Problem $\mathbf{1}$(a):

Let's start with the continuous global symmetries

$$G^{\text{classical}}_{\text{global}} = \Big(SU(4)\times U(1)_X\times U(1)_Y\Big)_1 \times \Big(SU(4)\times U(1)_X\times U(1)_Y\Big)_2 \times U(1)_R \tag{S.1}$$

which act on the A, B, and C fields according to the following charge table

QN / fields	$SU(4)_1$	$U(1)_{X1}$	$U(1)_{Y1}$	$SU(4)_2$	$U(1)_{X2}$	$U(1)_{Y2}$
A_1	$\mathbf{1}$	$+2$	0	$\mathbf{1}$	0	0
$B_1,\ldots,B_4$	$\mathbf{4}$	-1	$+1$	$\mathbf{1}$	0	0
$C_1,\ldots,C_4$	$\bar{\mathbf{4}}$	-1	-1	$\mathbf{1}$	0	0
A_2	$\mathbf{1}$	0	0	$\mathbf{1}$	$+2$	0
$B_5,\ldots,B_8$	$\mathbf{1}$	0	0	$\mathbf{4}$	-1	$+1$
$C_5,\ldots,C_8$	$\mathbf{1}$	0	0	$\bar{\mathbf{4}}$	-1	-1

(S.2)

Here QN — the quantum numbers — refer to the abelian charges or to the non-abelian multiplets (in boldface). As to the R symmetry,

$$\begin{aligned} \text{all scalars } \phi &\ \text{ have } R=+\tfrac{2}{3}\,,\\ \text{all matter fermions } \psi_\alpha &\ \text{ have } R=-\tfrac{1}{3}\,,\\ \text{all gauginos } \lambda_\alpha &\ \text{ have } R=+1,\\ \text{all vectors } V_\mu &\ \text{ have } R=0. \end{aligned} \tag{S.3}$$

In the quantum theory we must check the abelian classical symmetries for the gauge anomalies

$$\sum_{\text{fermions}} (\text{global charge}) \times \text{Index}\Big[\text{gauge } SU(2)_1 \text{ or } SU(2)_2\Big]. \tag{S.4}$$

It is easy to see that the Y_1 and Y_2 charges are anomalous WRT both $SU(2)$ factors but the combination $Y=Y_1-Y_2$ is anomaly-free. Also, the X_1, X_2, and R charges are non-anomalous.

Thus, the quantum theory has continuous global symmetry

$$G_{\text{global}}^{\text{quantum}} = \Big(SU(4)\times U(1)_X\Big)_1 \times \Big(SU(4)\times U(1)_X\Big)_2 \times \times U(1)_Y \times U(1)_R\,. \tag{S.5}$$

As to the discrete symmetries, we have the anomaly-free $\mathbf{Z}_8$ subgroup of the $U(1)_{Y1+Y2}$ which multiplies all the B_i fields by $e^{2\pi i/8}$ and all the C_i by $e^{-2\pi i/8}$. More importantly, we have a $\mathbf{Z}_2$ symmetry that swaps fields appearing in the first terms in the superpotential (2) with fields appearing in the second term,

$$\mathbf{Z}_2: \quad A_1 \leftrightarrow A_2\,, \quad B_i \leftrightarrow B_{i\pm4}\,, \quad C_i \leftrightarrow C_{i\pm4}\,. \tag{S.6}$$

This symmetry assures that the A_1 and the A_2 fields have the same anomalous dimension γ_A. Also, together with the $SU(4)_1 \times SU(4)_2$ it makes sure that all eight B_i fields have the same anomalous dimension γ_B and all eight C_i fields have the same anomalous dimension γ_C.

Problem **1**(b):

Thanks to SUSY, there is no counterterm δ_λ for the Yukawa coupling. Consequently, its renormalization follows solely from the wave-function renormalization of the chiral superfields, thus

$$\beta_\lambda = \lambda \times (\gamma_A + \gamma_B + \gamma_C). \tag{S.7}$$

As to the gauge couplings, their beta-function are given by the Novikov–Shifman–Vainshtein–Zakharov equations

$$\begin{aligned}\beta_{g1} &= \frac{g_1^3}{16\pi^2 - 4g_1^2} \times \Big[-3\times 2 \,+\, 2\times 2\times \tfrac{1}{2}\times(1-2\gamma_A) \,+\, 4\times 2\times \tfrac{1}{2}\times(1-2\gamma_B)\Big] \\ &= -\frac{g_1^3}{4\pi^2 - g_1^2}\times(\gamma_A + 2\gamma_B)\end{aligned} \tag{S.8}$$

and similarly

$$\begin{aligned}\beta_{g2} &= \frac{g_2^3}{16\pi^2 - 4g_2^2} \times \Big[-3\times 2 \,+\, 2\times 2\times \tfrac{1}{2}\times(1-2\gamma_A) \,+\, 4\times 2\times \tfrac{1}{2}\times(1-2\gamma_C)\Big] \\ &= -\frac{g_2^3}{4\pi^2 - g_2^2}\times(\gamma_A + 2\gamma_C).\end{aligned} \tag{S.9}$$

Equations (S.7), (S.8), and (S.9) give us exact beta-functions (true to all orders of the perturbation theory) in terms of the anomalous dimensions. Hence, the fixed points where all three beta–functions vanish, $\beta_\lambda = \beta_{g1} = \beta_{g2} = 0$ lie wherever

$$\gamma_A + \gamma_B + \gamma_C \;=\; \gamma_A + 2\gamma_B \;=\; \gamma_A + 2\gamma_C \;=\; 0, \tag{S.10}$$

or equivalently

$$\gamma_B(\lambda, g_1, g_2) \;=\; \gamma_C(\lambda, g_1, g_2) \;=\; -\tfrac{1}{2}\gamma_A(\lambda, g_1, g_2). \tag{S.11}$$

Note that the three eqs. (S.10) are not independent, hence only two eqs. (S.11).

Viewed as equations for the couplings λ, g_1 and g_2 — on which the anomalous dimensions depend — the two eqs. (S.11) define a line in the 4D coupling space. All point on this line are fixed points of the renormalization group.

Preamble to problem **1**(c):

A few weeks ago we saw in class that in the Wess–Zumino model the chiral superfield Φ has anomalous dimension

$$\gamma^{\rm WZ} \;=\; +\frac{|\lambda|^2}{32\pi^2} \;+\; O(|\lambda^4|) \tag{S.12}$$

where λ is the canonically normalized Yukawa coupling. This formula has a straightforward generalization to a theory of several chiral superfields Φ_i with Yukawa couplings λ_{ijk}, namely

$$\gamma^{\rm Yukawas}[\Phi_i] \;=\; +\frac{1}{32\pi^2}\sum_{jk}|\lambda_{ijk}|^2 \;+\; \text{higher loops}. \tag{S.13}$$

In this formula, we should sum over all quantum numbers hiding behind the indices j and k, such as the species (*e.g.*, A, B, or C), the 'flavor' (*e.g.*, i of A_i, B_i, or C_i) and the color — or whatever the gauge index may be appropriate for the field in question (*e.g.*, the colors of both $SU(2)$ groups for the A_i fields). Also, if $j \neq k$ we should include both (i, k) and (k, j) pairs into the sum — which gives us an overall factor of 2.

Now consider the effect of the gauge couplings. In SQED, we saw in class that

$$\gamma^{\rm SQED}[\Phi_i] \;=\; -\frac{e^2 q_i^2}{8\pi^2} \;+\; O(e^4). \tag{S.14}$$

Generalization to the non-abelian gauge theories is quite straightforward: For a simple gauge group G,

$$\gamma^{\rm gauge}[\Phi_i] \ = \ -\frac{g^2}{8\pi^2}\times C_2(\Phi_i)\ +\ \text{higher loops} \tag{S.15}$$

where C_2 is a quadratic Casimir of a gauge multiplet to which the Φ_i fields belongs (*e.g.*, $C_2 = I(I+1)$ for an $SU(2)$ multiplet of isospin I), and g is normalized conventionally (as in the Standard Model or GUTs, not as in the *Superspace* book). More generally, for a gauge symmetry $G = \prod_\nu G_\nu$ which has several simple (or abelian) factors G_ν with respective couplings g_ν, we have

$$\gamma^{\rm gauge}[\Phi_i] \ = \ -\frac{1}{8\pi^2}\sum_\nu g_\nu^2\times C_2^{(G_\nu)}(\Phi_i)\ +\ \text{higher loops.} \tag{S.16}$$

Finally, in a generic theory that has both gauge and Yukawa couplings, the one-loop anomalous dimensions are

$$\gamma^{\rm net}_{1\,\rm loop}[\Phi_i] \ = \ \gamma^{\rm Yukawa}_{1\,\rm loop}[\Phi_i] + \gamma^{\rm gauge}_{1\,\rm loop}[\Phi_i] \ = \ +\frac{1}{32\pi^2}\sum_{jk}|\lambda_{ijk}|^2 - \frac{1}{8\pi^2}\sum_\nu g_\nu^2\times C_2^{(G_\nu)}(\Phi_i). \tag{S.17}$$

Problem **1**(c):

Now let's apply eqs. (S.17) for the theory in question. The gauge group is $SU(2)\times SU(2)$, and WRT each factor, the A, B, C fields are either doublets ($C_2 = \frac{3}{4}$) or singlets ($C_2 = 0$). Specifically, $A_i \in (\mathbf{2},\mathbf{2})$, $B_i \in (\mathbf{2},\mathbf{1})$, $C_i \in (\mathbf{1},\mathbf{2})$, hence the gauge couplings' contributions to the one-loop anomalous dimensions are

$$\gamma^{\rm gauge}_{1\,\rm loop}[A] \ = \ -\frac{3}{32\pi^2}\times\left(g_1^2+g_2^2\right), \qquad \gamma^{\rm gauge}_{1\,\rm loop}[B] \ = \ -\frac{3}{32\pi^2}\times g_1^2, \qquad \gamma^{\rm gauge}_{1\,\rm loop}[C] \ = \ -\frac{3}{32\pi^2}\times g_2^2. \tag{S.18}$$

Now consider the Yukawa couplings' contributions (S.13). For the superpotential (2) all non-zero couplings have the same value $= \lambda$, hence

$$\gamma^{\rm Yukawa}_{1\,\rm loop}[\Phi_i] \ = \ +\frac{|\lambda|^2}{32\pi^2}\times\#(j,k)\ \text{pairs for which } \lambda_{ijk}\neq 0. \tag{S.19}$$

Let's start with Φ_i being an A field; for the sake of definiteness, let $\Phi_i = A_1$ of some fixed $SU(2)\times SU(2)$ colors (c_1, c_2). Then we get a non-zero Yukawa coupling for $\Phi_j = B_f$ of flavor

$f = 1, 2, 3, 4$ and color c_1 and $\Phi_k = C_f$ (same f) of color c_2. Summing over $f = 1, 2, 3, 4$ gives us four (j, k) pairs, times two for the $j \leftrightarrow k$ exchange. Altogether, there are eight (j, k) pairs, hence

$$\gamma_{1\,\text{loop}}^{\text{Yukawa}}[A] \;=\; +\frac{|\lambda|^2}{32\pi^2} \times 8. \tag{S.20}$$

Now let Φ_i be a B field, say $\Phi_i = B_1$ of color c_1. Then we get $\lambda_{ijk} \neq 0$ for $\Phi_j = C_1$ (note same flavor) and some color $c_2 = 1, 2$ and $\Phi_k = A_1$ of colors (c_1, c_2). All the flavors here are fixed, and so is the $SU(2)_1$ color c_1, but the $SU(2)_2$ color c_2 is not, hence summing over the $c_2 = 1, 2$ gives us two (j, k) pairs. Multiplying by two for the $j \leftrightarrow k$ exchange, we get four pairs, thus

$$\gamma_{1\,\text{loop}}^{\text{Yukawa}}[B] \;=\; +\frac{|\lambda|^2}{32\pi^2} \times 4. \tag{S.21}$$

Similarly,

$$\gamma_{1\,\text{loop}}^{\text{Yukawa}}[C] \;=\; +\frac{|\lambda|^2}{32\pi^2} \times 4. \tag{S.22}$$

Combining these Yukawa contributions with the gauge contributions (S.18), we arrive at the net one-loop anomalous dimensions

$$\begin{aligned}
\gamma_{1\,\text{loop}}^{\text{net}}[A] &= \frac{8|\lambda|^2 - 3g_1^2 - 3g_2^2}{32\pi^2}\,, \\
\gamma_{1\,\text{loop}}^{\text{net}}[B] &= \frac{4|\lambda|^2 - 3g_1^2}{32\pi^2}\,, \\
\gamma_{1\,\text{loop}}^{\text{net}}[C] &= \frac{4|\lambda|^2 - 3g_2^2}{32\pi^2}\,.
\end{aligned} \tag{S.23}$$

It remains to plug in these anomalous dimensions into the fixed-point equations

$$\gamma_B(\lambda, g_1, g_2) \;=\; \gamma_C(\lambda, g_1, g_2) \;=\; -\tfrac{1}{2}\gamma_A(\lambda, g_1, g_2). \tag{S.11}$$

Clearly, the first equation $\gamma_B = \gamma_C$ requires $g_1 = g_2$. In fact, by symmetry between the two $SU(2)$ gauge groups, this condition is exact to all orders of the perturbation theory. As to the second equation (S.11),

$$32\pi^2(\gamma_A+\gamma_B+\gamma_C) \;=\; 16|\lambda|^2 \;-\; 6g_1^2 \;-\; 6g_2^2 \;+\; \text{higher loops}, \tag{S.24}$$

hence the fixed points lie along the line

$$g_1^2 \;=\; g_2^2 \;=\; \frac{16}{12}|\lambda|^2 \;+\; O(|\lambda^4|/16\pi^2). \tag{3}$$

Quod erat demonstrandum.

Problem **1**(d):
Let's take a closer look at the beta-functions

$$\beta_{g1} \;=\; -\frac{g_1^3}{4\pi^2-g_1^2}\times(\gamma_A+2\gamma_B), \tag{S.8}$$
$$\beta_{g2} \;=\; -\frac{g_2^3}{4\pi^2-g_1^2}\times(\gamma_A+2\gamma_C), \tag{S.9}$$
$$\beta_\lambda \;=\; +\lambda\times(\gamma_A+\gamma_B+\gamma_C), \tag{S.7}$$

and the one-loop anomalous dimensions (S.23). Suppose we start at some point in the coupling space where $g_1>g_2$. In this case, eqs. (S.23) yield $\gamma_B<\gamma_C$, hence according to eqs. (S.8) and (S.9) $\beta_1>\beta_2$. This means that the difference g_1-g_2 increases in the UV direction but decreases in the IR direction. Likewise, had we started with $g_2>g_1$, the difference g_2-g_1 would also decrease in the IR direction. Thus, regardless of our starting point in the coupling space, the RG flow would bring the gauge couplings g_1 and g_2 closer to each other in the IR, and ultimately — in the very deep IR — we would end up with $g_1=g_2$.

Now the other direction in which we may deviate from the fixed line (3). Let start with the Yukawa coupling too weak compared to the gauge couplings, *i.e.* $g_1^2=g_2^2>\frac{4}{3}|\lambda|^2$. In this case,

$$\gamma_A\;+\;2\gamma_B \;=\; \gamma_A\;+\;2\gamma_C \;=\; \gamma_A\;+\;\gamma_B\;+\;\gamma_C \;<\; 0 \tag{S.25}$$

and hence $\beta_{g1}=\beta_{g2}>0$ while $\beta_\lambda<0$, which means that in the IR direction the Yukawa coupling becomes stronger while the gauge couplings become weaker. Likewise, if we start the

RG flow with the Yukawa coupling too strong relative to the gauge couplings, *i.e.* $g_1^2 = g_2^2 < \frac{4}{3}|\lambda|^2$, then

$$\gamma_A + 2\gamma_B \;=\; \gamma_A + 2\gamma_C \;=\; \gamma_A + \gamma_B + \gamma_C \;>\; 0 \tag{S.26}$$

and hence $\beta_{g1} = \beta_{g2} < 0$ while $\beta_\lambda > 0$: this time, the Yukawa coupling becomes weaker in the IR direction while the gauge couplings become stronger. In either case, the gauge and Yukawa coupling converge towards each other — or rather towards the fixed line (3) — in the IR direction. And that's what makes the fixed line (3) IR–attractive.

Problem **1**(e):

The line (3) or IR-attractive fixed points gives rise to a whole family of non-trivial SCFTs. At one end of the fixed line (3), all 3 couplings g_1, g_2, and λ are weak and all the anomalous dimensions are small. But apart from this weakly-coupled region, the fixed line (3) extends to strong, or at least $O(1)$ couplings. In this regime, the perturbation theory breaks down and we no longer have accurate formulae for the anomalous dimensions, or even for the $g_1 = g_2$ as a function of $|\lambda$ — the fixed line in the 3D coupling space probably isn't straight (outside the weak-couplings region), and we don't know its precise shape. But it certainly does go somewhere in the strongly-coupled region, so our family of the SCFTs includes both weakly-coupled and strongly-coupled theories.