1. Consider supersymmetric QCD with N_c colors and $N_f < N_c$ flavors. In matrix notations, the quark chiral superfields $A_i^{\,f}(y,\theta)$ form an $N_c \times N_f$ matrix A while the antiquark chiral superfields $B_f^{\,i}(y,\theta)$ form an $N_f \times N_c$ matrix B. Let all the flavors be exactly massless, so the Lagrangian is

$$\mathcal{L} \;=\; \frac{f}{2}\int d^2\theta\; \mathrm{tr}(W^\alpha W_\alpha) \;+\; \mathrm{H.c.} \;+\int d^4\theta\; \mathrm{tr}\left(\overline{A}\, e^{+2V} A \;+\; B\, e^{-2V}\overline{B}\right). \tag{1}$$

(a) Show that the classical scalar potential of the theory has form

$$V_{\text{scalar}} \;=\; \frac{g^2}{8}\sum_{a=1}^{N^2-1}\left[\mathrm{tr}\left(\lambda^a\left(AA^\dagger - B^\dagger B\right)\right)\right]^2 \tag{2}$$

where $g^2 = 1/Ref$.

(b) Show that this potential vanishes if and only if

$$AA^\dagger \;-\; B^\dagger B \;=\; c\times \mathbf{1}_{\mathrm{N_c}\times\mathrm{N_c}} \tag{3}$$

for some real number c. Also show that for $N_f < N_c$ this matrix relation implies $c = 0$ and hence

$$AA^\dagger \;=\; B^\dagger B. \tag{4}$$

⋆ **Lemma:** Any complex square matrix X can be written as a product of two unitary matrices U_1, U_2 and a real diagonal matrix D with non-negative eigenvalues.

$$X \;=\; U_1 D U_2\,. \tag{5}$$

Proving this lemma is an optional exercise for the students.

(c) Generalize this lemma to the rectangular matrices such as A and B. Then show that the matrices A and B which obey eq. (4) may be written as

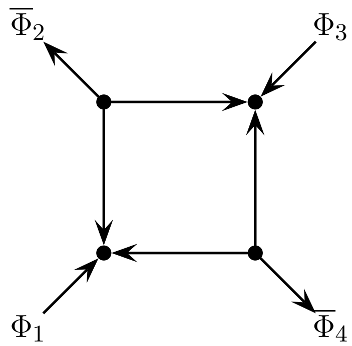
$$A \;=\; U_C \times \left(\begin{array}{c} \mathbf{D}_{N_f\times N_f} \\ \hline \mathbf{0}_{(N_c-N_f)\times N_f} \end{array}\right) \times V_A\,, \qquad B \;=\; V_B \times \left(\mathbf{D}_{N_f\times N_f} \,\middle|\, \mathbf{0}_{N_F\times(N_c-N_f)}\right) \times U_C^{-1} \tag{6}$$

where U_C is an $SU(N_c)$ matrix (same gauge symmetry for A and B), V_A and V_B are $N_F \times N_F$ unitary matrices, and $\mathbf{D}$ is a real ≥ 0 diagonal $N_f \times N_F$ matrix, same $\mathbf{D} = \mathrm{diag}(d_1, \ldots d_{N_f})$ for both A and B.

(d) The independent holomorphic moduli of the flat directions form an $N_f \times N_f$ matrix $\mathcal{M} = BA$. (We assume $N_f < N_c$.) Use eqs. (5) and (6) to show that any given complex $N_f \times N_f$ matrix $\mathcal{M}$ can be written as a product BA of rectangular matrices obeying eqs. (4). Also, argue that any two such decompositions $\mathcal{M} = BA = B'A'$ of the same matrix $\mathcal{M}$ are gauge-equivalent to each other, $A' = UA$, $B' = AU^{-1}$ for the same $U \in SU(N_c)$.

In other words, there is a one-to-one correspondence between the classical moduli space of SQCD and the space of complex $N_f \times N_f$ matrices $\mathcal{M}$.

2. Next, an exercise in superfield Feynman rules. Evaluate the 1-loop diagram



(7)

(a) Dress the graph with the fermionic derivatives $-\frac{1}{4}D^2$ and $-\frac{1}{4}\overline{D}^2$ for the propagators, then eliminate some of these derivatives according to the vertices. Then show that after integrating $\int d^4\theta$ for 3 out of 4 vertices of the graph, the Feynman amplitude takes form

$$
\begin{aligned}
i\mathcal{M} &= \int \frac{d^4q_4}{(2\pi)^4}\,\frac{y^2y^{*2}}{(q_1^2+i0)(q_2^2+i0)(q_3^2+i0)(q_4^2+i0)}\;\times \\
&\quad\times\;\frac{1}{256}\int d^4\theta_4\,\Phi_1(\theta_4)\overline{\Phi}_2(\theta_4)D^2\overline{D}^2\Phi_3(\theta_4)\overline{\Phi}_4(\theta_4)D^2\overline{D}^2\,\delta^{(4)}(\theta_4-\theta_1)\Big|_{\theta_1=\theta_4}\,.
\end{aligned}
\tag{8}
$$

Note: The derivatives in this formula are WRT θ_4, and each derivative acts on everything to its right.

(b) Work out the action of derivatives on the $\delta^{(4)}(\theta_4-\theta_1)$ factor and show that the integral on the second line of the amplitude (8) evaluates to

$$
\int d^4\theta\,\Phi_1\overline{\Phi}_2\times\left(\frac{1}{16}\,D^2\bigl(\Phi_3\overline{D}^2\overline{\Phi}_4\bigr)\;+\;\frac{(q_4^\mu\bar{\sigma}_\mu)^{\dot\alpha\alpha}}{2}\times D_\alpha\bigl(\Phi_3\overline{D}_{\dot\alpha}\overline{\Phi}_4\bigr)\;+\;(q_4^2)\times\Phi_3\overline{\Phi}_4\right) \tag{9}
$$

where all the superfields depend on the same superspace coordinates θ^α and $\bar{\theta}^{\dot\alpha}$, and q_4^μ is the momentum carried by the bottom propagator in the loop.

This exercise shows that the momentum-integral part of a superfield Feynman graph can have both numerator and denominator factors. The denominators come from the propagators, while the numerators follow from the D-derivative algebra in the superfield part of the amplitude.

3. Now, let's count the fermionic derivatives and the powers of loop momenta in a generic superfield Feynman graph. For simplicity, let's focus on a Wess–Zumino-like model of one chiral superfield Φ with a generic polynomial superpotential of degree $n>3$.

$$
\begin{aligned}
\mathcal{L} &= \int d^4\theta\,\overline{\Phi}\Phi\;+\;\int d^2\theta\,W(\Phi)\;+\;\int d^2\bar{\theta}\,W^*(\overline{\Phi}),\\
W &= \frac{m}{2}\,\Phi^2\;+\;\frac{y}{6}\,\Phi^3\;+\;\cdots\;+\;\frac{c_n}{n!}\,\Phi^n.
\end{aligned}
\tag{10}
$$

(a) Write down the superfield Feynman rules (the vertices and the propagators) for this theory.

Now consider a generic Feynman graph with L loops, V vertices (of all kinds), P propagators of the $\overline{\Phi}\Phi$ type, and P' propagators of the $\Phi\Phi$ or $\overline{\Phi\Phi}$ types.

(b) Count the fermionic derivative operators D^α and $\overline{D}^{\dot\alpha}$ for the graph *before* you do the Grassmannian integrals and use up some derivatives to close the loops via $D^2\overline{D}^2\delta^{(4)}\,(\theta_1-\theta_2)|_{\theta_1=\theta_2}=16$, *etc.*. Show that

$$\text{net}\quad \#(D^\alpha)\ +\ \#(\overline{D}^{\dot\alpha})\ =\ 2L\ +\ 2P\ -\ 2. \tag{11}$$

(c) Now let's do the Grassmannian integrals $\int d^4\theta_2\ \cdots \int d^4\theta_V$. This process soaks up some of the fermionic derivatives, while the remaining derivatives either end up acting on the external-leg fields in the remaining $\int d^4\theta_1$ integral, or else they may produce powers of the loop momenta in the numerator via the anticommutators $\{D_\alpha,\overline{D}_{\dot\alpha}\}=2\sigma^\mu_{\alpha\dot\alpha}q_\mu$.

Calculate the maximal power of loop momenta in the numerator that may result from this process.

(d) Calculate the superficial degree of divergence Δ of the momentum integral of the Feynman graph.

(e) Finally, show that

$$\Delta\ \leq\ 2\ -\ E\ -\ P'\ +\sum_{k=3}^{n}(k-3)V_k \tag{12}$$

where E is the number of the graph's external legs and V_k is the number of vertices with k legs. Then use eq. (12) to argue that the WZ model with a cubic superpotential is renormalizable while theories with higher-order $n\geq 4$ superpotentials are not renormalizable.

4. The last problem is about supersymmetric QED,

$$\mathcal{L}\ =\int d^4\theta\left(\overline{A}e^{+2eV}A\ +\ \overline{B}e^{-2eV}B\ +\ \tfrac{1}{8}VD^\alpha\overline{D}^2D_\alpha V\right)+\int d^2\theta\,mAB\ \int d^2\bar\theta\,m^*\overline{AB}. \tag{13}$$

The superfield Feynman rules for SQED will be explained in class next week. For now, please take them for granted:

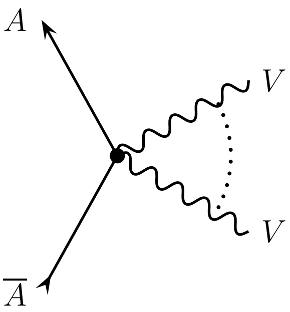
- Chiral propagators:

$$
\begin{aligned}
\overline{A} \succ\!\!\!\longrightarrow A \;&=\; \frac{i}{p^2 - mm^* + i0} \times \frac{\overline{D}^2 D^2}{16}\,\delta^{(4)}(\theta_1 - \theta_2),\\
\overline{A} \succ\!\!\!\longrightarrow\!\!\prec \overline{B} \;&=\; \frac{i}{p^2 - mm^* + i0} \times \frac{m\overline{D}^2}{4}\,\delta^{(4)}(\theta_1 - \theta_2),\\
B \longleftarrow\!\!\!\longrightarrow A \;&=\; \frac{i}{p^2 - mm^* + i0} \times \frac{m^* D^2}{4}\,\delta^{(4)}(\theta_1 - \theta_2),\\
B \longleftarrow\!\!\prec \overline{B} \;&=\; \frac{i}{p^2 - mm^* + i0} \times \frac{D^2\overline{D}^2}{16}\,\delta^{(4)}(\theta_1 - \theta_2),
\end{aligned}
\tag{14}
$$

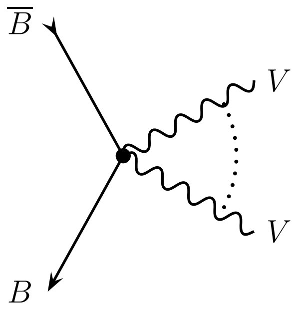
- Vector propagator in the Feynman gauge:

$$
V \sim\!\sim\!\sim\!\sim\!\sim V \;=\; \frac{i}{k^2 + i0} \times \delta^{(4)}(\theta_1 - \theta_2). \tag{15}
$$

- Vertices: One incoming chiral line, one outgoing chiral line of the same species, any number $n = 1, 2, 3, \ldots$ of vector lines,



$= i(+2e)^n,$



$= i(-2e)^n,$ (16)

without any superderivative factors in the numerator or denominator.

Count the superderivatives and powers of momenta in a general Feynman diagram and show that a diagram with E_C external legs of chiral superfields (A, B, $\overline{A}$, or $\overline{B}$), E_V external legs of vectors, and any numbers of loops, vertices, and internal lines has superficial degree of divergence

$$
\Delta \;\leq\; 2 \;-\; E_C\,. \tag{17}
$$

In class, I shall use this formula to prove that SQED is renormalizable.