1. In class I showed how to integrate out one heavy quark flavor from the low-energy effective field theory. In this problem you should generalize the formula to SQCD with several heavy flavors, and ultimately to any gauge theory with any kings of multiplets of heavy chiral superfields.

   (a) Let's start with the ordinary QCD (no SUSY) with N_c colors and N_f massive flavors. Consider a chiral re-definition of the quark and antiquark flavors; in Weyl fermion terms,

$$\psi^{fc}_\alpha \;\mapsto\; U^f{}_{f'}\psi^{cf'}_\alpha, \qquad \tilde\psi_{fc\alpha} \;\mapsto\; \tilde U_f{}^{f'}\tilde\psi_{f'c\alpha} \tag{1}$$

   for two independent unitary $N_f \times N_f$ matrices U and $\tilde U$. In general, such redefinition is not a symmetry of QCD; instead, it must be accompanied by the appropriate redefinition of the quark mass matrix m, and also by shifting the Θ angle to compensate for the axial anomaly.

   Write down the new mass matrix and the Θ angle and show that the combination

$$\overline{\Theta} \;=\; \Theta \;+\; \text{phase}\big(\det(m)\big) \tag{2}$$

   remains invariant. Note: in QCD it's the $\overline{\Theta}$ which governs the strong CP violation. The experimental limits on such violations (such as neutron's electric dipole moment) imply $|\overline{\Theta}| < 10^{-10}$

   (b) In SQCD we may generalize eqs. (1) to arbitrary linear redefinitions of quark flavors,

$$Q^{fc} \;\mapsto\; U^f{}_{f'}Q^{f'c}, \qquad \tilde Q \;\mapsto\; \tilde U_f{}^{f'}\tilde Q_{f'c} \tag{3}$$

   where the complex $N_f \times N_f$ matrices do not need to be unitary, only invertible. A non-unitary field redefinition should be accompanied by the appropriate redefinition of the kinetic energy matrices Z and $\tilde Z$ for the quarks and the antiquarks as well as the mass matrix m; also, both real and imaginary parts of the Wilsonian gauge coupling f should be adjusted to compensate for the Konishi anomaly.

Write down the transformation rules for the Z, $\tilde{Z}$, and m matrices and of the f, and show that the holomorphic combination

$$8\pi^2 \bar{f} \;=\; 8\pi^2 f \;-\; log\,\det(m) \tag{4}$$

remains invariant. Note: The imaginary part of this formula is eq. (2).

(c) Now, suppose all the quark flavors are very heavy, all eigenvalues$(m) \gg \Lambda_{\mathrm{SQCD}}$. Let's integrate them out from the low-energy effective theory — which is therefore a quark-less SYM. Use eq. (4) and holomorphy to argue that

$$\Lambda_{\mathrm{SYM}}^{3N_c} \;=\; \Lambda_{\mathrm{SQCD}}^{3N_c-N_f} \times \det(m) \times \text{a numeric constant.} \tag{5}$$

(d) Let's run the renormalization group flow to verify eq. (5) for the magnitude $|\Lambda_{\mathrm{SYM}}|$. For simplicity, suppose that the matrices m, Z, and $\tilde{Z}$ are diagonal and that the physical quark masses

$$M_f \;=\; \frac{|m_f|}{\sqrt{Z_f \tilde{Z}_f}} \tag{6}$$

have hierarchically different values, say $M_1 \gg M_2 \gg \cdots \gg M_{N_f}$. In this case, the RG flow has N_f well-separated thresholds.

Analyzing each threshold just like I did in class, show that after the last threshold we obtain $|\Lambda_{\mathrm{SYM}}|$ exactly as in eq. (5).

(e) Now consider SQCD that has both heavy and light flavors. Let's integrate out the heavy flavors only, so the low-energy effective theory includes both the gauge fields and the light flavors. Generalize the result of part (d) to this case and write down a holomorphic formula for the Λ_{low} in terms of Λ_{high} and the mass matrix for the heavy quarks.

(f) Optional exercise:

Generalize from SQCD to a SUSY gauge theory with any kind of a simple gauge group G and "quarks" and "antiquarks" in some generic multiplets $R_1 + R_2 + \cdots$

of G. Suppose that some of these "quarks" or "antiquarks" are heavy so we may integrate them out from the low-energy effective theory.

Show that the resulting low energy theory has

$$-b_{\text{low}} \times \log \Lambda_{\text{low}} \;=\; -b^{\text{high}} \times \Lambda_{\text{high}} \;+\; \sum_i \text{Index}(R_i) \times \log m_i\,, \tag{7}$$

where the sum is over heavy multiplets only, and b_{high} and b_{low} are the one-loop beta-function coefficient of the respective high-energy and low-energy gauge theories.

2. Now consider the Higgs regime of QCD. Suppose several quark flavors have large VEVs $\langle\Phi\rangle \gg \Lambda$ so we may treat the Higgs mechanism perturbatively. As we saw earlier in class, in general VEVs of N_f flavors break the $SU(N_c)$ gauge symmetry down to $SU(N_c - N_f)$, while the VEVs themselves may be parametrized in a gauge-invariant form by the $N_f \times N_f$ matrix

$$\mathcal{M}_f^{\;f'} \;=\; \left\langle \tilde{Q}_{cf} Q^{cf'} \right\rangle. \tag{8}$$

This *moduli matrix* affects the Λ_{low} of the un-broken $SU(N_c - N_f)$ gauge group according to

$$\lambda_{\text{low}}^{3(N_c-N_f)} \;=\; \frac{\Lambda_{\text{high}}^{3N_c-N_f}}{\det(\mathcal{M})} \times \text{a numeric constant.} \tag{9}$$

In class I derived this formula for $N_f = 1$, your task is to generalize it to any $N_f \leq N_c - 2$.

(a) Let's start by deriving eq. (9) from holomorphy and Konishi anomaly of a linear redefinition (3) of quark superfields. Show that the holomorphic combination

$$8\pi^2 f \;+\; \det(\mathcal{M}) \tag{10}$$

is invariant under all redefinitions, then use this formula and the holomorphy of the Λ_{low} to derive eq. (9).

(b) Now, let's verify eq. (9) by running the RG flow through several thresholds due to massive vector multiplets. For simplicity, assume diagonal VEVs of squark and anti-squark matrices in some gauge with hierarchically different eigenvalues,

$$\left\langle Q^{fc}\right\rangle \;=\; \delta^{fc}\times\phi_f \quad\text{and}\quad \left\langle \tilde{Q}_{fc}\right\rangle \;=\; \delta_{fc}\times\phi_f\,, \qquad \phi_1\gg\phi_2\gg\cdots\gg\phi_{N_f}\,, \tag{11}$$

so that the RG flow has well-separated thresholds.

Write down the physical masses of the vector superfields that become massive due to each ϕ_f, then work out the matching condition for the RG flow at each threshold just like I did it in class. Show that below the last threshold, the effective $SU(N_c-N_f)$ SYM theory has $|\Lambda_{\text{low}}|$ which agrees with eq. (9).

(c) Optional exercise:
Generalize eq. (9) to any SUSY gauge theory G Higgsed down to a subgroup G' by VEVs of chiral superfields belonging to any multiplets of G.
In general, the massive vector superfields in $G-G'$ form several multiplets of G'; let's label such multiplets by v and let $\text{Index}'(v)$ denote the index of such a multiplet WRT the unbroken G'. Show that

$$-b_{\text{low}}\times\log\Lambda_{\text{low}} \;=\; -b^{\text{high}}\times\Lambda_{\text{high}} \;-\; 2\sum_V \text{Index}'(v)\times\log\left\langle H_v\right\rangle \;+\; \text{a numeric constant} \tag{12}$$

where $\langle H_v\rangle$ is the VEV of the Higgs field that gives the vector fields in v their masses.

3. Finally, consider the non-perturbative superpotential due to gaugino condensation. Suppose we have $N_f\le N_c-2$ low-mass flavors with large VEVs, so the $SU(N_c)$ is Higgsed down to $SU(N_c-N_f)$ as in the previous problem.

(a) Show that the gaugino condensate in the un-broken $SU(N_c-N_f)$ subgroup gives rise to the non-perturbative effective superpotential for the $\mathcal{M}_f^{f'}$ moduli fields

$$W_{NP}(\mathcal{M}) \;=\; -\frac{N_c-N_f}{16\pi^2}\times\Lambda^3_{\text{low}}(\mathcal{M}) \;=\; \sqrt[N_c-N_f]{\frac{\Lambda_{\text{high}}^{3N_c-N_f}}{\det(\mathcal{M})}}\times\text{a numeric constant.} \tag{13}$$

(b) At the tree level, the mass term for the quarks give rise to a linear superpotential for the moduli, $W_{\text{tree}} = -\operatorname{tr}(m\mathcal{M})$ where m is the quark mass matrix. Adding the non-perturbative term (13) we obtain

$$W_{\text{net}}(\mathcal{M}) \;=\; -\operatorname{tr}(m\mathcal{M}) \;+\; W_{NP}(\mathcal{M}). \tag{14}$$

In a SUSY vacuum, this W_{net} should have zero derivatives with respect of all N_f^2 matrix elements of $\mathcal{M}$. Show that these conditions lead to

$$\begin{aligned} \langle\mathcal{M}\rangle\times m \;&=\; m\times\langle\mathcal{M}\rangle \;=\; \mathbf{1}_{N_f\times N_f}\times\frac{\langle S\rangle}{16\pi^2} \\ \text{where}\quad \langle S\rangle \;&=\; \Lambda^3_{\text{low}}(\langle\mathcal{M}\rangle) \;=\; \sqrt[N_c-N_f]{\frac{\Lambda_{\text{high}}^{3N_c-N_f}}{\det(\mathcal{M})}}\times\text{a numeric constant.} \end{aligned} \tag{15}$$

(c) Show that the Veneziano–Yankielowicz–Taylor effective superpotential

$$W_{\text{VYT}}(S,\mathcal{M}) \;=\; -\operatorname{tr}(m\mathcal{M}) \;+\; \frac{S}{16\pi^2}\times\left(\log\frac{S^{N_c-N_f}\times\det(\mathcal{M})}{\Lambda_{\text{high}}^{3N_c-N_f}}\;+\;\text{const}\right) \tag{16}$$

leads to the same equations (15) for the VEVs $\langle S\rangle$ and $\langle\mathcal{M}\rangle$.

(d) Assume that none of the quarks is massless, thus $\det(m)\neq 0$. Solve the equations (15) and show that they have N_c distinct solutions.

(e) Show that when some flavors' masses become very small, the corresponding squarks get very large VEVs. For simplicity, assume a diagonal mass matrix m.

(f) Finally, suppose $m=0$ and all the flavors are massless. For simplicity, assume the moduli matrix is diagonal, $\mathcal{M} = \operatorname{diag}(\phi_1^2,\phi_2^2,\ldots,\phi_{N_f}^2)$; semiclassically, this corresponds to diagonal VEVs of the squark and antisquark matrices, $\left\langle Q^{fc}\right\rangle = \phi_f\delta^{fc}$, $\left\langle\tilde{Q}_{fc}\right\rangle = \phi_f\delta_{fc}$. Assuming all the diagonal VEVs are large, all $\phi_g\gg\Lambda$, we may approximate the the Kähler function by the tree-level

$$K_{\rm tree} \;=\; 2\sum_f \phi_f^*\phi_f \tag{17}$$

which leads to the scalar potential

$$V_s \;=\; \frac{1}{2}\sum_f \left|\frac{\partial W_{\rm NP}}{\partial\phi_f}\right|^2 . \tag{18}$$

Show that this potential decreases monotonically for $\phi_f \to \infty$, so all the VEVs run away to infinity — there are no stable vacua, supersymmetric or otherwise.