1. First, a reading assignment: Gerard 't Hooft's lecture notes on *Monopoles, Instantons, and Confinement*, arXiv:hep-th/0010225. These are the same notes I have assigned back in the homework#9, but this time please focus on chapter 3 about the magnetic monopoles, and also §4.8 about the Θ angle in QED.

⋆ The rest of this homework is about Seiberg duality for SQCD. As explained in class, this duality involves two distinct UV theories A and B with the same IR physics. Theory A is pure SQCD with N_c^A colors and $N_f \geq N_c^A + 2$ flavors. It has quarks $Q^{c,f}$, antiquarks $\widetilde{Q}_c^f$, and the gauge fields V^a; there is nothing else. Theory B is also SQCD-like but with extra singlets. Specifically, it has the same number N_f of flavors as (A) but a different number of colors, $N_c^B = N_f - N_c^A$. Besides the gauge fields V^a, it has quarks q_f^c, antiquarks $\tilde{q}_{c,f}$, and N_f^2 gauge-singlet fields $\Phi^{ff'}$ with Yukawa couplings to the quarks and antiquarks,

$$W_{\rm tree}^B \;=\; \lambda \sum_{c,f,f'} q_f^c \Phi^{ff'} \tilde{q}_{c,f}\,. \tag{1}$$

Note that the quarks and antiquarks have upper flavors indices in theory A but lower flavor indices in theory B. This denotes their opposite quantum numbers with respect to the $SU(N_f)_L \times SU(N_f)_R$ flavor symmetry group: The A-quarks $Q^{c,f}$ form the $\mathbf{N_f}$ multiplet of $SU(N_f)_L$ while the B-quarks q_f^c form the conjugate $\overline{\mathbf{N_f}}$ multiplet. Likewise, the A-antiquarks $\widetilde{Q}_f^c$ form the $\mathbf{N_f}$ of the $SU(N_f)_R$ while the B-antiquarks $\tilde{q}_{c,f}$ form the $\overline{\mathbf{N_f}}$.

2. Let's check 't Hoof's anomaly matching conditions for the Seiberg duality. Clearly, if any two distinct gauge theories A and B have the same IR physics, they must have exactly the same un-broken flavor symmetries. Moreover, because the massless composite fermions are the same in both cases, their flavor anomalies must match those of the elementary fermions of either UV theory, hence the UV–UV anomaly matching conditions,

$$\begin{aligned} \mathrm{tr}_{(A)}\bigl(F\{F',F''\}\bigr) \;&=\; \mathrm{tr}_{(B)}\bigl(F\{F',F''\}\bigr) \quad \forall \text{ unbroken flavor charges } F, F', F'', \\ \text{and also } \mathrm{tr}_{(A)}(F) \;&=\; \mathrm{tr}_{(B)}(F) \quad \forall \text{ abelian } F, \end{aligned} \tag{2}$$

where $\mathrm{tr}_{(A)}$ means the trace over all the elementary LH Weyl fermions of the UV theory A, and likewise $\mathrm{tr}_{(B)}$ is the trace over all the elementary LH Weyl fermions of the UV theory B.

(a) Check that the A and B theories in SQCD Seiberg duality have the same flavor symmetry groups

$$G_F \;=\; SU(N_f)_L \times SU(N_f)_R \times U(1)_B \times U(1)_R \tag{3}$$

where B is the (vector-like) baryon number while $U(1)_R$ is the color-anomaly-free combination of the pure-R and the axial symmetries. Make sure that the B theory does not have additional symmetries.

(b) Tabulate the G_F quantum numbers of all the elementary fermions of the A and B theories.

(c) Calculate all the flavor anomalies of each theory and check that they match.

⋆ In class I had identified the $\Phi^{ff'}$ fields of the B theory with the mesons $\mathcal{M}^{ff'} = Q^{cf}\widetilde{Q}_c^{f'}$ of the A theory, but the literal identification $\mathcal{M}^{ff'} = \Phi^{ff'}$ works only for the renormalized fields of the same non-canonical dimension $\Delta = 3N_c^B/N_f$. In terms of the un-renormalized fields which keep their canonical dimensions — 1 for the elementary $\Phi^{ff'}$ but 2 for the mesons $\mathcal{M}^{ff'} = \widetilde{Q}_c^f Q^{cf'}$ — the duality has form

$$\mathcal{M}^{ff'} \;=\; \Phi^{ff'} \times \mu \tag{4}$$

for some constant μ of dimension mass^{+1}. In the next problem, you will see that the value of this constant should be

$$\mu \;=\; \frac{\lambda}{16\pi^2} \times \left((-1)^{N_c^A}\Lambda_A^{3N_c^A-N_f} \times \Lambda_B^{3N_c^B-N_f}\right)^{1/N_f} \tag{5}$$

where λ is the un-renormalized Yukawa coupling in the bare superpotential (1) of the B theory. Note that for $N_c^A + N_c^B = N_f$ this μ indeed has dimension 1.

3. This problem concerns *deformations* of the A and B theories by relevant terms in the superpotential.

Suppose the color-to-flavor ratio of the A theory is in the conformal window but all the quark flavors are massive,

$$W^A_{\text{tree}} \;=\; -m_{ff'}\widetilde{Q}^f_c Q^{cf'}, \qquad \text{rank}(m) = N_f\,. \tag{6}$$

The masses spoil the conformal invariance of the IR physics; instead, we have a mass gap (no massless particles at all) due to quark confinement and gaugino condensation.

(a) Calculate the gaugino condensate $\langle S\rangle$ and the meson VEVs $\left\langle \mathcal{M}^{ff'}\right\rangle$ in the massive theory A. Note: when all quark flavors are massive, quantum corrections lead to $\text{rank}(\mathcal{M}) = N_f$ contrary to the classical limit $\text{rank}(\mathcal{M}_{\text{cl}}) \leq N_c$.

Under Seiberg duality, the quark masses (6) of the A theory are dual to the O'Raifeartaigh terms for the Φ fields of the B theory. In light of eq. (4), the un-normalized superpotential of the B theory is

$$W^B_{\text{tree}} \;=\; -\mu m_{ff'}\Phi^{ff'} \;+\; \lambda\Phi^{ff'}\tilde{q}_{fc}\,q^c_{f'}\,. \tag{7}$$

(b) Find the supersymmetric vacua of the B theory and calculate the VEVs of the gaugino condensate S_B, the B-mesons $M^B_{ff'} = \tilde{q}_{fc}\,q^c_{f'}$, and the $\Phi^{ff'}$ fields.
Hint: first show that in the B theory

$$\begin{aligned}
\lambda M^B \;&\overset{\text{c.r.}}{=}\; \mu m \quad \text{as } N_f\times N_f \text{ matrices},\\
\lambda M^B\times\Phi \;&\overset{\text{c.r.}}{=}\; -\frac{S_b}{16\pi^2}\times\mathbf{1}_{N_f\times N_f}\,,\\
\left(S^{N_c}\right)_B \;&\overset{\text{c.r.}}{=}\; \left(\Lambda^{3N_c-N_f}\right)_B\times\det(-\lambda\Phi),
\end{aligned} \tag{8}$$

then solve these chiral ring equations.

(c) Show that if we take μ as in eq. (5) then

$$\langle S\rangle_B \;=\; -\langle S\rangle_A \quad\text{and}\quad \mu\times\langle\Phi\rangle_B \;=\; \langle\mathcal{M}\rangle_A \quad (\text{as } N_f\times N_f \text{ matrices}). \tag{9}$$

Now, suppose some of the quark flavors of the A theory remain massless, $i.e.$, the mass matrix has $\mathrm{rank}(m) < N_f$.

(d) Show that for $N_f - \mathrm{rank}(m) < N_c^A$ both theories do not have any stable supersymmetric vacua.

(e) Now suppose $N_f - \mathrm{rank}(m) > N_c^A + 1$ but the non-zero quark masses are much larger than Λ_A, so that we may integrate out the massive flavors perturbatively. Show that in this regime, the B theory suffers a partial Higgsing down, and that when we integrate out all the massive vector multiplets, the remaining low-energy theory is dual to the low-energy limit of the A theory.

(f) Optional exercise:
Explain what happens for $N_f - \mathrm{rank}(m) = N_c^A$ or $N_c^A + 1$.