1. First of all, refresh your memory of basic SUSY. And if you have not finished reading about it during the summer, please do it now.

   Warning: to follow this class, you absolutely have to be familiar with at least the first 3 items on the required self-study list: (1) The SUSY algebra, the supermultiplets, and the $\mathcal{N}=1$ superspace in 4 dimensions; (2) The chiral superfields and their interactions, the superpotential, and the Kähler function. (3) The abelian vector superfields and SQED. The remaining items can wait a bit, but these three are needed from day one: If you do not grok the superfield formalism, you are SOL.

2. Second, an exercise in D-operator algebra.

   (a) Calculate the commutators $[D_\alpha, \overline{D}^2]$, $[\overline{D}_{\dot\alpha}, D^2]$, and $[D^2, \overline{D}^2]$.

   (b) Show that $D^\alpha \overline{D}^2 D_\alpha = \overline{D}_{\dot\alpha} D^2 \overline{D}^{\dot\alpha}$.

   (c) Show that $2D^\alpha \overline{D}^2 D_\alpha - D^2 \overline{D}^2 - \overline{D}^2 D^2 = 16\partial^2$.

   (d) Now, let's define 3 projector operators,

   $$\Pi_C = \frac{-1}{16\partial^2}\,\overline{D}^2 D^2, \qquad \Pi_A = \frac{-1}{16\partial^2}\,D^2\overline{D}^2, \qquad \Pi_L = \frac{+1}{8\partial^2}\,D^\alpha\overline{D}^2 D_\alpha. \tag{1}$$

   Verify that they are indeed projectors, *i.e.* satisfy $\Pi_C^2 = \Pi_C$, $\Pi_A^2 = \Pi_A$, and $\Pi_L^2 = \Pi_L$. Also, verify that they all commute with each other and add up to 1. Hint: Use $DDD = 0$ (for any indices of the 3 D operators) and likewise $\overline{DDD} = 0$.

   These 3 projector operators correspond to the three kinds of linearly-constrained superfields: A chiral superfield Φ satisfies $\overline{D}_{\dot\alpha}\Phi = 0$, and anti-chiral superfield $\overline{\Phi}$ satisfies $D_\alpha\overline{\Phi} = 0$, and a linear superfield L satisfies $D^2 L = \overline{D}^2 L = 0$.

   (e) Verify the correspondence: show that for any superfield $X(x,\theta,\bar\theta)$, $\Pi_C X$ is chiral, $\Pi_A X$ is anti-chiral, $\Pi_L X$ is linear, and also that $\Pi_C\Phi = \Phi$ for any chiral superfield Φ, $\Pi_A\overline{\Phi} = \overline{\Phi}$ for any anti-chiral $\overline{\Phi}$, and $\Pi_L L = L$ for any linear superfield L.

3. Finally, consider a free massive vector superfield: a real general superfield $V(x, \theta, \bar{\theta})$ with action

$$S = \int d^4x\, d^2\theta\, d^2\bar{\theta}\, V(m^2 + \tfrac{1}{8} D^\alpha \overline{D}^2 D_\alpha)V. \tag{2}$$

Off-shell, $V(x, \theta, \bar{\theta})$ is not subject to any constraints except $V^\dagger = V$, but on-shell it satisfies the equation of motion

$$(m^2 + \tfrac{1}{8} D^\alpha \overline{D}^2 D_\alpha)V = 0. \tag{3}$$

(a) Show that eq. (3) implies the Klein–Gordon equation $(m^2 + \partial^2)V = 0$ as well as $D^2V = \overline{D}^2 V = 0$. Consequently, *on-shell* V is a linear superfield.

(b) Expand $V(x, \theta, \bar{\theta})$ into component fields. Focus on the bosonic components C, f, f^*, A^μ, and $\mathcal{D}$ and show that $D^2V = \overline{D}^2 V = 0$ implies $f = f^* = 0$, $\mathcal{D} = -\partial^2 C$ (and hence $\mathcal{D} = m^2 C$), and $\partial_\mu A^\mu = 0$. Consequently, the independent bosonic fields comprise a massive vector field $A^\mu(x)$ and a single real scalar $C(x)$.

(c) Now focus on the fermionic component fields χ_α, λ_α and their conjugates $\bar{\chi}_{\dot{\alpha}}$ and $\bar{\lambda}_{\dot{\alpha}}$ and show that they satisfy the Weyl equations. This is equivalent to a Dirac spinor field Ψ and its conjugate $\overline{\Psi}$, each satisfying the appropriate Dirac equation.