1. First, an exercise in off-shell chiral rings. Consider a $U(N)$ gauge theory with a single adjoint multiplet Φ of chiral fields; there are no quarks or antiquarks or any other superfields besides the $\Phi(y,\theta)$, $\overline{\Phi}(\bar{y},\bar{\theta})$, and $V(x,\theta,\bar{\theta})$.

The chiral ring of this theory is made out of gauge-invariant combinations of the scalar and gaugino fields — or in superfield notations, out of Φ and W^α chiral superfields. In matrix notations, all such gauge invariant combinations are traces — or products of traces — of matrix products of the Φ and W_α matrices,

$$ T_k \;=\; \mathrm{tr}\big(\Phi^k\big), \quad P_k^\alpha \;=\; \mathrm{tr}\big(\Phi^k W^\alpha\big), \quad R_{k,\ell}^{\alpha\beta} \;=\; \mathrm{tr}\big(\Phi^k W^\alpha \Phi^\ell W^\beta\big), \quad etc., \tag{1} $$

or in other words, the traces (1) *generate* the chiral ring.

Actually, many of the traces (1) are equivalent to each other as members of the chiral ring because their difference is a total $\overline{D}^{\dot\alpha}$ derivative of some gauge-invariant composite superfield. Your task is to show that the only independent generators are the T_k, P_k^α, and $R_k = R_{k,0}^{12}$, and there are no others.

(a) Show that for any matrix product X of Φ and W^α matrices,

$$ \mathrm{tr}\big(X[W^\alpha,\Phi]\big) \;=\; -\tfrac{1}{8}\overline{D}^2\,\mathrm{tr}\big(X\nabla^\alpha\Phi\big) \;\overset{\text{c.r.}}{=}\; 0 \tag{2} $$

and hence

$$ \mathrm{tr}\big(XW^\alpha\Phi\big) \;\overset{\text{c.r.}}{=}\; \mathrm{tr}\big(X\Phi W^\alpha\big). \tag{3} $$

In eq. (2), ∇^α is the gauge-covariant spinor derivative, $\nabla^\alpha\Phi = D^\alpha\Phi + [\Gamma^\alpha,\Phi]$ for $\Gamma^\alpha = e^{-2V}D^\alpha e^{2V}$, or equivalently $\nabla^\alpha\Phi = e^{-2V}D^\alpha\big(e^{+2V}\Phi e^{-2V}\big)e^{+2V}$.

Note that eq. (3)allows us to re-order the Φ and the W^α matrices in the traces (1), thus

$$ \mathrm{tr}\big(\Phi\cdots\Phi W^\alpha\Phi\cdots\Phi W^\beta\cdots\text{more matrices}\cdots W^\gamma\big) \;\overset{\text{c.r.}}{=}\; \mathrm{tr}\big(\Phi\cdots\Phi \times W^\alpha W^\beta\cdots W^\gamma\big). \tag{4} $$

In particular, $R_{k,\ell}^{\alpha\beta} \overset{\text{c.r.}}{=} R_{k+\ell,0}^{\alpha\beta}$.

(b) Now show that

$$\mathrm{tr}\big(X\{W^\alpha, W^\beta\}\big) \;=\; -\tfrac{1}{8}\overline{D}^2\,\mathrm{tr}\big(X\nabla^\alpha W^\beta\big) \tag{5}$$

and hence

$$\mathrm{tr}\big(\Phi^k W^\alpha W^\beta\big) \;\overset{\text{c.r.}}{=}\; \epsilon^{\alpha\beta}\,\mathrm{tr}\big(\Phi^k W^1 W^2\big) \;=\; \tfrac{1}{2}\epsilon^{\alpha\beta} \times R_k\,. \tag{6}$$

(c) Finally, show that all traces including 3 or more gaugino fields W^α are equivalent to zero,

$$\mathrm{tr}\big(\Phi^k W^\alpha W^\beta W^\gamma \cdots\big) \;\overset{\text{c.r.}}{=}\; 0. \tag{7}$$

Thus, the only independent generators of the chiral ring are the

$$T_k \;=\; \mathrm{tr}\big(\Phi^k\big), \quad P_k^\alpha \;=\; \mathrm{tr}\big(\Phi^k W^\alpha\big), \quad \text{and} \quad R_k \;=\; \mathrm{tr}\big(\Phi^k W^\alpha W_\alpha\big). \tag{8}$$

Moreover, for finite N, the traces involving more then N Φ matrices are polynomial functions of the traces with fewer Φ's. For example, for $N = 2$

$$\begin{aligned}
2T_3 &= 3T_2 T_1 - T_1^3, \\
2P_3^\alpha &= P_2^\alpha \times T_1 + P_1^\alpha \times T_2 - P_0^\alpha \times (T_1^2 - T_2)T_1, \\
2R_3 &= R_2 \times T_1 + R_1 \times T_2 - R_0 \times (T_1^1 - T_2)T_1,
\end{aligned} \tag{9}$$

and similar (but more complicated) relations for $k = 4, 5, \dots$. In general, such relations are corrected by the instanton effects, but that goes beyond the scope of this exercise.

⋆ Optional reading assignment:

If you are interested in chiral rings and chiral-ring equations, read arXiv:hep-th/0211170, *Chiral Rings and Anomalies in Supersymmetric Gauge Theory* by Freddy Cachazo, Michael Douglas, Nathan Seiberg, and Edward Witten.

Note: this paper covers many related subjects, and the math is not easy to follow. So take your time, don't try to finish the whole paper this week. Meanwhile, make sure to do the rest of this homework on time!

2. And now a required reading assignment: Philip Argyres's 2001 lecture notes, sections §3.3–5 about SQCD with $N_f \geq N_c$, conformal and superconformal symmetry, and Seiberg duality.

   Focus on §3.4 about the conformal and superconformal symmetries and their representations, as I am going to explain these subject this week. But read the other 2 sections as well, since I shall start explaining the Seiberg duality as soon as I explain the superconformal field theories.

3. Finally, an exercise in conformal symmetry and radial quantization. I shall explain the radial quantization and the state-operator correspondence in CFT on Tuesday 11/12, so you can do this problem after the Tuesday lecture. Meanwhile, do the rest of this homework set!

   Suppose a CFT (supersymmetric or otherwise) has an abelian gauge symmetry and let $|F^{\mu\nu}\rangle$ be the primary state (in the Hilbert space on the S^3) corresponding to the $F^{\mu\nu}$ operator at $x = 0$. Consider the descendant states

   $$|J_{\text{el}}^{\nu}\rangle \;=\; P^{\mu}|F^{\mu\nu}\rangle \quad \text{and} \quad \left|J_{\text{mag}}^{\nu}\right\rangle \;=\; P^{\mu}\left|\tilde{F}^{\mu\nu}\right\rangle \;=\; \tfrac{1}{2}\epsilon^{\mu\nu\kappa\lambda}P_{\mu}\left|F^{\kappa\lambda}\right\rangle. \qquad (10)$$

   Use $P_{\mu}^{\dagger} = -iK_{\mu}$ (in radial quantization) and the conformal algebra to calculate the Hilbert norms $\langle J_{\text{el}}^{\nu}|J_{\text{el}}^{\nu}\rangle$ and $\langle J_{\text{mag}}^{\nu}|J_{\text{mag}}^{\nu}\rangle$ of these states in terms of the scaling dimension Δ of the $F^{\mu\nu}$ field. Show that both norms are zero if $\Delta = 2$ and both are positive if $\Delta > 2$.

   Use this result to argue that either the abelian gauge field is free (in the IR limit where the theory is conformal) or else it must couple to both electric and magnetic charges.