This homework is about Ward–Takahashi identities in supersymmetric QED,

$$\mathcal{L} \;=\; \int d^4\theta \left(\overline{A} e^{+2V} A \;+\; \overline{B} e^{-2V} B \;+\; \frac{1}{8g^2}\, V D^\alpha \overline{D}^2 D_\alpha V \right). \tag{1}$$

For simplicity I take the charged chiral superfields A and B to be massless. This is not important for the Ward identities themselves — they hold just as well for the massive charged fields — but it simplifies the proofs.

Note notations: in the following, Φ stands for either A or B and $\overline{\Phi}$ for the *corresponding* $\overline{A}$ or $\overline{B}$; $q = \pm 1$ is the electric charge of the chiral field in question, $q = +1$ for the A and $q = -1$ for the B. The vector field are normalized non-canonically, $V = gV_{\rm can}$. Consequently, the vector propagators carry a factor $(g^2/2)$ while the vertices do not carry power of g (vertex $= i(2q)^n$).

1. Consider the correlation function of a charged chiral field Φ and its conjugate $\overline{\Phi}$ in the background of a vector supefiled $V(x,\theta)$,

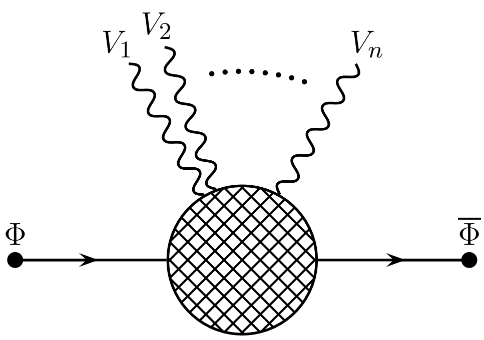
$$\langle\Omega\rangle\, \mathbf{T}\Phi(p,\theta)\overline{\Phi}(-p',\theta')\,|\Omega\rangle \;=\; \mathcal{S}[V]\delta^{(4)}(\theta-\theta') \tag{2}$$

where $\mathcal{S}$ is an operator involving spinor derivatives D and $\overline{D}$, vector superfield V, and functions of momenta. Expanding it in powers of the vector superfield, we have

$$\mathcal{S} \;=\; \sum_{n=0}^{\infty} (2q)^n \mathcal{S}_n(V_1,\ldots,V_n) \tag{3}$$

where the $\mathcal{S}_n$ may be obtained diagrammatically as Feynman amplitudes with appropriate external legs — a Φ, a $\overline{\Phi}$, and n vectors $V_1,\ldots,V_n$,

$$(2q)^n \mathcal{S}_n(V_1,\ldots,V_n) \;=\; \tag{4}$$



To be precise, these amplitudes are amputated with respect to the vector fields $V_1,\ldots,V_n$ but not the charged fields Φ and $\overline{\Phi}$; in other words, they include the external lines for the

Φ and $\overline{\Phi}$ but not for the vectors. On the other hand, these amplitudes include the external vector fields V_i themselves but not the external Φ or $\overline{\Phi}$, and there is no overall $\int d^4\theta$, just the operator between the Φ and the $\overline{\Phi}$,

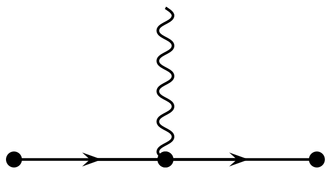
$$(2q)^n \int d^4\theta\, \Phi \times \boxed{\mathcal{S}_n(V_1,\ldots,V_n)} \times \overline{\Phi} \quad \leftarrow \text{This part only!} \tag{5}$$

For example, at the tree level

$$\mathcal{S}_0^{\text{tree}} =$$

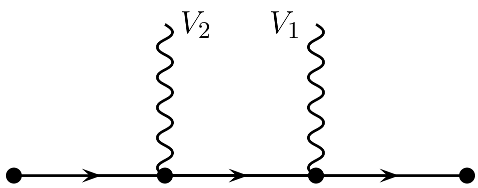
$$= \frac{iD^2\overline{D}^2}{16p^2},$$

$$2q \times \mathcal{S}_1^{\text{tree}}(V_1) =$$



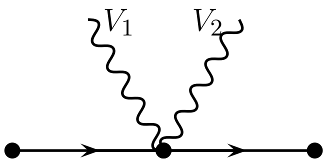
$$= \frac{iD^2\overline{D}^2}{16p_1^2} \times (2iqV_1) \times \frac{iD^2\overline{D}^2}{16p_2^2},$$

$$(2q)^2 \times \mathcal{S}_2^{\text{tree}}(V_1,V_2) =$$



$$+ \big(V_1 \leftrightarrow V_2\big)$$

$$+$$



$$= \frac{iD^2\overline{D}^2}{16p_1^2} \times (2iqV_2) \times \frac{iD^2\overline{D}^2}{16p_2^2} \times (2iqV_1) \times \frac{iD^2\overline{D}^2}{16p_3^2} + \big(V_1 \leftrightarrow V_2\big)$$
$$+ \frac{iD^2\overline{D}^2}{16p_1^2} \times 4q^2 iV_2V_1 \times \frac{iD^2\overline{D}^2}{16p_2^2},$$

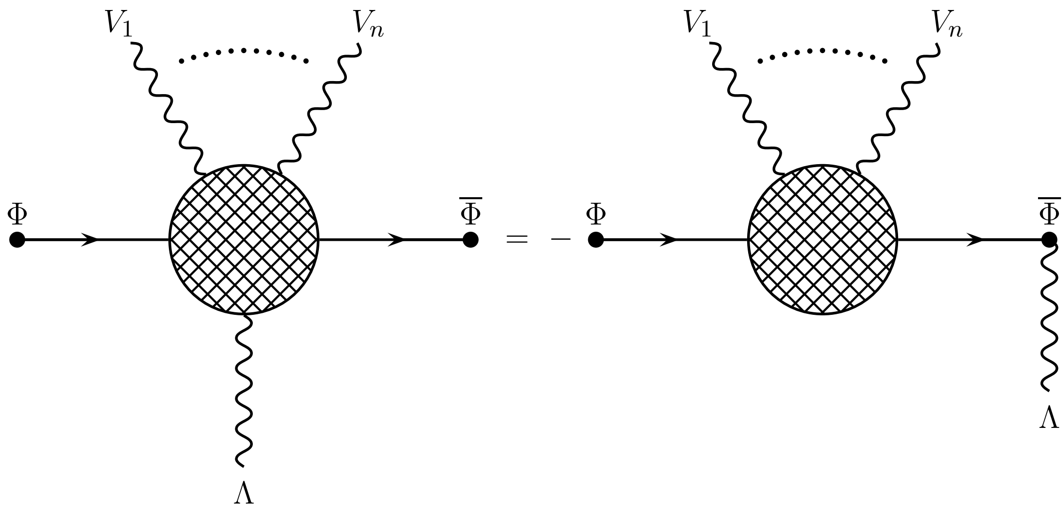
etc., etc. (6)

Note: the propagators and the vertices are spelled in the order of the line from $\overline{\Phi}$ on the right and Φ on the left because of the way $\mathcal{S}$ is sandwiched between the $\overline{\Phi}$ and Φ.

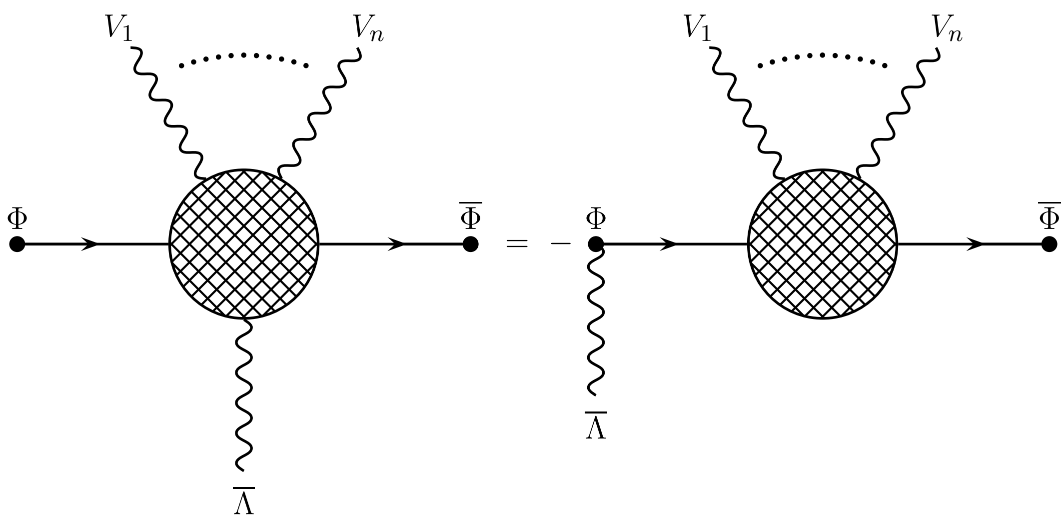
Your task is to show that **if** any of the vector fields happen to be chiral or antichiral, $V_i = \Lambda(y, \theta)$ or $V_i = \overline{\Lambda}(\bar{y}, \bar{\theta})$, **then**

$$\begin{aligned}\mathcal{S}_{n+1}(V_1, \ldots, V_i = \Lambda, \ldots, V_{n+1}) &= -\mathcal{S}_n(V_1, \ldots \cancel{V_i} \ldots, V_{n+1}) \times \Lambda, \\ \mathcal{S}_{n+1}(V_1, \ldots, V_i = \overline{\Lambda}, \ldots, V_{n+1}) &= -\overline{\Lambda} \times \mathcal{S}_n(V_1, \ldots \cancel{V_i} \ldots, V_{n+1}),\end{aligned} \tag{7}$$

or graphically (suppressing powers of $2q$)



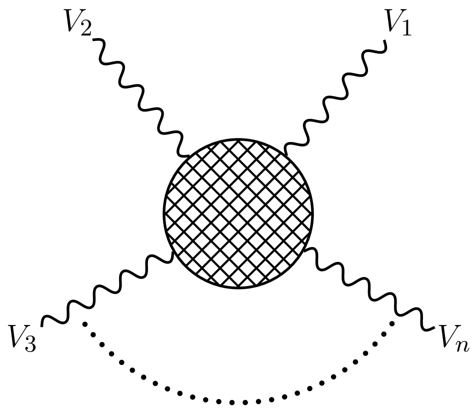
(8)



(9)

(a) Prove the relations (7) at the tree level. Note: this does not work diagram-by-diagram. Instead, you have to some over all the places the $(n+1)^{\text{st}}$ "photon" $V_{n+1} = \Lambda$ or $V_{n+1} = \overline{\Lambda}$ can be inserted into an amplitude that already has n other photons.

Now consider the n–vector amputated amplitudes without any external Φ or $\overline{\Phi}$ lines,



$$= \ i(2q)^n \int d^4\theta \, \mathcal{V}_n(V_1, \ldots, V_n). \tag{10}$$

A very important Ward–Takahashi identity says that all these amplitudes vanish when any one of the vectors V_i is chiral or antichiral,

$$\int d^4\theta \, \mathcal{V}(V_1, \ldots, V_n) \ = \ 0 \quad \text{when any } V_i = \Lambda \text{ or } V_i = \overline{\Lambda} \tag{11}$$

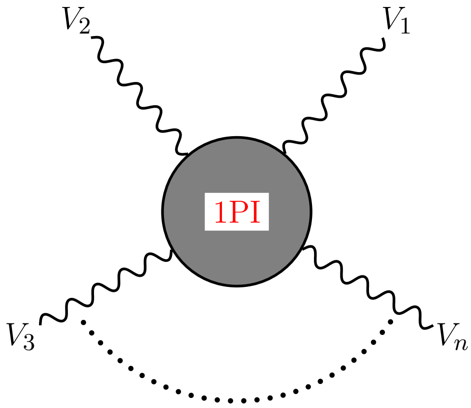
(b) Prove this identity at the one-loop level. Note: this involves cancellation between diagrams where that bad vector $V_n = \Lambda$ or $V_n = \overline{\Lambda}$ is inserted into the charged loop relative to the other $n-1$ vectors.

Assume that all the loop-momentum integrals either converge or else may be regulated in a way that does not affect the vertices or the chiral propagators. This assumption allows us to cancel diagrams graphically without worrying about shifting the loop momenta $q^\mu \to q^\mu + p^\mu$ in divergent $\int d^4q$ integrals.

(c) Finally, use (a) and (b) to prove the relations (7) and (11) to all orders of the perturbation theory.

2. Thanks to the Ward–Takahashi identities, SQED is renormalizable in superspace. In this exercise, you shall see how this works.

   Our first step is to restate the Ward Identities in terms of the one-particle-irreducible (1PI) amplitudes: The all–vector 1PI amplitudes

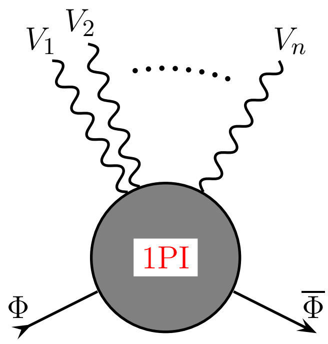


$$= i(2q)^n \int d^4\theta\, \mathcal{V}_n^{\rm 1PI}(V_1,\ldots,V_n) \tag{12}$$

   vanish when any of the vectros superfields is chiral or antichiral,

$$\int d^4\theta\, \mathcal{V}_n^{\rm 1PI}(V_1,\ldots,V_n) \;\to\; 0 \quad \text{when any } V_i=\Lambda \text{ or } V_i=\overline{\Lambda}, \tag{13}$$

   while the two-scalars-plus–n–vectors 1PI amplitudes



$$= i(2q)^n \int d^4\theta\, \Phi\, \Gamma_n(V_1,\ldots,V_n)\, \overline{\Phi} \tag{14}$$

   obey recursive relations: for $n>1$ we have

$$\Gamma_1(V=\Lambda) \;=\; \Lambda\times(1+\Gamma_0), \qquad \Gamma_1(V=\overline{\Lambda}) \;=\; (1+\Gamma_0)\times\overline{\Lambda}, \tag{15}$$

   while for $n>1$

$$\begin{aligned} \Gamma_n(V_1,\ldots,V_{n-1},V_n=\Lambda) &= \Lambda\times\Gamma_{n-1}(V_1,\ldots,V_{n-1}),\\ \Gamma_n(V_1,\ldots,V_{n-1},V_n=\overline{\Lambda}) &= \Gamma_{n-1}(V_1,\ldots,V_{n-1})\times\overline{\Lambda}. \end{aligned} \tag{16}$$

Note that the $1+\Gamma_0$ combination in eq. (15) is related to the *dressed* chiral propagator

$$\equiv\ \mathcal{S}_0\ =\ \frac{1}{1+\Gamma_0(p)}\times\frac{iD^2\overline{D}^2}{16p^2}\,. \tag{17}$$

(a) Use the identities (7) and (11) from problem 1 to prove the relations (13), (15), and (16). Note: (13) is trivial and (15) is easy but (16) takes work.

(b) In an earlier homework (set#3, problem 4) we saw that all the 1PI amplitudes Γ_n have logarithmic divergences (superficial degree of divergence = 0). Use eqs. (15) and (16) to show that all these divergences have exactly the same coefficient δ_Z,

$$\Gamma_n(V_1,\ldots,V_n)\ =\ \delta_Z\times V_1\cdots V_n\ +\ \text{finite},\quad \text{same } \delta_{\rm Z}\ \forall\text{n}. \tag{18}$$

Consequently,

$$\sum_{n=1}^{\infty}\frac{(2q)^n}{n!}\Gamma_n(V,\ldots,V)\ =\ \delta_Z\times\exp(2qV)\ +\ \text{finite} \tag{19}$$

and the renormalized SQED Lagrangian terms for the charged fields

$$\mathcal{L}^{\rm ren}\ \supset\ \int d^4\theta\,(1+\delta_Z)\times\Big(\overline{A}\,e^{+2V}A\ +\ \overline{B}\,e^{-2V}B\Big) \tag{20}$$

have exactly the same gauge symmetry as in the classical Lagrangian.

Now consider the 1PI amplitudes (11) for n vectors and no external charged fields. By the charge conjugation $A\leftrightarrow B$, $V\to -V$, all the amplitudes with odd n vanish, so let's consider the even n only.

(c) Use eq. (13) to show that the all-vector amplitudes $\mathcal{V}_n^{\rm 1PI}$ must involve several spinor derivatives D^α and $\overline{D}^{\dot\alpha}$. The number of such derivatives should be at least 4 for $n=2$ vectors and more than 4 for $n=4,6,8,\ldots$.

(d) Explain how these derivatives acting on the V_i superfields reduce the degree of divergence of the momentum integral. Show that that the 2-vector amplitude $\mathcal{V}_2^{\rm 1PI}$

diverges logarithmically while the n-vector 1PI amplitudes for $n > 2$ do not diverge at all. Consequently, the renormalized Lagrangian for the vector superfield is simply

$$\mathcal{L}_V^{\rm ren} \;=\; \int d^4\theta\, (g^{-2} + \delta_3) \times V\, \frac{D^\alpha \overline{D}^2 D_\alpha}{8}\, V \;\equiv\; (g^{-2} + \delta_3) \times \mathcal{L}_V^{\rm tree}. \qquad (21)$$

Note: together, eqs. (20) and (21) prove that in the superspace, SQED is renormalizable despite having an infinite number of vertex types.