightarrow -\ell$  while everything else in that integral is even. Consequently, *in the context of*

eq. (15) we may neglect the linear terms, thus

$$\begin{aligned}
\mathcal{N}^\mu &= -2m^2\gamma^\mu + 4m(p' + p + 2\ell - 2xp - 2yp')^\mu \\
&\quad - 2(\not{p} + \not{\ell} - x\not{p} - y\not{p}')\gamma^\mu(\not{p}' + \not{\ell} - x\not{p} - y\not{p}') \\
&\quad + (4 - D)(\not{p}' + \not{\ell} - x\not{p} - y\not{p}' - m)\gamma^\mu(\not{p} + \not{\ell} - x\not{p} - y\not{p}' - m) \\
&\quad \langle\langle \text{skipping terms linear in } \ell \rangle\rangle \\
&\cong -2m^2\gamma^\mu + 4m(p + p' - 2xp - 2yp')^\mu \\
&\quad - 2\not{\ell}\gamma^\mu\not{\ell} - 2(\not{p} - x\not{p} - y\not{p}')\gamma^\mu(\not{p}' - x\not{p} - y\not{p}') \\
&\quad + (4 - D)\not{\ell}\gamma^\mu\not{\ell} + (4 - D)(\not{p}' - y\not{p}' - x\not{p} - m)\gamma^\mu(\not{p} - x\not{p} - y\not{p}' - m).
\end{aligned} \tag{18}$$

Next, we make use of  $p' - p = q$  and  $1 - x - y = z$  to rewrite

$$\begin{aligned}
2xp + 2yp' &= (x + y) \times (p + p') + (x - y) \times (p - p'), \\
p + p' - 2xp - 2yp' &= z \times (p' + p) + (x - y) \times q, \\
p - xp - yp' &= z \times p - y \times q \\
&= z \times p' - (1 - x) \times q, \\
p' - xp - yp' &= z \times p' + x \times q \\
&= z \times p + (1 - y) \times q,
\end{aligned} \tag{19}$$

and consequently

$$\begin{aligned}
\mathcal{N}^\mu &\cong -2m^2\gamma^\mu + 4mz(p' + p)^\mu + 4m(x - y)q^\mu \\
&\quad + (-2 + 4 - D) \times \not{\ell}\gamma^\mu\not{\ell} \\
&\quad - 2(z\not{p}' + (x - 1)\not{q})\gamma^\mu(z\not{p} + (1 - y)\not{q}) \\
&\quad + (4 - D)(z\not{p}' + x\not{q} - m)\gamma^\mu(z\not{p} - y\not{q} - m).
\end{aligned} \tag{20}$$

The third step is to make use of the external fermions being on-shell. This means more than just  $p^2 = p'^2 = m^2$ : We also sandwich the vertex  $ie\Gamma^\mu$  between the Dirac spinors  $\bar{u}(p')$  on the left and  $u(p)$  on the right. The two spinors satisfy the appropriate Dirac equations  $\not{p}u(p) = mu(p)$  and  $\bar{u}(p')\not{p}' = \bar{u}(p')m$ , so in the context of  $\bar{u}(p')\Gamma^\mu u(p)$ ,

$$A \times \not{p} \cong A \times m \quad \text{and} \quad \not{p}' \times B \cong m \times B \tag{21}$$

for any terms in  $\Gamma^\mu$  that look like  $A \times \not{p}$  or  $\not{p}' \times B$  for some  $A$  or  $B$ . Consequently, the terms on

the last two lines of eq. (20) are equivalent to

$$\begin{aligned}
(z \not{p}' + (x-1) \not{q}) \gamma^\mu (z \not{p}' + (1-y) \not{q}) &\cong (zm + (x-1) \not{q}) \gamma^\mu (zm + (1-y) \not{q}) \\
&= z^2 m^2 \times \gamma^\mu - (1-x)(1-y) \times \not{q} \gamma^\mu \not{q} \\
&\quad + z(x-y)m \times \left( \frac{1}{2} \{ \gamma^\mu, \not{q} \} = q^\mu \right) \\
&\quad + z(2-x-y)m \times \left( \frac{1}{2} [ \gamma^\mu, \not{q} ] = -i \sigma^{\mu\nu} q_\nu \right) \\
(z \not{p}' + x \not{q} - m) \gamma^\mu (z \not{p}' - y \not{q} - m) &\cong ((z-1)m + x \not{q}) \gamma^\mu ((z-1)m - y \not{q}) \\
&= (1-z)^2 m^2 \times \gamma^\mu - xy \times \not{q} \gamma^\mu \not{q} \\
&\quad - (1-z)(x-y)m \times \left( \frac{1}{2} \{ \gamma^\mu, \not{q} \} = q^\mu \right) \\
&\quad + (1-z)(x+y)m \times \left( \frac{1}{2} [ \gamma^\mu, \not{q} ] = -i \sigma^{\mu\nu} q_\nu \right).
\end{aligned} \tag{22}$$

Let's plug these expressions back into eq. (20), collect similar terms together, and make use of  $1-x-y=z$ . This gives us

$$\begin{aligned}
\mathcal{N}^\mu &\cong -(D-2) \not{q} \gamma^\mu \not{q} + 4mz(p' + p)^\mu \\
&\quad + m^2 \gamma^\mu \times \left( -2 - 2z^2 + (4-D)(1-z)^2 \right) \\
&\quad + \not{q} \gamma^\mu \not{q} \times \left( 2(z+xy) - (4-D)xy \right) \\
&\quad + mq^\mu \times (x-y) \left( 4 - 2z - (4-D)(1-z) \right) \\
&\quad + im\sigma^{\mu\nu} q_\nu \times \left( 2z(1+z) - (4-D)(1-z)^2 \right).
\end{aligned} \tag{23}$$

Furthermore, in the context of the Dirac sandwich  $\bar{u}(p') \Gamma^\mu u(p)$  we have

$$\not{q} \gamma^\mu \not{q} = 2q^\mu \not{q} - q^2 \gamma^\mu \cong -q^2 \gamma^\mu \tag{24}$$

because  $\bar{u}(p') \not{q} u(p) = 0$ , and also

$$(p' + p)^\mu \cong 2m\gamma^\mu - i\sigma^{\mu\nu} q_\nu \tag{25}$$

(the Gordon identity). Plugging these formulae into eq. (23), we arrive at

$$\begin{aligned}
\mathcal{N}^\mu &\cong -(D-2) \not{q} \gamma^\mu \not{q} + m^2 \gamma^\mu \times \left( 8z - 2(1+z^2) + (4-D)(1-z)^2 \right) \\
&\quad - q^2 \gamma^\mu \times \left( 2(z+xy) - (4-D)xy \right) - im\sigma^{\mu\nu} q_\nu \times (1-z) \left( 2z + (4-D)(1-z) \right) \\
&\quad + mq^\mu \times (x-y) \left( 4 - 2z - (4-D)(1-z) \right).
\end{aligned} \tag{26}$$

To further simplify this expression, let us go back to the symmetries of the integral (15). The integral over the Feynman parameters, the integral  $\int d^D \ell$ , and the denominator  $[l^2 - \Delta]^3$  are all invariant under the parameter exchange  $x \leftrightarrow y$ . In eq. (26) for the numerator, the first two lines are invariant under this symmetry, but the last line changes sign. Consequently, only the first two lines contribute to the integral (15) while the third line integrates to zero and may be disregarded, thus

$$\begin{aligned} \mathcal{N}^\mu &\cong -(D-2) \not{\ell} \gamma^\mu \not{\ell} + m^2 \gamma^\mu \times \left( 8z - 2(1+z^2) + (4-D)(1-z)^2 \right) \\ &\quad - q^2 \gamma^\mu \times \left( 2(z+xy) - (4-D)xy \right) - im\sigma^{\mu\nu} q_\nu \times (1-z) \left( 2z + (4-D)(1-z) \right). \end{aligned} \quad (27)$$

Finally, thanks to the Lorentz invariance of the  $\int d^D \ell$  integral,

$$\ell_\lambda \ell_\nu \cong g_{\lambda\nu} \times \frac{\ell^2}{D}, \quad (28)$$

and hence

$$\not{\ell} \gamma^\mu \not{\ell} = \gamma^\lambda \gamma^\mu \gamma^\nu \times \ell_\lambda \ell_\nu \cong \gamma^\lambda \gamma^\mu \gamma^\nu \times g_{\lambda\nu} \frac{\ell^2}{D} = -(D-2) \gamma^\mu \times \frac{\ell^2}{D}. \quad (29)$$

Plugging this formula into eq. (26) and grouping terms according to their  $\gamma$ -matrix structure, we arrive at

$$\mathcal{N}^\mu = \mathcal{N}_1 \times \gamma^\mu - \mathcal{N}_2 \times \frac{i\sigma^{\mu\nu} q_\nu}{2m} \quad (30)$$

where

$$\begin{aligned} \mathcal{N}_1 &\cong \frac{(D-2)^2}{D} \times \ell^2 + \left( 8z - 2(1+z^2) + (4-D)(1-z)^2 \right) \times m^2 \\ &\quad - \left( 2(z+xy) - (4-D)xy \right) \times q^2 \\ &= \frac{(D-2)^2}{D} \times \ell^2 - (D-2) \times \Delta + 2z \times (2m^2 - q^2), \end{aligned} \quad (31)$$

$$\mathcal{N}_2 \cong (1-z) \left( 4z + 2(4-D)(1-z) \right) \times m^2. \quad (32)$$

Note that splitting the numerator according to eq. (30) is particularly convenient for calculating the electron's form factors:

$$\Gamma_{1\text{loop}}^\mu = F_1^{1\text{loop}}(q^2) \times \gamma^\mu + F_2^{1\text{loop}}(q^2) \times \frac{i\sigma^{\mu\nu} q_\nu}{2m}, \quad (33)$$

$$F_1^{1\text{loop}}(q^2) = -2ie^2 \int_0^1 \int \int dx dy dz \delta(x+y+z-1) \int \frac{d^D \ell}{(2\pi)^D} \frac{\mathcal{N}_1}{[\ell^2 - \Delta + i0]^3}, \quad (34)$$

$$F_2^{1\text{loop}}(q^2) = +2ie^2 \int_0^1 \int \int dx dy dz \delta(x+y+z-1) \int \frac{d^D \ell}{(2\pi)^D} \frac{\mathcal{N}_2}{[\ell^2 - \Delta + i0]^3}. \quad (35)$$

## Electron's Gyromagnetic Moment

As explained earlier in class, electron's spin couples to the static magnetic field as

$$\hat{H} \supset \frac{-eg}{2m_e} \mathbf{S} \cdot \mathbf{B} \quad \text{where} \quad g = 2 \left( F_{\text{mag}} = F_1 + F_2 \right) \Big|_{q^2=0}. \quad (36)$$

The electric form factor  $F_1 \equiv F_{el}$  for  $q^2 = 1$  is constrained by the Ward identity,

$$F_1^{\text{tot}} = F_1^{\text{tree}} + F_1^{\text{loops}} + F_1^{\text{counter-terms}} \xrightarrow{q^2 \rightarrow 0} 1. \quad (37)$$

Therefore, the gyromagnetic moment is

$$g = 2 + 2F_2(q^2 = 0) \quad (38)$$

where  $F_2 = F_2^{\text{loops}}$  because there are no tree-level or counter-term contributions to the  $F_2$ , only to the  $F_1$ . Thus, to calculate the  $g - 2$  at the one-loop level, all we need is to evaluate the integral (35) for  $q^2 = 0$ .

Let's start with the momentum integral

$$\int \frac{d^D \ell}{(2\pi)^D} \frac{\mathcal{N}_2}{[\ell^2 - \Delta + i0]^3} \quad (39)$$

where  $\Delta = (1-z)^2 m^2$  for  $q^2 = 0$  and  $\mathcal{N}_2$  is as in eq. (32). Because the numerator here does not depend on the loop momentum  $\ell$ , this integral converges in  $D = 4$  dimensions and there is no



need for dimensional regularization. All we need is to rotate the momentum into Euclidean space,

$$\begin{aligned}
\int \frac{d^4 \ell}{(2\pi)^4} \frac{\mathcal{N}_2}{[\ell^2 - \Delta + i0]^3} &= \mathcal{N}_2 \times \int \frac{i d^4 \ell_E}{(2\pi)^4} \frac{1}{-(\ell_E^2 + \Delta)^3} \\
&= \frac{-i \mathcal{N}_2}{16\pi^2} \times \int_0^\infty d\ell_E^2 \frac{\ell_E^2}{(\ell_E^2 + \Delta)^3} \\
&= \frac{-i \mathcal{N}_2}{16\pi^2} \times \frac{1}{2\Delta} \\
&= \frac{-i}{32\pi^2} \times \frac{\mathcal{N}_2 = 4z(1-z)m^2 \quad \langle\langle \text{for } D = 4 \rangle\rangle}{\Delta = (1-z)^2 m^2 \quad \langle\langle \text{for } q^2 = 0 \rangle\rangle} \\
&= \frac{-i}{32\pi^2} \times \frac{4z}{1-z}.
\end{aligned} \tag{40}$$

Substituting this formula into eq. (35), we have

$$F_2^{1\text{loop}}(q^2 = 0) = \frac{e^2}{16\pi^2} \iiint_0^1 dx dy dz \delta(x + y + z - 1) \times \frac{4z}{1-z}. \tag{41}$$

The integrand here depends on  $z$  but not on the other two Feynman parameters, so we can immediately integrate over  $x$  and  $y$  and obtain

$$\iint_0^1 dx dy \delta(x + y + z - 1) = \int_0^{1-z} dx = 1 - z. \tag{42}$$

Consequently,

$$F_2^{1\text{loop}}(q^2 = 0) = \frac{e^2}{16\pi^2} \times \int_0^1 dz (1-z) \times \frac{4z}{1-z} = \frac{e^2}{16\pi^2} \times 2 = \frac{\alpha}{2\pi} \tag{43}$$

and the gyromagnetic moment is

$$g = 2 + \frac{\alpha}{\pi} + O(\alpha^2). \tag{44}$$

Higher-loop calculations are more complicated because the number of diagrams grows very rapidly with the number of loops; at 4-loop order there are thousands of diagrams, and one needs a computer just to count them! Also, at higher orders one has to include the effects of strong and weak interactions because the photons interact with hadrons and  $W^\pm$  particles, which in turn interact with other hadrons,  $Z^0$ , Higgs, *etc.*, *etc.* Nevertheless, people have calculated the electron's and muon's  $g$  factors up to the order  $\alpha^4$  back in the 1970s, and more recent calculations are good up to the order  $\alpha^5$ . Meanwhile, the experimentalists have measured  $g_e$  to a comparable accuracy of 12 significant digits and  $g_\mu$  to 9 significant digits

$$g_e = 2.00231930436\overline{22}(15), \quad g_\mu = 2.002331841\overline{4}(12). \quad (45)$$

The theoretical value of  $g_e$  is in good agreement with the experimental value, while for the muon there is a small discrepancy  $g_\mu^{\text{exp}} - g_\mu^{\text{theory}} \approx (59 \pm 13 \pm 12) \cdot 10^{-10}$ . This discrepancy indicates some physics beyond the Standard Model, maybe supersymmetry, maybe something else. In general, effect of heavy particles on  $g_\mu$  is proportional to  $(m_\mu/M_{\text{heavy}})^2$ , that's why  $g_\mu$  is much more sensitive to new physics than  $g_e$ .

For a recent review of high-precision  $g_e$  and  $g_\mu$  — both the experiments and the theory — see <http://arxiv.org/abs/hep-ph/0703049>.

I would like to complete this section of the notes by calculating the  $F_2^{1\text{loop}}(q^2)$  form factor for  $q^2 \neq 0$ . Proceeding as in eq. (40) but letting  $\Delta = (1-z)^2 m^2 - xyq^2$ , we have

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{\mathcal{N}_2}{[\ell^2 - \Delta + i0]^3} = \frac{-i}{32\pi^2} \times \frac{4z(1-z)m^2}{(1-z)^2 m^2 - xyq^2} \quad (46)$$

and hence

$$F_2^{1\text{loop}}(q^2) = \frac{e^2}{16\pi^2} \int_0^1 \int_0^1 \int_0^1 dx dy dz \delta(x+y+z-1) \times \frac{4z(1-z)m^2}{(1-z)^2 m^2 - xyq^2}. \quad (47)$$

To evaluate this integral over Feynman parameters, we change variables from  $x, y, z$  to  $w = 1 - z$

and  $\xi = x/(x + y)$ ,

$$x = w\xi, \quad y = w(1 - \xi), \quad z = 1 - w, \quad dx dy dz \delta(x + y + z - 1) = w dw d\xi. \quad (48)$$

Consequently,

$$\begin{aligned} F_2^{1\text{loop}}(q^2) &= \frac{e^2}{16\pi^2} \int_0^1 d\xi \int_0^1 dw w \times \frac{4(1-w)w \times m^2}{w^2 \times m^2 - w^2 \xi(1-\xi) \times q^2} \\ &= \frac{e^2}{16\pi^2} \int_0^1 d\xi \frac{m^2}{m^2 - \xi(1-\xi)q^2} \times \int_0^1 dw w \times \frac{4w(1-w)}{w^2} \\ &= \frac{e^2}{8\pi^2} \times \int_0^1 d\xi \frac{m^2}{m^2 - \xi(1-\xi)q^2} \\ &= \frac{\alpha}{2\pi} \times \frac{4m^2}{\sqrt{q^2 \times (4m^2 - q^2)}} \times \arctan \sqrt{\frac{q^2}{4m^2 - q^2}} \\ &= \frac{\alpha}{2\pi} \times \frac{4m^2}{\sqrt{(-q^2) \times (4m^2 - q^2)}} \times \log \frac{\sqrt{4m^2 - q^2} + \sqrt{-q^2}}{2m}. \end{aligned} \quad (49)$$

For  $q^2 < 0$  and  $-q^2 \gg m^2$ ,

$$F_2^{1\text{loop}}(q^2) \approx \frac{\alpha}{2\pi} \times \frac{2m^2}{-q^2} \times \log \frac{-q^2}{m^2}. \quad (50)$$

## The Electric Form Factor

Now consider the electric form factor  $F_1(q^2)$ . In the first section we have obtained

$$F_1^{1\text{loop}}(q^2) = -2ie^2 \iiint_0^1 dx dy dz \delta(x+y+z-1) \int \frac{d^D \ell}{(2\pi)^D} \frac{\mathcal{N}_1}{[\ell^2 - \Delta + i0]^3}, \quad (34)$$

for

$$\mathcal{N}_1^\mu \cong \frac{(D-2)^2}{D} \times \ell^2 - (D-2) \times \Delta + 2z \times (2m^2 - q^2) \quad (31)$$

and  $\Delta = (1-z)^2 m^2 - xyq^2$ .

Let's start by calculating the momentum integral in eq. (34). The numerator  $\mathcal{N}_1$  depends on  $\ell$  as  $a\ell^2 + b$ , so there is a logarithmic UV divergence for  $\ell \rightarrow \infty$ ; to regularize this divergence, we work in  $D = 4 - 2\epsilon$  dimensions. Thus,

$$\begin{aligned} -i \int_{\text{reg}} \frac{d^4 \ell}{(2\pi)^4} \frac{a\ell^2 + b}{[\ell^2 - \Delta + i0]^3} &\equiv -i\mu^{4-D} \int \frac{d^D \ell}{(2\pi)^D} \frac{a\ell^2 + b}{[\ell^2 - \Delta + i0]^3} = \\ &= -i\mu^{4-D} \int \frac{id^D \ell_E}{(2\pi)^D} \frac{-a\ell_E^2 + b}{-[\ell_E^2 + \Delta]^3} \\ &= \mu^{4-D} \int \frac{d^D \ell_E}{(2\pi)^D} \times \left[ \frac{a\ell_E^2 - b}{(\ell_E^2 + \Delta)^3} = \frac{a}{(\ell_E + \Delta)^2} - \frac{a\Delta + b}{(\ell_E^2 + \Delta)^3} \right] \\ &= \mu^{4-D} \int \frac{d^D \ell_E}{(2\pi)^D} \int_0^\infty dt \left( a \times t - (a\Delta + b) \times \frac{1}{2}t^2 \right) \times e^{-t(\Delta + \ell_E^2)} \\ &= \int_0^\infty dt \left( a \times t - (a\Delta + b) \times \frac{1}{2}t^2 \right) e^{-t\Delta} \times \mu^{4-D} \int \frac{d^D \ell_E}{(2\pi)^D} e^{-t\ell_E^2} \\ &= \int_0^\infty dt \left( a \times t - (a\Delta + b) \times \frac{1}{2}t^2 \right) e^{-t\Delta} \times \frac{\mu^{4-D}}{(4\pi t)^{D/2}} \\ &= \frac{\mu^{4-D}}{(4\pi)^{D/2}} \int_0^\infty dt e^{-t\Delta} \times \left( a \times t^{1-(D/2)} - \frac{1}{2}(a\Delta + b) \times t^{2-(D/2)} \right) \\ &= \frac{\mu^{4-D}}{(4\pi)^{D/2}} \left\{ a \times \Gamma\left(2 - \frac{D}{2}\right) \times \Delta^{\frac{D}{2}-2} - \frac{1}{2}(a\Delta + b) \times \Gamma\left(3 - \frac{D}{2}\right) \times \Delta^{\frac{D}{2}-3} \right\} \\ &\rightarrow \frac{(4\pi\mu^2)^\epsilon}{16\pi^2} \times \frac{\Gamma(1+\epsilon)}{\Delta^\epsilon} \times \left\{ \frac{a}{\epsilon} - \frac{a\Delta + b}{2\Delta} \right\}. \end{aligned} \quad (51)$$

In light of eq. (31),

$$a = \frac{(D-2)^2}{D}, \quad b = 2z \times (2m^2 - q^2) - (D-2) \times \Delta, \quad (52)$$

so on the last line of eq. (51)

$$\frac{a}{\epsilon} - \frac{a\Delta + b}{2\Delta} = \frac{1-\epsilon}{\epsilon} - \frac{z(2m^2 - q^2)}{\Delta}. \quad (53)$$

Consequently, the momentum integral in eq. (34) for the electric form factors evaluates to

$$\begin{aligned} -2i\epsilon^2 \mu^{4-D} \int \frac{d^D \ell}{(2\pi)^D} \frac{\mathcal{N}_1}{[\ell^2 - \Delta + i0]^3} &= \\ &= \frac{\alpha}{2\pi} \left( \frac{4\pi\mu^2}{\Delta} \right)^\epsilon \left\{ \Gamma(\epsilon) \times (1-\epsilon) - \Gamma(1+\epsilon) \times \frac{z \times (2m^2 - q^2)}{\Delta} \right\}, \end{aligned} \quad (54)$$

and now we need to integrate this expression over the Feynman parameters.

Changing the integration variables from  $x, y, z$  to  $w$  and  $\xi$  according to eq. (48), we have

$$F_1^{1\text{loop}}(q^2) = \frac{\alpha}{2\pi} (4\pi\mu^2)^\epsilon \int_0^1 d\xi \int_0^1 dw w \times \left\{ \begin{aligned} &(1-\epsilon)\Gamma(\epsilon) \times \frac{1}{[\Delta(w, \xi)]^\epsilon} \\ &- \Gamma(1+\epsilon) \times \frac{(1-w)(2m^2 - q^2)}{[\Delta(w, \xi)]^{1+\epsilon}} \end{aligned} \right\} \quad (55)$$

where

$$\Delta(w, \xi) = (1-z)^2 m^2 - xyq^2 = w^2 \times (m^2 - \xi(1-\xi)q^2), \quad (56)$$

or equivalently,

$$\Delta(w, \xi) = w^2 \times H(\xi) \quad \text{where} \quad H(\xi) \stackrel{\text{def}}{=} m^2 - \xi(1-\xi)q^2. \quad (57)$$

The form (57) is particularly convenient for evaluating the  $\int dw$  integral in eq. (55), which becomes

$$\int_0^1 dw \left\{ \frac{2(1-\epsilon)\Gamma(\epsilon)}{H^\epsilon} \times \frac{w}{w^{2\epsilon}} - 2\Gamma(1+\epsilon) \times \frac{2m^2 - q^2}{H^{1+\epsilon}} \times \frac{w(1-w)}{w^{2+2\epsilon}} \right\}. \quad (58)$$

Near the lower limit  $w \rightarrow 0$ , the integrand is dominated by the second term, which is proportional

to  $w^{-1-2\epsilon}$ . But for any  $\epsilon \geq 0$  — *i.e.*, for any dimension  $D \leq 4$  — the integral

$$\int_0^{\text{positive}} \frac{dw}{w^{1+2\epsilon}} \quad (59)$$

diverges: For  $D = 4$  the divergence is logarithmic while for  $D < 4$  it becomes power-like.

#### INFRARED DIVERGENCE.

Physically, the divergence (59) is *infrared* rather than ultraviolet, that's why it gets worse as we lower the dimension  $D$ . Indeed, let's go back to the diagram (4) and look at the denominator  $\mathcal{D}$  in eqs. (5) and (7). Taking the electron's momenta  $p$  and  $p'$  on-shell before introducing the Feynman parameters, we have

$$(p+k)^2 - m^2 = k^2 + 2kp \quad \text{and likewise} \quad (p'+k)^2 - m^2 = k^2 + 2kp'. \quad (60)$$

Therefore, for  $k \rightarrow 0$  the denominator behaves as  $\mathcal{D} \propto |k|^4$  while the numerator  $\mathcal{N}^\mu$  remains finite, which makes the integral

$$\int d^D k \frac{\mathcal{N}^\mu}{\mathcal{D}} \propto \int d^D k \frac{1}{|k|^4} \quad (61)$$

diverge for  $k \rightarrow 0$ . In  $D = 4$  dimensions, the infrared divergence here is logarithmic, while in lower dimensions  $D < 4$  it becomes power-like, *i.e.*  $O((1/k_{\min})^{4-D})$  — precisely as in eqs. (59) and (58).

We can regularize the infrared divergence (61) — and also (59) — by analytically continuing spacetime dimension to  $D > 4$ . Such dimensional regularization of the IR divergences is used in many situations in both QFT and condensed matter. However, taking  $D > 4$  makes the ultraviolet divergences worse, so if some amplitude has both UV and IR divergences, we cannot cure both of them at the same time by analytically continuing to  $D \neq 4$ . In particular, when calculating the electric form factor  $F_1(q^2)$  of the electron, we need  $D < 4$  to regulate the momentum integral  $\int d^D \ell$ , but then we need  $D > 4$  to regulate the integral over the Feynman parameters.

A common *dirty trick* is to first continue to  $D < 4$  and evaluate the  $\int d^D \ell$  momentum integral, then analytically continue the result to  $D > 4$  and integrate over the Feynman parameters, and then continue the final result to  $D = 4$ . However, in this kind of dimensional regularization it's

hard to disentangle the  $1/\epsilon$  poles coming from the UV divergence  $\log(\Lambda^2/\mu^2)$  from the  $1/\epsilon$  poles coming from the IR divergence  $\log(\mu^2/k_{\min}^2)$ , so we are not going to use it here.

Instead, we are going to use DR for the UV divergence only, while the IR divergence is regulated by a tiny but not-quite-zero photon mass  $m_\gamma^2 \ll m_e^2$ . Strictly speaking, a massive vector particle has three polarization states and its propagator is

$$\text{wavy line} = \frac{-i}{k^2 - m_\gamma^2 + i0} \times \left( g^{\mu\nu} - \frac{k^\mu k^\nu}{m_\gamma^2} \right). \quad (62)$$

However, the longitudinal polarization of the massive but ultra-relativistic photon does not couple to a conserved current, so we are going to disregard the  $k^\mu k^\nu$  terms in the propagator (62) and use

$$\text{wavy line} = \frac{-ig^{\mu\nu}}{k^2 - m_\gamma^2 + i0}. \quad (63)$$

In other words, we use the Feynman gauge in spite of the photon's mass; this is not completely consistent, but the inconsistencies go away in the  $m_\gamma \rightarrow 0$  limit.

Using this infrared regulator for the internal photon line in the one-loop diagram (4), we get the vertex amplitude that looks exactly like eq. (5) except for one factor in the denominator,

$$\frac{1}{k^2 + i0} \text{ becomes } \frac{1}{k^2 - m_\gamma^2 + i0}. \quad (64)$$

In terms of the integral (15), this change has no effect on the numerator  $\mathcal{N}^\mu$  or the loop momentum  $\ell$  (which remains exactly as in eq. (13)), but the  $\Delta$  in the denominator becomes

$$\Delta'(x, y, z) = \Delta(x, y, z) + z \times m_\gamma^2. \quad (65)$$

Consequently, the electric form factor is

$$F_1^{\text{1loop}}(q^2) = \int d(FP) \int \mu^{4-D} \frac{d^D \ell}{(2\pi)^D} \frac{-2ie^2 \times \mathcal{N}_1}{[\ell^2 - \Delta' + i0]^3}, \quad (66)$$

exactly as in eq. (34), except for the  $\Delta'$  instead of the  $\Delta$  in the denominator. The momentum integral here converges for any  $D < 4$  and it evaluates exactly as in eq. (51). The only subtlety

here is that in the numerator, the  $\ell$ -independent term  $b$  involves the un-modified  $\Delta$  instead of  $\Delta'$  (cf. eq. (52)), but we can fix that by writing

$$b = 2z \times (2m_e^2 - q^2 + (1 - \epsilon)m_\gamma^2) - 2(1 - \epsilon) \times \Delta'. \quad (67)$$

Hence, instead of eq. (55) we get

$$F_1^{\text{1 loop}}(q^2) = \frac{\alpha}{2\pi} (4\pi\mu^2)^\epsilon \int_0^1 d\xi \int_0^1 dw w \times \left\{ \begin{array}{l} (1 - \epsilon)\Gamma(\epsilon) \times \frac{1}{[\Delta'(w, \xi)]^\epsilon} \\ - \Gamma(1 + \epsilon) \times \frac{(1 - w)(2m_e^2 - q^2 + (1 - \epsilon)m_\gamma^2)}{[\Delta'(w, \xi)]^{1+\epsilon}} \end{array} \right\} \quad (68)$$

where

$$\Delta'(w, \xi) = (1 - z)^2 m_e^2 - xyq^2 + zm_\gamma^2 = w^2 \times H(\xi) + (1 - w) \times m_\gamma^2. \quad (69)$$

Note that the photon's mass is tiny,  $m_\gamma^2 \ll m_e^2, q^2$ ; were it not for the IR divergences, we would have used  $m_\gamma^2 = 0$ . This allows us to neglect various  $O(m_\gamma^2)$  terms in eq. (68) except when it would cause a divergence for  $w \rightarrow 0$ ; in particular, we may neglect the  $(1 - \epsilon)m_\gamma^2$  term in the numerator of the second term in the integrand. As to the denominators, in eq. (69) the second term containing the photon's mass becomes important only in the  $w \rightarrow 0$  limit, and in that limit  $(1 - w)m_\gamma^2 \rightarrow m_\gamma^2$ . Thus, we approximate

$$\Delta'(w, \xi) \approx w^2 \times H(\xi) + m_\gamma^2 \quad (70)$$

and the  $\int dw$  integral in eq. (68) becomes

$$\begin{aligned} & \int_0^1 dw w \times \left\{ (1 - \epsilon)\Gamma(\epsilon) \times \frac{1}{[w^2 H(\xi) + m_\gamma^2]^\epsilon} - \Gamma(1 + \epsilon) \times \frac{(1 - w)(2m_e^2 - q^2)}{[w^2 H(\xi) + m_\gamma^2]^{1+\epsilon}} \right\} \\ &= \frac{(1 - \epsilon)\Gamma(\epsilon)}{H^\epsilon} \times \int_0^1 \frac{dw w}{[w^2 + (m_\gamma^2/H)]^\epsilon} \\ &+ \Gamma(1 + \epsilon) \frac{2m_e^2 - q^2}{H^{1+\epsilon}} \times \int_0^1 \frac{dw w^2}{[w^2 + (m_\gamma^2/H)]^{1+\epsilon}} \\ &- \Gamma(1 + \epsilon) \frac{2m_e^2 - q^2}{H^{1+\epsilon}} \times \int_0^1 \frac{dw w}{[w^2 + (m_\gamma^2/H)]^{1+\epsilon}}. \end{aligned} \quad (71)$$



For  $0 < \epsilon < \frac{1}{2}$  — *i.e.*, for  $3 < D < 4$  — the integrals on the second and third lines here converge even for  $m_\gamma^2 = 0$ ,

$$\begin{aligned}\int_0^1 \frac{dw w}{[w^2]^\epsilon} &= \frac{1}{2-2\epsilon} \quad \text{for } \epsilon < 1, \\ \int_0^1 \frac{dw w^2}{[w^2]^{1+\epsilon}} &= \frac{1}{1-2\epsilon} \quad \text{for } \epsilon < \frac{1}{2},\end{aligned}\tag{72}$$

so we may just as well evaluate them without the photon's mass. Only on the last line of eq. (71) we do need  $m_\gamma^2 \neq 0$  to make the integral converge for some  $D \leq 4$ :

$$\int_0^1 \frac{dw w}{[w^2 + (m_\gamma^2/H)]^{1+\epsilon}} = \frac{-1}{2\epsilon} \frac{1}{[w^2 + (m_\gamma^2/H)]^\epsilon} \Big|_0^1 = \frac{1}{2\epsilon} \left[ \left( \frac{H}{m_\gamma^2} \right)^\epsilon - 1 \right].\tag{73}$$

Combining all these  $\int dw$  integrals together, we get

$$\begin{aligned}\int_0^1 dw \{ \dots \} &= \frac{\Gamma(\epsilon)}{2H^\epsilon} + \frac{\Gamma(1+\epsilon)}{1-2\epsilon} \times \frac{2m_e^2 - q^2}{H^{1+\epsilon}} - \frac{\Gamma(1+\epsilon)}{2\epsilon} \times \frac{2m_e^2 - q^2}{H^{1+\epsilon}} \times \left[ \left( \frac{H}{m_\gamma^2} \right)^\epsilon - 1 \right] \\ &= \frac{\Gamma(\epsilon)}{2H^\epsilon} \times \left\{ 1 + \frac{2m_e^2 - q^2}{H} \times \left[ \frac{1}{1-2\epsilon} - \left( \frac{H}{m_\gamma^2} \right)^\epsilon \right] \right\}\end{aligned}\tag{74}$$

and hence

$$F_1^{\text{1 loop}}(q^2) = \frac{\alpha}{4\pi} \int_0^1 d\xi \Gamma(\epsilon) \left( \frac{4\pi\mu^2}{H(\xi)} \right)^\epsilon \times \left\{ 1 + \frac{2m_e^2 - q^2}{H(\xi)} \times \left[ \frac{1}{1-2\epsilon} - \left( \frac{H(\xi)}{m_\gamma^2} \right)^\epsilon \right] \right\}\tag{75}$$

where

$$H(\xi) = m_e^2 - \xi(1-\xi)q^2.\tag{57}$$

Before we even try to perform this last integral, let's remember that

$$\Gamma_{\text{net}}^\mu = \gamma_{\text{tree}}^\mu + \Gamma_{\text{loops}}^\mu + \delta_1 \times \gamma^\mu\tag{76}$$

and hence

$$F_1^{\text{net}}(q^2) = 1^{\text{tree}} + F_1^{\text{loops}}(q^2) + \delta_1. \quad (77)$$

Also, the net electric charge does not renormalize, so we must have

$$F_1^{\text{net}}(q^2) \rightarrow 1 \quad \text{for } q^2 \rightarrow 0 \quad (78)$$

and hence

$$\delta_1 = -F_1^{\text{loops}}(q^2 = 0). \quad (79)$$

To calculate the counterterm  $\delta_1$  to order  $\alpha$  we use eq. (75) for  $q^2 = 0$ , in which case  $H(\xi) \equiv m_e^2$  and the  $\int d\xi$  becomes trivial (the integrand does not depend on  $\xi$  at all). Thus,

$$\delta_1 = -\frac{\alpha}{4\pi} \Gamma(\epsilon) \left( \frac{4\pi\mu^2}{m_e^2} \right)^\epsilon \times \left\{ 1 + \frac{2}{1-2\epsilon} - 2 \left( \frac{m_e^2}{m_\gamma^2} \right)^\epsilon \right\} + O(\alpha^2). \quad (80)$$

This formula holds for any dimension  $D$  between 3 and 4 (*i.e.*,  $0 < \epsilon < \frac{1}{2}$ ). In the  $D \rightarrow 4$  limit, it becomes

$$\delta_1 = -\frac{\alpha}{4\pi} \times \left\{ \frac{1}{\epsilon} - \gamma_E + \log \frac{4\pi\mu^2}{m_e^2} + 4 - 2 \log \frac{m_e^2}{m_\gamma^2} \right\} + O(\alpha^2). \quad (81)$$

Now let's go back to the electric form factor  $F_1^{\text{net}}(q^2)$  for  $q^2 \neq 0$ . According to eqs. (77) and (79), at the one-loop level

$$F_1^{\text{net}}(q^2) - 1 = F_1^{\text{1loop}}(q^2) - F_1^{\text{1loop}}(0) + O(\alpha^2) \quad (82)$$

where  $F_1^{\text{1loop}}(q^2)$  is given by eq. (75). Taking the  $\epsilon \rightarrow 0$  limit of that formula, we arrive at

$$F_1^{\text{1loop}}(q^2) = \frac{\alpha}{4\pi} \int_0^1 d\xi \left\{ \frac{1}{\epsilon} - \gamma_E + \log \frac{4\pi\mu^2}{H(\xi)} + \frac{2m_e^2 - q^2}{H(\xi)} \times \left[ 2 - \log \frac{H(\xi)}{m_\gamma^2} \right] \right\}, \quad (83)$$

and now we should subtract a similar a similar expression for  $q^2 = 0$ . This subtraction cancels the UV divergence and the associated  $1/\epsilon$  pole but not the IR divergence. Moreover, not only the

subtracted one-loop amplitude depends on the IR regulators, but the coefficient of the  $\log m_\gamma^2$  has a non-trivial momentum dependence. Indeed,

$$\begin{aligned}
F_1^{1\text{loop}}(q^2) - F_1^{1\text{loop}}(0) &= \\
&= \frac{\alpha}{4\pi} \int_0^1 d\xi \left\{ \log \frac{m_e^2}{H(\xi)} + \frac{2m_e^2 - q^2}{H(\xi)} \times \left[ 2 - \log \frac{H(\xi)}{m_\gamma^2} \right] - 2 \left[ 2 - \log \frac{m_e^2}{m_\gamma^2} \right] \right\} \\
&= \frac{\alpha}{4\pi} \int_0^1 d\xi \left\{ \left( 1 + \frac{2m_e^2 - q^2}{H(\xi)} \right) \times \log \frac{m_e^2}{H(\xi)} + \left( \frac{2m_e^2 - q^2}{H(\xi)} - 2 \right) \times \left[ 2 - \log \frac{m_e^2}{m_\gamma^2} \right] \right\} \\
&= -\frac{\alpha}{4\pi} \times \left\{ h(q^2/m_e^2) + f_{\text{IR}}(q^2/m_e^2) \times \log \frac{m_e^2}{m_\gamma^2} \right\}
\end{aligned} \tag{84}$$

where  $h(q^2/m_e^2)$  and  $f_{\text{IR}}(q^2/m_e^2)$  are finite (in the limit  $m_\gamma \rightarrow 0$ ) functions of the  $q^2/m_e^2$  ratio; both of them vanish for  $q^2 = 0$ . Specifically,

$$f_{\text{IR}}(q^2/m_e^2) = \int_0^1 d\xi \left( \frac{2m_e^2 - q^2}{H(\xi)} - 2 = \frac{-q^2 \times (1 - 2\xi + 2\xi^2)}{m_e^2 - q^2 \times \xi(1 - \xi)} \right), \tag{85}$$

which happens to be the same function that governs the IR divergence of the the soft-photon bremsstrahlung. In terms of §6.1 of the Peskin & Schroeder textbook,

$$f_{\text{IR}}(q^2/m_e^2) = \mathcal{I}(\mathbf{v}, \mathbf{v}') = \int \frac{d^2\Omega_{\mathbf{n}}}{4\pi} \left[ - \left( \frac{p'^\mu}{(np')} - \frac{p^\mu}{(np)} \right)^2 \right]^{n^0=|\mathbf{n}|=1}, \tag{86}$$

see textbooks eqs. (6.69–70) for the proof. Note: my definition of the  $i_{\text{IR}}$  differs from the textbook's by a factor of 2.

Altogether, the electric form factor of the electron is

$$F_1^{\text{net}}(q^2) = 1 - \frac{\alpha}{4\pi} \times \left\{ g(q^2/m_e^2) + f_{\text{IR}}(q^2/m_e^2) \times \log \frac{m_e^2}{m_\gamma^2} \right\} + O(\alpha^2). \tag{87}$$

Implications of this formula will be discussed in class; see also §6.4 of the textbook.