1. First, a couple of reading assignments, or rather catch-up-on-your-summer-reading assignments.

   (a) The non-abelian vector superfield. Make sure you understand how various superfields — Φ, $\overline{\Phi}$, V, W^α, and $\overline{W}^{\dot\alpha}$ — transform under the non-abelian gauge symmetries and why they should transform that way.

   (b) The supersymmetric sigma models and the basics of Kähler geometry: the metric and its relation to the Kähler function, the Christoffel symbols, and the curvature tensor $R_{\bar\imath j\bar k\ell}$. Make sure you understand how

$$\mathcal{L} \;=\; \int d^4\theta\, K(\Phi^1,\ldots,\Phi^n,\overline{\Phi}^1,\ldots,\overline{\Phi}^n) \tag{1}$$

   expands to

$$\begin{aligned}\mathcal{L} \;=\; & g_{\bar\imath j}(\phi,\bar\phi)\times\partial^\mu\bar\phi^{\bar\imath}\partial_\mu\phi^j \;+\; g_{\bar\imath j}(\phi,\bar\phi)\times\tfrac{i}{2}\left(\bar\psi^{\bar\imath}_{\dot\alpha}\bar\sigma^{\mu\,\dot\alpha\alpha}\overset{\leftrightarrow}{\partial}_\mu\,\psi^j_\alpha\right)\\ &+\;\tfrac12\bar\Psi^{\bar\imath}_{\dot\alpha}\bar\sigma^{\mu\,\dot\alpha\alpha}\psi^j_\alpha\times\left(i\{j,\bar\imath,k\}(\phi,\bar\phi)\partial_\mu\phi_k\;-\;i\{\bar\imath,j,\bar k\}(\phi,\bar\phi)\partial_\mu\bar\phi^{\bar k}\right)\\ &+\;R_{\bar\imath j\bar k\ell}(\phi,\bar\phi)\times\bar\psi^{\dot\alpha}_{\bar\imath}\bar\psi^{\dot\alpha\bar k}\,\psi^{\alpha j}\psi^\ell_\alpha\,.\end{aligned} \tag{2}$$

2. Now consider the SUSY gauge theory with $G=SU(2)$ and two doublets of chiral superfields. Assume the scalar fields have non-zero VEVs

$$\langle A\rangle \;=\; \begin{pmatrix}h\\0\end{pmatrix},\quad \langle B\rangle \;=\; \begin{pmatrix}0\\h\end{pmatrix},\quad h\neq 0, \tag{3}$$

   and the gauge symmetry is Higgsed down to nothing.

   (a) Impose a superfield unitary gauge, show that in that gauge

$$\mathcal{L} \;=\; \frac{-i\tau}{8\pi}\int d^2\theta\;\mathrm{tr}(W^\alpha W_\alpha)+\frac{+i\tau^*}{8\pi}\int d^2\bar\theta\;\mathrm{tr}(\overline{W}_{\dot\alpha}\overline{W}^{\dot\alpha})+\int d^4\theta\,\overline{\Phi}\Phi\times\mathrm{tr}\left(e^{2V}\right), \tag{4}$$

   and argue that this gives equal masses to the three vector superfields $V^{1,2,3}$.

(b) The reason all vector masses are equal is the un-broken global $SU(2)$ symmetry. Identify the action of this symmetry, show that the scalar VEVs break

$$SU(2)_{\rm global} \times SU(2)_{\rm gauge} \;\to\; SU(2)_{\rm global}\,. \tag{5}$$

and that the 3 massive vector superfield form a triplet of the un-broken $SU(2)$ — that's why their masses have to be equal.

(c) Now go to the Wess–Zumino gauge, expand the Lagrangian

$$\mathcal{L} \;=\; \frac{-i\tau}{8\pi}\int d^2\theta\ {\rm tr}(W^\alpha W_\alpha) \;+\; \frac{i\tau^*}{8\pi}\int d^2\bar\theta\ {\rm tr}(\overline{W}_{\dot\alpha}\overline{W}^{\dot\alpha}) \;+\int d^4\theta\left(\overline{A}e^{2V}A + \overline{B}e^{2V}B\right) \tag{6}$$

into component fields, and explain how all the massive components get their masses.

3. Finally, consider the $SU(2)$ SUSY gauge theory with a triplet $\Phi = (\Phi^1, \Phi^2, \Phi^3)$ of chiral superfields. This triplet may be written in a matrix form as a traceless 2×2 matrix $\Phi = \sum_a \Phi^a(y,\theta)\times\frac{\tau^a}{2}$.

(a) Explain how the matrices Φ and $\overline{\Phi}$ transform under the superfield gauge symmetries and show that the Lagrangian

$$\mathcal{L} \;=\; \mathcal{L}_{\rm SYM} \;+\int d^4\theta\ {\rm tr}\left(\overline{\Phi}e^{+2V}\Phi e^{-2V}\right) \tag{7}$$

is gauge invariant.

(b) Calculate the scalar potential for this theory and show that it vanishes iff $[\phi^\dagger,\phi]=0$, or in $SO(3)$ notations, iff $\vec\phi^*\times\vec\phi = 0$.

(c) Argue that while generic $\langle\phi\rangle$ and $\langle\phi^\dagger\rangle$ would break the $SU(2)$ symmetry down to nothing, along the flat directions of the potential there is an unbroken $U(1)$ subgroup of the $SU(2)$.

(d) Count the fields to show that the theory has one complex modulus. Also, show that there is only one holomorphic gauge invariant combination of the chiral fields, namely $U = {\rm tr}(\Phi^2) = \frac{1}{2}\sum_a(\Phi^a)^2$.