Conformal and Superconformal Symmetries

The conformal symmetry group is generated by the following operators:

1. The momenta $P^\mu = i\partial^\mu$ generate spacetime translations,

$$\exp(ia_\mu P^\mu) X^\lambda \exp(-ia_\mu P^\mu) \;=\; X^\lambda - a^\lambda. \tag{1}$$

2. The angular momenta $J^{\mu\nu} = X^\mu P^\nu - X^\nu P^\mu + S^{\mu\nu}$ generate rotations of space and Lorentz boosts,

$$\exp(\tfrac{i}{2} r_{\mu\nu} J^{\mu\nu}) X^\lambda \exp(-\tfrac{i}{2} r_{\mu\nu} J^{\mu\nu}) \;=\; L^\lambda{}_\rho X^\rho, \qquad L^*{}_* = \exp(r^*{}_*) \in SO^+(1,3). \tag{2}$$

3. The dilatation operator $D = X^\mu P_\mu + D_{\text{intrinsic}}$ generates uniform rescaling

$$\exp(icD) X^\lambda \exp(-icD) \;=\; e^{-c} X^\lambda, \qquad \exp(icD)\Phi(X)\exp(-icD) \;=\; e^{c\Delta} \times \Phi(e^c X). \tag{3}$$

4. Finally, the special conformal operators $K^\mu = -2X^\mu D + X^2 P^\mu + 2S^{\mu\nu} X_\nu$ generate the "inverted translations" (translations of the inverted spacetime coordinates X^λ / X^2),

$$\begin{aligned} \exp(i\alpha_\mu K^\mu) X^\lambda \exp(-i\alpha_\mu K^\mu) \;&=\; \frac{X^\lambda - (X^2)\alpha^\lambda}{1 - 2(\alpha_\nu X^\nu) + \alpha^2 X^2}\,, \\ i.\,e., \quad \exp(i\alpha_\mu K^\mu) \left(\frac{X^\lambda}{X^2}\right) \exp(-i\alpha_\mu K^\mu) \;&=\; \left(\frac{X^\lambda}{X^2}\right) - \alpha^\lambda. \end{aligned} \tag{4}$$

Note: The sign conventions I used above are for the Minkowski spacetime with signature $(+---)$. In the Euclidean space, some signs are different, for example $P^\mu = -i\partial^\mu$ (instead of $P^\mu = +i\partial^\mu$).

The commutator algebra of the conformal generators includes the usual Poincaré algebra for the Lorentz and translation generators

$$[J_{\mu\nu}, J^{\rho,\sigma}] \;=\; i\delta_{[\mu}^{[\rho}\, J_{\nu]}^{\;\sigma]}, \qquad [J_{\mu\nu}, P^{\lambda}] \;=\; i\delta_{[\mu}^{\lambda}\, P_{\nu]}\,, \qquad [P^{\mu}, P^{\nu}] \;=\; 0. \tag{5}$$

We also have

$$[J_{\mu\nu}, K^{\lambda}] \;=\; i\delta_{[\mu}^{\lambda}\, K_{\nu]}\,, \qquad [J_{\mu\nu}, D] \;=\; 0 \tag{6}$$

because K^{λ} is a 4–vector while D is a 4–scalar,

$$[D, P^{\lambda}] \;=\; -iP^{\lambda}, \qquad [D, K^{\lambda}] \;=\; +iK^{\lambda} \tag{7}$$

because the momenta have scaling dimension $+1$ while the special conformal generators K^{μ} have dimension -1, and finally

$$[K^{\mu}, K^{\nu}] \;=\; 0, \qquad [K^{\mu}, P^{\nu}] \;=\; 2ig^{\mu\nu}D \,+\, 2iJ^{\mu\nu}. \tag{8}$$

The Lie algebra of the commutation relations (5)–(8) is $SO(2,4)$, a Lorentz-like symmetry of four space and two time coordinates. To see how this works, let's define $J^{ab} = -J^{ba}$ for $a, b = -1, 0, \ldots, 4$ according to

$$\begin{aligned} &\text{the usual } J^{\mu\nu} && \text{for } \mu,\nu = 0,1,2,3,\\ J^{4\mu} \;&=\; -J^{\mu 4} \;&&=\; \frac{P^{\mu} + K^{\mu}}{2}\,,\\ J^{-1,\mu} \;&=\; -J^{4,-1} \;&&=\; \frac{P^{\mu} - K^{\mu}}{2}\,,\\ J^{4,-1} \;&=\; -J^{-1,4} \;&&=\; D. \end{aligned} \tag{9}$$

In terms of J^{ab}, the commutators (5)–(8) become

$$[J^{ab}, J^{cd}] \;=\; -ig^{bc}J^{ad} \;-\; ig^{ac}J^{bd} \;-\; ig^{bd}J^{ac} \;+\; ig^{ad}J^{bc} \tag{10}$$

provided we extend the metric matrix from 4 to 6 dimensions according to

$$\begin{aligned} &\text{diagonal } g^{-1,-1} \;=\; g^{0,0} \;=\; +1, \quad g^{4,4} \;=\; g^{1,1} \;=\; g^{2,2} \;=\; g^{3,3} \;=\; -1,\\ &\text{off-diagonal } g^{a,b} \;=\; 0 \text{ for } a \neq b. \end{aligned} \tag{11}$$

In other words, the $J^{a,b}$ algebra is the Lorentz-like algebra $SO(2,4)$ in a "spacetime" of dimension $d = 2+4$: two times and 4 space dimensions. Or if you don't like multiple time dimensions, it's the symmetry algebra of the anti-de-Sitter space with one time and 4 space dimensions, which can be embedded into the $2+4$ dimensional space as a hypersurface $g_{ab}X^aX^b = R^2$.

More generally, the conformal symmetry group in the Minkowski spacetime with $d-1$ space dimensions and one time dimension is $SO(2,d)$. The anti-de-Sitter space with one time-like and d spacelike dimensions has the same symmetry, which facilitates the AdS/CFT correspondence. (Many conformal field theories are dual to supergravity theories on AdS spaces of one more dimension than the gauge theory.)

In the d-dimensional Euclidean space, the conformal symmetry group is $SO^+(d+1,1)$. For example, in 2 Euclidean dimensions, the conformal symmetry group is $SO^+(3,1)$, which is isomorphic to the Lorentz symmetry group in 4D. The simplest way to see that is in terms of the spin group $\mathrm{Spin}(3,1) = SL(2,\mathbf{C})$ — the group of 2×2 complex matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - bc = 1. \tag{12}$$

Each such matrix defines a fractional linear function

$$z' \;=\; \frac{az+b}{cz+d}\,, \tag{13}$$

which is a conformal map of the complex sphere $\mathbf{C}^*$ onto itself. Conversely, all one-to-one conformal maps of the complex sphere are meromorphic functions with a single pole, so they have to have form (13) for some $SL(2,\mathbf{C})$ matrix (12) and hence correspond to some $SO^+(3,1)$ Lorentz transform.

In two dimensions, one can define a much bigger conformal symmetry by including the higher-order meromorphic functions, which are conformal but not one-to one. The infinitesimal transformations of this bigger symmetry have form

$$\delta Z \;=\; \sum_{n=-\infty}^{+\infty} \alpha_n Z^{n+1} \;+\; O(\alpha^2), \tag{14}$$

and the corresponding generators L_n satisfy the Virasoro algebra. Any textbook on string theory will describe this algebra in painful detail.

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Now consider the supersymmetric conformal theories. Any theory that has both the conformal symmetry and the ordinary supersymmetries Q^α and $\overline{Q}^{\dot\alpha}$ must also have the *special conformal supersymmetries*

$$S_\alpha \;=\; -\tfrac{i}{2}[K_{\alpha\dot\beta},\overline{Q}^{\dot\beta}], \qquad \overline{S}^{\dot\alpha} \;=\; -\tfrac{i}{2}[K^{\dot\alpha\beta},Q_\beta]. \tag{15}$$

The supercharges form spinor multiplets of the conformal symmetry group Spin(2,4), which is isomorphic to $SU(2,2)$ — a non-compact cousin of the $SU(4)$, made of complex matrices with det $=1$ preserving an hermitian metric of signature $(++--)$. In $SU(2,2)$ terms,

$$F_A \;=\; \begin{pmatrix} Q_\alpha \\ \overline{S}^{\dot\alpha} \end{pmatrix} \text{ comprise } \mathbf{4}, \qquad \overline{F}^A \;=\; \left(\overline{Q}^{\dot\alpha}, S_\alpha\right) \text{ comprise } \bar{\mathbf{4}}, \tag{16}$$

while the generators of the conformal symmetry comprise a hermitian traceless 4×4 matrix

$$J_A^{\ B} \;=\; \begin{pmatrix} J_\alpha^{\ \beta} + \frac12\delta_\alpha^\beta D & P_{\alpha\dot\beta} \\ K^{\dot\alpha\beta} & J^{\dot\alpha}_{\ \dot\beta} - \frac12\delta^{\dot\alpha}_{\dot\beta} D \end{pmatrix}, \qquad J_A^{\ A} \;=\; 0. \tag{17}$$

The commutators between these generators and the supercharges (16) follow from the $SU(2,2)$ algebra,

$$\left[J_A^{\ B},F_C\right] \;=\; i\delta_C^B F_A \,-\, \tfrac{i}{4}\delta_A^B F_C\,, \qquad \left[J_A^{\ B},\overline{F}^C\right] \;=\; -i\delta_A^C\overline{F}^B \,+\, \tfrac{i}{4}\delta_A^B\overline{F}^C. \tag{18}$$

In 4D terms, these commutators become

$$
\begin{aligned}
[J^{\mu\nu}, Q_\alpha] &= \tfrac{i}{2}(\sigma^\mu\bar\sigma^\nu)_\alpha^{\ \beta} Q_\beta\,,\\
[J^{\mu\nu}, S_\alpha] &= \tfrac{i}{2}(\sigma^\mu\bar\sigma^\nu)_\alpha^{\ \beta} S_\beta\,,\\
[J^{\mu\nu}, \overline{Q}^{\dot\alpha}] &= \tfrac{i}{2}(\bar\sigma^\mu\sigma^\nu)^{\dot\alpha}_{\ \dot\beta}\overline{Q}^{\dot\beta},\\
[J^{\mu\nu}, \overline{S}^{\dot\alpha}] &= \tfrac{i}{2}(\bar\sigma^\mu\sigma^\nu)^{\dot\alpha}_{\ \dot\beta}\overline{S}^{\dot\beta},
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
[D, Q_\alpha] &= -\tfrac{i}{2} Q_\alpha,\\
[D, \overline{Q}^{\dot\alpha}] &= -\tfrac{i}{2}\overline{Q}^{\dot\alpha},\\
[D, S_\alpha] &= +\tfrac{i}{2} S_\alpha\,,\\
[D, \overline{S}^{\dot\alpha}] &= +\tfrac{i}{2}\overline{S}^{\dot\alpha},
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
[P^\mu, Q_\alpha] &= [P^\mu, \overline{Q}^{\dot\alpha}] \ = \ 0,\\
[K^\mu, S_\alpha] &= [K^\mu, \overline{S}^{\dot\alpha}] \ = \ 0,
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
[K^\mu, Q_\alpha] &= i\sigma^\mu_{\alpha\dot\beta}\overline{S}^{\dot\beta},\\
[K^\mu, \overline{Q}^{\dot\alpha}] &= i\bar\sigma^{\mu\dot\alpha\beta} S_\beta\,,\\
[P^\mu, S_\alpha] &= i\sigma^\mu_{\alpha\dot\beta}\overline{Q}^{\dot\beta},\\
[P^\mu, \overline{S}^{\dot\alpha}] &= i\bar\sigma^{\mu\dot\alpha\beta} Q_\beta
\end{aligned}
\tag{22}
$$

In particular, eqs. (20) tell us that the ordinary supercharges Q_α and $\overline{Q}_{\dot\alpha}$ have scaling dimension $+\frac{1}{2}$ while the special conformal supercharges S_α and $\overline{S}_{\dot\alpha}$ have scaling dimension $-\frac{1}{2}$.

The ordinary SUSY algebra includes the anticommutation relations

$$
\{Q_\alpha, Q_\beta\} \ = \ \{\overline{Q}_{\dot\alpha}, \overline{Q}_{\dot\beta}\} \ = \ 0, \qquad \{Q_\alpha, \overline{Q}_{\dot\beta}\} \ = \ 2P_{\alpha\dot\beta}\,. \tag{23}
$$

Applying $SU(2,2)$ symmetries to both sides of these relations, we promote them to

$$
\{F_A, F_B\} \ = \ \{\overline{F}^A, \overline{F}^B\} \ = \ 0, \tag{24}
$$

$$
\{F_A, \overline{F}^B\} \ = \ 2J_A^{\ B} \ + \ \delta_A^B \times Z, \tag{25}
$$

where Z is some kind of a central charge — a bosonic operator which commutes with the whole conformal algebra $SO(2,4)$ but does not commute with the supercharges F_A and $\overline{F}^A$.

To work out the commutator $[Z, F_A]$, let's use the Jacobi identity

$$\big[\{F_A, \overline{F}^B\}, F_C\big] \;+\; (A \leftrightarrow C) \;=\; \big[\overline{F}^B, \{F_A, F_C\}\big] \;=\; 0. \tag{26}$$

On the left hand side here,

$$\big[\{F_A, \overline{F}^B\}, F_C\big] \;=\; \big[(2J_A{}^B + \delta_A^B Z), F_C\big] \;=\; 2i\delta_C^B F_A \;-\; \tfrac{i}{2}\delta_A^B F_C \;+\; \delta_A^B [Z, F_C]. \tag{27}$$

hence

$$0 \;=\; 2i\delta_C^B F_A - \tfrac{i}{2}\delta_A^B F_C + \delta_A^B [Z, F_C] + (A \leftrightarrow C) \;=\; \delta_A^B \big(\tfrac{3}{2} i F_C + [Z, F_C]\big) + \delta_C^B \big(\tfrac{3}{2} i F_A + [Z, F_A]\big) \tag{28}$$

and therefore

$$[Z, F_A] \;=\; -\tfrac{3}{2} i F_A\,. \tag{29}$$

Similarly,

$$[Z, \overline{F}^A] \;=\; +\tfrac{3}{2} i \overline{F}^A. \tag{30}$$

Physically, eqs. (29) and (30) mean the supercharges Q_α and $\overline{S}^{\dot\alpha}$ have central charges $Z = +\frac{3}{2}$ while the supercharges $\overline{Q}^{\dot\alpha}$ and S_α have central charges $Z = -\frac{3}{2}$. In other words,

$$Z \;=\; \tfrac{3}{2} R \tag{31}$$

and the central charge generates the R–symmetry of a superconformal theory. Note that in non-conformal supersymmetric theories, the R–symmetry is optional: in some theories, the interactions respect the R-symmetry, in other theories they don't. However, in the conformal supersymmetric theories, the R–charge is a part of the superconformal algebra, without it the algebra would not close. So if a theory has both SUSY and conformal symmetry, it must also have an R–symmetry. It does not have to be a pure–R symmetry under which all scalar are neutral. Instead, it could be an anomaly-free combination of the pure-R symmetry and the axial symmetry of a SQCD-like theory, or with some other global symmetry of a more general theory. Generically, the R-charge of the superconformal algebra generates a global $U(1)$ symmetry which gives different phases to bosons and fermions,

$$A^\mu \;\to\; A^\mu, \qquad \lambda^\alpha \;\to e^{+i\rho}\lambda^\alpha, \qquad \phi_i \;\to\; e^{ir_i\rho}\phi_i, \qquad \psi_i^\alpha \;\to\; e^{i(r_i-1)\rho}\psi_i^\alpha\,. \tag{32}$$

In an exactly superconformal theory, this R-symmetry must be exact. But suppose a supersymmetric theory is not conformal at high energies, although at low energies the RG flows to a non-trivial fixed point. In the deep IR, all the irrelevant operators become negligible, so the effective theory without them becomes conformally invariant end hence superconformal. The effective IR theory must have an exact R-symmetry, but we have more options at higher energies: The UV theory may have R-symmetry-breaking interactions, as long as all such interactions become irrelevant in the deep IR limit.

For the future reference, let me spell out the anticommutation relations (24) and (25) in conventional 4D notations:

$$\{Q_\alpha, Q_\beta\} \;=\; \{Q_\alpha, \overline{S}^{\dot\beta}\} \;=\; \{\overline{S}^{\dot\alpha}, \overline{S}^{\dot\beta}\} \;=\; 0, \tag{33}$$

$$\{\overline{Q}_{\dot\alpha}, \overline{Q}_{\dot\beta}\} \;=\; \{S^\alpha, \overline{Q}_{\dot\beta}\} \;=\; \{S^\alpha, S^\beta\} \;=\; 0, \tag{34}$$

$$\{Q_\alpha, \overline{Q}_{\dot\beta}\} \;= 2P_{\alpha\dot\beta}\,, \tag{35}$$

$$\{\overline{S}^{\dot\alpha}, S^\beta\} \;= 2K^{\dot\alpha\beta}\,, \tag{36}$$

$$\{Q_\alpha, S^\beta\} \;=\; (\sigma^\mu\bar\sigma^\nu)_\alpha^{\ \beta}\times J_{\mu\nu} \;+\; \delta_\alpha^\beta\times\left(\tfrac{3}{2}R\,+\,D\right). \tag{37}$$

$$\{\overline{S}^{\dot\beta}, \overline{Q}_{\dot\alpha}\} \;=\; (\bar\sigma^\mu\sigma^\nu)^{\dot\beta}_{\ \dot\alpha}\times J_{\mu\nu} \;+\; \delta^{\dot\beta}_{\dot\alpha}\times\left(\tfrac{3}{2}R\,-\,D\right). \tag{38}$$

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The superconformal symmetries generated by the $J_A^{\ B}$, R, F_A, and $\overline{F}^A$ form a super-group called $PSU\big(2,2\,\big|\,1\big)$. Mathematically, it's a group of pseudo-unitary matrices

$$\exp\begin{pmatrix} J_A^{\ B} & \overline{F}^B \\ F_A & \frac{3}{2}R \end{pmatrix} \tag{39}$$

transforming four complex bosons (two timelike and two spacelike) and one complex fermion into each other. The bosonic part of this super-group is $SU(2,2)_{\text{conformal}} \times U(1)_R$ while the fermionic part is SUSY (ordinary plus conformal).

When an extended $\mathcal{N} = 2$ or $\mathcal{N} = 4$ supersymmetry is combined with conformal invariance (automatic for the $\mathcal{N} = 4$ SYM theories), we get a larger superconformal group $PSU\big(2,2 \,\big|\, \mathcal{N}\big)$. This time we have pseudounitary matrices

$$\exp \begin{pmatrix} J_A^{\ B} & \overline{F}_i^B \\ F_A^i & \frac{3}{2} R^i_{\ j} \end{pmatrix} \tag{40}$$

acting on 4 complex bosons and $\mathcal{N}$ complex fermions. The bosonic part of this symmetry is a direct product of the conformal $SU(2,2)$ symmetry and the extended R-symmetry $U(\mathcal{N})_R$ while the fermionic part comprises $\mathcal{N}$ ordinary supersymmetries $\mathcal{N}$ conformal supersymmetries. The fermionic anticommutation relations of the extended superconformal algebra are

$$\{F_A^i, F_B^j\} \;=\; 0, \qquad \{\overline{F}_i^A, \overline{F}_j^B\} \;=\; 0, \qquad \{F_A^i, \overline{F}_j^B\} \;=\; \delta_j^i \times J_A^{\ B} \;+\; \tfrac{3}{2}\delta_A^B \times R^i_{\ j}\,. \tag{41}$$