1. First, a reading assignment: Gerard 't Hooft's lecture notes on *Monopoles, Instantons, and Confinement*, arXiv:hep-th/0010225. Focus on chapter 4 about the instantons and their effects on the fermions.

   This week you may skim over all the other chapters, but you should read them later when you you have time. In particular, chapter 3 about the monopoles will be a reading assignment in a future homework.

2. The rest of this homework concerns SQCD with $N_c = N$ colors and massless $N_f = N + 1$ flavors. Let's start with the classical moduli space of this theory.

   (a) Show that the classical moduli space has N_f^2 complex dimensions.

   The holomorphic gauge invariants of the quark Q^c_f and antiquark $\widetilde{Q}_{f,c}$ chiral superfields include

$$\begin{aligned} \text{mesons } \mathcal{M}_{ff'} &= \widetilde{Q}_{f,c} Q^c_{f'}\,, \\ \text{baryons } \mathcal{B}^f &= \frac{1}{N!}\epsilon^{ff_1\cdots f_N}\epsilon_{c_1\cdots c_N} Q^{c_1}_{f_1}\cdots Q^{c_N}_{f_N}\,, \\ \text{and antibaryons } \widetilde{\mathcal{B}}^f &= \frac{1}{N!}\epsilon^{ff_1\cdots f_N}\epsilon^{c_1\cdots c_N}\widetilde{Q}_{f_1,c_1}\cdots\widetilde{Q}_{f_N.c_N}\,. \end{aligned} \tag{1}$$

   But these invariants are not independent:

   (b) Show that *classically*, the invariants (1) satisfy several constraints, namely

$$\det(\mathcal{M}) = 0, \quad \mathcal{M}_{ff'}\mathcal{B}^{f'} = 0, \quad \widetilde{\mathcal{B}}^f \mathcal{M}_{ff'} = 0, \quad \operatorname{minor}(\mathcal{M})^{ff'} = \widetilde{\mathcal{B}}^f \mathcal{B}^{f'}. \tag{2}$$

   (c) Show that the space of mesons, baryons, and antibaryons which satisfy these constraints has precisely N_f^2 dimensions. Consequently, it may be identified with the classical moduli space of the SQCD with $N_f = N_c + 1$.

   In the low-energy effective theory for the moduli superfields we may treat the moduli $\mathcal{M}_{ff'}$, $\mathcal{B}^f$, $\widetilde{\mathcal{B}}^f$ as independent superfields,

$$\mathcal{L}_{\text{eff}} = \int d^4\theta\, K(\mathcal{M},\mathcal{B},\widetilde{\mathcal{B}};\overline{\mathcal{M}},\overline{\mathcal{B}},\overline{\widetilde{\mathcal{B}}}) + \int d^2\theta\, W(\mathcal{M},\mathcal{B},\widetilde{\mathcal{B}}) + \text{H. c.}, \tag{3}$$

   but the VEVs $\langle M\rangle$, $\langle B\rangle$, $\langle\widetilde{\mathcal{B}}\rangle$ must satisfy constraints $\partial W/\partial(\text{any modulus}) = 0$.

(d) Show that constraints on the mesonic and baryonic VEVs due to effective superpotential

$$W_{\text{tree}} \;=\; C\left(\widetilde{\mathcal{B}}^f \mathcal{M}_{ff'} \mathcal{B}^f \;-\; \det(\mathcal{M})\right), \qquad C = \text{const} \tag{4}$$

are precisely the classical constraints (2).

Note: C has dimension $1 - 2N_c$, so we expect $C = \Lambda^{1-2N_c} \times$ a numerical constant.

In the chiral ring language, $\mathcal{M}_{ff'}$, $\mathcal{B}^f$, and $\widetilde{\mathcal{B}}^f$ are generators of the SQCD's off-shell chiral ring and eqs. (2) are operatorial identities for those generators. In the low-energy effective field theory, there are no operatorial identities; instead, eqs. (2) are the on-shell chiral ring equations which follow from the superpotential (4). Thus, the off-shell chiral ring of SQCD becomes the on-shell chiral ring of the effective theory.

In the effective theory of the $N_f^2 + 2N_f$ chiral superfields $\mathcal{M}_{ff'}$, $\mathcal{B}^f$, $\widetilde{\mathcal{B}}^f$, at a generic point of the moduli space, the superpotential (4) makes $2N_f$ superfields massive while the remaining N_f^2 remain massless. But at at some special subspaces of the moduli space there are more than N^2 massless superfields. In particular, at the special point $\mathcal{M} = \mathcal{B} = \widetilde{\mathcal{B}} = 0$ — which corresponds to zero squark VEVs $\langle Q \rangle = \left\langle \widetilde{Q} \right\rangle = 0$, — all the $N_f^2 + 2N_f$ low-energy superfields remain massless.

⋆ Optional exercise: prove this.

To study the quantum corrections to the superpotential (4) — and hence to the complex structure of the moduli space — consider the flavor symmetries of the SQCD,

$$G_F \;=\; SU(N_f)_L \times SU(N_f)_R \times U(1)_B \times U(1)_A \times U(1)_R \tag{5}$$

(e) Describe how all these symmetries act on the moduli fields $\mathcal{M}_{ff'}$, $\mathcal{B}^f$, and $\widetilde{\mathcal{B}}^f$ and on the $\Lambda^{3N_c - N_f}$.

Note: The $U(1)_A$ and the $U(1)_R$ symmetries are anomalous, but an appropriate adjustment of the Θ angle — and hence of the phase of the complex $\Lambda^{3N_c - N_f}$ — would cancel the anomaly.

The exact superpotential $W(\mathcal{M}, \mathcal{B}, \widetilde{\mathcal{B}}; \Lambda^{3N_c - N_f})$ for the moduli fields of the quantum theory must be invariant under all the flavor symmetries (5).

(f) Show that this implies

$$W(\mathcal{M},\mathcal{B},\widetilde{\mathcal{B}};\Lambda^{3N_c-N_f}) \;=\; \Lambda^{1-2N_c}\times F\left(\left(\widetilde{\mathcal{B}}^f\mathcal{M}_{ff'}\mathcal{B}^{f'}\right),\det(\mathcal{M})\right) \tag{6}$$

where $F(x,y)$ is a holomorphic homogeneous function of degree 1, *i.e.*, $F(\alpha x,\alpha y) = \alpha F(x,y)$.

Note that the classical effective superpotential (4) is indeed of the form (6) for $F(x,y) = x-y$, provided we identify the overall coefficient C as Λ^{1-2N_C}.

In general, the quantum corrections due to instantons or other non-perturbative effects should carry higher powers of the $\Lambda^{3N_c-N_f}$ than the classical superpotential. But for the superpotential (6), the power of Λ is completely fixed by the R–symmetry, which means that there are no non-perturbative corrections at all! Instead

$$W(\mathcal{M},\mathcal{B},\widetilde{\mathcal{B}};\Lambda^{3N_c-N_f}) \;=\; W_{\text{tree}}\;+\;0\;=\;\Lambda^{1-2N_c}\times\left((\widetilde{\mathcal{B}}^f\mathcal{M}_{ff'}\mathcal{B}^{f'}\;-\;\det(\mathcal{M})\right), \tag{7}$$

and there are no quantum corrections to the classical constraints (2).

3. The classical moduli space of SQCD with $N_f = N_c+1$ has a singular point $\langle Q\rangle = \langle\widetilde{Q}\rangle = 0$ where none of the symmetries are broken. In problem 1 we saw that the quantum moduli space of the theory has the same complex structure, so it has a similar singular point $\mathcal{M} = \mathcal{B} = \widetilde{\mathcal{B}} = 0$ where all the flavor symmetries remain unbroken despite the color confinement. Or rather, all the flavor symmetries free from the color (CCF) anomalies remain unbroken.

(a) Show that a combination of the axial symmetry $U(1)_A$ and the R–symmetry $U(1)_R$ which acts on the quarks, antiquarks, gluinos, and their superpartners as

$$\begin{aligned}\lambda^\alpha \;\to\; e^{i\rho}\lambda^\alpha,\quad \Psi^\alpha \;\to\; e^{-i(N_c/N_f)\rho}\Psi^\alpha,\quad \widetilde{\Psi}^\alpha \;\to\; e^{-i(N_c/N_f)\rho}\widetilde{\Psi}^\alpha,\\ A^\mu \;\to\; A^\mu,\quad Q\;\to\; e^{i\rho/N_f}Q,\quad \widetilde{Q}\;\to e^{i\rho/N_f}\widetilde{Q}\end{aligned} \tag{8}$$

is free from the color anomaly. Consequently, the net color-anomaly-free flavor symmetry is

$$\hat{G}_f \;=\; SU(N_f)_L\times SU(N_f)_R\times U(1)_B\times U(1)_{RA} \tag{9}$$

where $U(1)_B$ is the vector-like baryon number and $U(1)_{RA}$ is the symmetry (8).

At the singular point of the moduli space, the entire flavor symmetry (9) of SQCD remains unbroken, which calls for 't Hoof's anomaly matching condition between the elementary and composite fermions. The elementary fermions here are the quarks, the antiquarks, and the gluinos, while the massless composite fermions are the fermionic superpartners of the massless moduli $\mathcal{M}_{ff'}$, $\mathcal{B}^f$, and $\widetilde{\mathcal{B}}^f$.

(b) List the flavor (9) quantum numbers of all the massless composite fermions. For comparison, list the flavor and color quantum numbers of the elementary fermions.

(c) And now comes the hard part: Calculate all the non-trivial flavor anomalies $\mathrm{tr}(F)$ and $\mathrm{tr}(F\{F',F''\})$ over the elementary fermions and over the massless composite fermions and verify that in all cases

$$\mathrm{tr}_{\mathrm{elem}}(F) \;=\; \mathrm{tr}_{\mathrm{comp}}(F), \quad \mathrm{tr}_{\mathrm{elem}}(F\{F',F''\}) \;=\; \mathrm{tr}_{\mathrm{comp}}(F\{F',F''\}) \quad \forall F,F',F'' \in \hat{G}_f\,. \tag{10}$$