Problem **3**(a):

The classical flavor symmetry of the A theory is

$$G^A_{\rm classical} \;=\; SU(N_f)_L\times SU(N_f)_R\times U(1)_B\times U(1)_A\times U(1)_{R0} \tag{S.1}$$

where $U(1)_B$ charge is the vectorlike baryon number, the $U(1)_A$ is the axial symmetry, and $U(1)_{R0}$ is the pure-R symmetry of SUSY. In the quantum theory, the axial and the pure-R symmetries are destroyed by the color anomaly, but one combination of the two symmetries — henceforth denoted $U(1)_R$ — is anomaly free. Altogether, the flavor symmetry of the quantum A theory is (3).

As discussed in class, the quarks and the antiquarks of the quantum B theory have the same flavor symmetry (3), albeit with unusual 'charge' assignments: the quarks q^c_f belong to the $\overline{\mathbf{N_f}}$ multiplet of the $SU(N_f)_L$ instead of the $\mathbf{N_f}$, and likewise the antiquarks $\tilde q_{c,f}$ belong to the $\overline{\mathbf{N_f}}$ rather than the $\mathbf{N_f}$ of the $SU(N_f)_R$. As to the quantum numbers of the gauge-singlets $\Phi^{ff'}$, they are completely determined by the invariance of the Yukawa couplings (1). Specifically, $\Phi\in(\mathbf{N_f},\mathbf{N_f})$ of the $SU(N_f)_L\times SU(N_f)_R$, the baryon number of M is zero, and the R–charge is

$$R(\Phi) \;=\; 2\;-\;R(q)\;-\;R(\tilde q) \tag{S.2}$$

because the R–charge of the superpotential is 2. Also, Φ cannot transform under any symmetry that does not act on the other fields, which rules out any additional global symmetries of the B theory.

Problem **2**(b):

Here is the table of the color and flavor quantum numbers of all the elementary fields of the A theory:

fields \ QN	$SU(N_c^A)$	$SU(N_f)_L$	$SU(N_f)_R$	B	R(boson)	R(fermion)
A^μ, λ^α	Adjoint	$\mathbf{1}$	$\mathbf{1}$	0	0	+1
Q, Ψ_Q	$\mathbf{N_c^A}$	$\mathbf{N_f}$	$\mathbf{1}$	$+\frac{1}{N_c^A}$	$1 - \frac{N_c^A}{N_f}$	$-\frac{N_c^A}{N_f}$
$\widetilde{Q}, \widetilde{\Psi}_Q$	$\overline{\mathbf{N_c^A}}$	$\mathbf{1}$	$\mathbf{N_f}$	$-\frac{1}{N_c^A}$	$1 - \frac{N_c^A}{N_f}$	$-\frac{N_c^A}{N_f}$

(S.3)

Similarly, for the B theory:

fields \ QN	$SU(N_c^B)$	$SU(N_f)_L$	$SU(N_f)_R$	B	R(boson)	R(fermion)
A^μ, λ^α	Adjoint	$\mathbf{1}$	$\mathbf{1}$	0	0	+1
q, ψ_q^α	$\mathbf{N_c^B}$	$\overline{\mathbf{N_f}}$	$\mathbf{1}$	$+\frac{1}{N_c^B}$	$1 - \frac{N_c^B}{N_f}$	$-\frac{N_c^B}{N_f}$
$\tilde{q}, \tilde{\psi}_q^\alpha$	$\overline{\mathbf{N_c^B}}$	$\mathbf{1}$	$\overline{\mathbf{N_f}}$	$-\frac{1}{N_c^B}$	$1 - \frac{N_c^B}{N_f}$	$-\frac{N_c^B}{N_f}$
Φ, ψ_ϕ^α	$\mathbf{1}$	$\mathbf{N_f}$	$\mathbf{N_f}$	0	$+2\frac{N_c^B}{N_f}$	$2\frac{N_c^B}{N_f} - 1$

(S.4)

Problem **2**(c):

⋆ First, a little group theory reminder. A triangle anomaly involving 3 generators T^a, T^b, T^c of the same non-abelian symmetry factor has form $\mathrm{tr}(T^a\{T^b, T^c\}) = \mathcal{A} \times d^{abc}$ where d^{abc} is a cubic invariant of the group in question while

$$\mathcal{A} \;=\; \#\text{fundamentals} \;-\; \#\text{antifundamentals} \;+\; \cdots \tag{S.5}$$

where the $\cdots$ stand for terms counting various tensor multiplets (which we fortunately do not have in this homework). To compare anomalies of different sets of fermions, all we need is to compare the respective coefficients $\mathcal{A}$.

Likewise, an anomaly involving two non-abelian generators T^a, T^b and one abelian charge X has form $\mathrm{tr}(T^aT^bX) = \mathcal{A} \times \delta^{ab}$ where

$$\mathcal{A} = \sum_{\psi} X(\psi) \times \mathrm{Index}(\psi) \times \#(\psi) \tag{S.6}$$

where the Index is with respect to the non-abelian factor in question while # counts multiplicity with respect to the other symmetries, if any.

Finally, for a purely abelian anomaly

$$\mathrm{tr}(XYZ) \;=\; \sum_{\psi} X(\psi) \times Y(\psi) \times Z(\psi) \times \#\psi. \tag{S.7}$$

And now let's compute and compare all the non-trivial favor anomalies of the two theories.

- The $[SU(N_f)_L]^3$ anomaly.
  The A theory has N_c^A fundamentals (the quarks Ψ_Q) of the $SU(N_f)_L$ and no antifundamentals, thus $\mathcal{A}(A) = N_c^A$. The B theory has N_f fundamentals (the gauge singlets ψ_ϕ) and N_c^B antifundamentals (the quarks ψ_q), hence $\mathcal{A}(B) = N_f - N_c^B = N_c^a = \mathcal{A}(A)$.
- The $[SU(N_f)_R]^3$ anomaly: the anomalies match in the same way.
- The $[SU(N_f)_L]^2 \times U(1)_B$ anomaly.
  In the either theory, the only fermions charged under both $SU(N_f)_L$ and $U(1)_B$ symmetries are the quarks, thus

$$\begin{aligned} \mathcal{A}(A) \;&=\; \mathcal{A}(\Psi_Q) \;=\; \frac{+1}{N_c^A} \times \frac{1}{2} \times N_c^A \;=\; +\tfrac{1}{2}\,, \\ \mathcal{A}(B) \;&=\; \mathcal{A}(\psi_q) \;=\; \frac{+1}{N_c^B} \times \frac{1}{2} \times N_c^B \;=\; +\tfrac{1}{2}\,, \end{aligned} \tag{S.8}$$

  and the anomalies match.
- The $[SU(N_f)_R]^2 \times U(1)_B$ anomaly: similar matching.

- The $[SU(N_f)_L]^2 \times U(1)_R$ anomaly.

  This time, in the A theory only the quarks carry both $SU(N_f)_L$ and $U(1)_R$ charges, but in the B theory, both the quarks ψ_q and the gauge-singlets ψ_ϕ have the requisite quantum numbers. Thus,

$$\begin{aligned}
\mathcal{A}(A) &= \mathcal{A}(\Psi_Q) \;=\; -\frac{N_c^A}{N_f} \times \frac{1}{2} \times N_c^A \\
&= -\frac{(N_c^A)^2}{2N_f}, \\
\mathcal{A}(B) &= \mathcal{A}(\psi_q) \;+\; \mathcal{A}(\psi_\phi) \\
&= -\frac{N_c^B}{N_f} \times \frac{1}{2} \times N_c^B \;+\; \left(2\frac{N_c^B}{N_f} - 1\right) \times \frac{1}{2} \times N_f \\
&= -\frac{(N_c^B)^2}{2N_f} \;+\; \frac{2N_c^B - N_f}{2} \;=\; -\frac{(N_f - N_c^B)^2}{2N_f} \\
&= -\frac{(N_c^A)^2}{2N_f},
\end{aligned} \tag{S.9}$$

  and the anomalies match.

- The $[SU(N_f)_R]^2 \times U(1)_R$ anomaly: similar matching.

- The abelian $[U(1)_B]^2 \times U(1)_R$ anomaly.

  In both theories, this anomaly comes from the quarks and the antiquarks because the gauginos and the ψ_ϕ fermions (in B theory) have zero baryon numbers. Thus,

$$\begin{aligned}
\mathcal{A}(A) &= \mathcal{A}(\Psi_Q) \;+\; \mathcal{A}(\widetilde{\Psi}_Q) \\
&= \left(\frac{+1}{N_c^A}\right)^2 \times \left(-\frac{N_c^A}{N_f}\right) \times N_c^A N_f \;+\; \left(\frac{-1}{N_c^A}\right)^2 \times \left(-\frac{N_c^A}{N_f}\right) \times N_c^A N_f \\
&= -2, \\
\mathcal{A}(B) &= \mathcal{A}(\psi_q) \;+\; \mathcal{A}(\tilde{\psi}_q) \\
&= \left(\frac{+1}{N_c^B}\right)^2 \times \left(-\frac{N_c^B}{N_f}\right) \times N_c^B N_f \;+\; \left(\frac{-1}{N_c^B}\right)^2 \times \left(-\frac{N_c^B}{N_f}\right) \times N_c^B N_f \\
&= -2,
\end{aligned} \tag{S.10}$$

  and the anomalies match.

- The abelian $[U(1)_R]^3$ anomaly.

  This time we must account for all the elementary fermions of each theory since all of them have non-zero R–charges. Thus, for the A theory,

$$\begin{aligned}\mathcal{A}(A) &= \mathcal{A}(\Psi_Q) \;+\; \mathcal{A}(\widetilde{\Psi}_Q) \;+\; \mathcal{A}(\lambda) \\ &= \left(-\frac{N_c^A}{N_f}\right)^3 \times N_c^A N_f \times 2 \;+\; (+1)^3 \times \left((N_c^A)^2 - 1\right) \\ &= -\frac{2(N_c^A)^4}{N_f^2} \;+\; (N_c^A)^2 \;-\; 1,\end{aligned} \tag{S.11}$$

  while for the B theory,

$$\begin{aligned}\mathcal{A}(B) &= \mathcal{A}(\psi_q) \;+\; \mathcal{A}(\tilde{\psi}_q) \;+\; \mathcal{A}(\psi_\phi) \;+\; \mathcal{A}(\lambda) \\ &= \left(-\frac{N_c^B}{N_f}\right)^3 \times N_c^B N_f \times 2 \;+\; \left(2\frac{N_c^B}{N_f} - 1\right)^3 \times N_f^2 \;+\; (+1)^3 \times \left((N_c^B)^2 - 1\right) \\ &= -\frac{2(N_c^B)^4}{N_f^2} \;+\; 8\frac{(N_c^B)^3}{N_f} \;-\; 12(N_c^B)^2 \;+\; 6N_c^B N_f \;-\; N_f^2 \;+\; (N_c^B)^2 -\; 1 \\ &= \frac{2(N_f - N_c^B)^4}{N_f^2} \;+\; (N_f - N_c^B)^2 \;-\; 1,\end{aligned} \tag{S.12}$$

  and the anomalies match because $N_f - N_c^B = N_c^A$.

- The trace anomaly $\mathrm{tr}(R)$.

  Again, in each theory all fermions contribute to this anomaly, thus

$$\begin{aligned}\mathcal{A}(A) &= \mathcal{A}(\Psi_Q) \;+\; \mathcal{A}(\widetilde{\Psi}_Q) \;+\; \mathcal{A}(\lambda) \\ &= \left(-\frac{N_c^A}{N_f}\right) \times N_c^A N_f \times 2 \;+\; (+1) \times \left((N_c^A)^2 - 1\right) \\ &= -(N_c^A)^2 \;-\; 1, \\ \mathcal{A}(B) &= \mathcal{A}(\psi_q) \;+\; \mathcal{A}(\tilde{\psi}_q) \;+\; \mathcal{A}(\lambda) \;+\; \mathcal{A}(\psi_M) \\ &= \left(-\frac{N_c^B}{N_f}\right)^{\times} N_c^B N_f \times 2 \;+\; (+1) \times \left((N_c^B)^2 - 1\right) \;+\; \left(2\frac{N_c^B}{N_f} - 1\right) \times N_f^2 \\ &= -N_f^2 \;+\; 2N_f N_c^B \;-\; (N_c^B)^2 \;-\; 1 \\ &= -(N_f - N_c^B = N_c^A)^2 \;-\; 1,\end{aligned} \tag{S.13}$$

  and the trace anomalies match too. *Quod erat demonstrandum.*

Problem **2**(a):

The chiral ring of SQCD with any numbers of colors and flavors obeys on-shell equations

$$\mathcal{M}\times m \ \stackrel{\text{c.r.}}{=}\ m\times\mathcal{M}\ \stackrel{\text{c.r.}}{=}\ \frac{S}{16\pi^2}\times\mathbf{1}_{N_f\times N_f}\,. \tag{S.14}$$

Consequently, in any supersymmetric vacuum of the theory, the mesonic VEVs and the gaugino condensate are related as

$$\langle\mathcal{M}\rangle\times m\ =\ m\times\langle\mathcal{M}\rangle\ =\ \langle S\rangle\,16\pi^2\times\mathbf{1}_{N_f\times N_f}\,. \tag{S.15}$$

In particular, when any of the quark flavors are massless, the gaugino condensate $\langle S\rangle$ must vanish, and hence all the mesons involving the massive flavors also have zero VEVs. On the other hand, when $\operatorname{rank}(m)=N_f$ so that all the flavors are massive, the theory develops a non-zero gaugino condensate, which according to eq. (S.15) implies $\operatorname{rank}(\langle\mathcal{M}\rangle)=N_f$. Although classically, the rank of squark VEV matrices is limited by the $N_c^A<N_f$, the quantum correctons due to $\langle S\rangle\neq 0$ provide for $\langle\mathcal{M}\rangle\neq 0$ even in the absence of the Higgs mechanism.

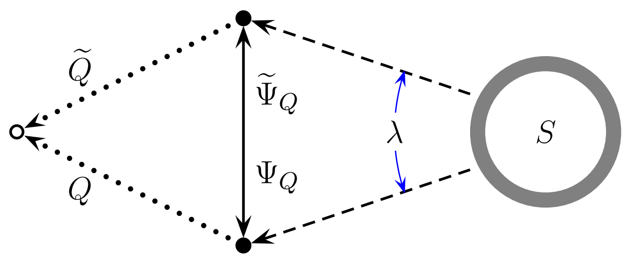
To see how this works, suppose all the masses are much larger then Λ_A. In this case, the gaugino condensate can be obtained from an effective IR theory from which all the quarks are integrated out perturbatively. Specifically, the IR theory is an SYM with

$$\Lambda_{\text{SYM}}^{3N_c^A}\ =\ \Lambda_A^{3N_c^A-N_f}\times\det(m), \tag{S.16}$$

so the gaugino condensate is

$$S\ =\ \Lambda^3_{\text{SYM}}\times\sqrt[N_c^A]{1}\ =\ \left(\Lambda_A^{3N_c^A-N_f}\times\det(m)\right)^{1/N_c^A}. \tag{S.17}$$

And once we have $\langle\lambda\lambda\rangle\neq 0$, the superpartner of the anomaly diagram in the massive theory



(S.18)

provides for quantum corrections

$$\langle \widetilde{Q} Q \rangle \;=\; \frac{1}{\text{mass}} \times \frac{\langle \text{tr}(\lambda\lambda) \rangle}{16\pi^2} \tag{S.19}$$

to the mesonic VEVs even though $\langle Q \rangle = \langle \widetilde{Q} \rangle = 0$. For multiple flavors we have the anomaly equations (S.15), hence

$$\left\langle \mathcal{M}^{ff'} \right\rangle \;=\; (m^{-1})^{ff'} \times \left(\Lambda_A^{3N_c^A - N_f} \times \det(m) \right)^{1/N_c^A}. \tag{S.20}$$

By holomorphy, we may analytically continue eqs. (S.17) and (S.20) from large quark masses to small masses. Earlier in class we did it for $N_f < N_c$ and found that mesonic VEVs grow large when masses become small. But for $N_f > N_c$ we have the opposite behavior: when all the masses become small, the mesonic VEVs become small rather then large,

$$\langle \mathcal{M} \rangle \;\propto\; m^{(N_f/N_c)-1} \quad\Longrightarrow\quad \langle \mathcal{M} \rangle \to 0 \text{ when } m \to 0. \tag{S.21}$$

Problem **3**(b):

Under Seiberg duality, the quark masses of the A theory become O'Raifeartaigh terms of the B theory, which require the non-zero VEVs of the B's squark-antisquark bilinears,

$$M^B_{ff'} \;\equiv\; \left\langle \tilde{q}_{fc} q^c_{f'} \right\rangle \;= \frac{\mu}{\lambda}\, m_{ff'}\,. \tag{8.a}$$

Classically, the matrix of such bilinears has rank $\le N_c^B$, but in the quantum theory, the Konishi anomaly allows for full rank $= N_f$ provided all the flavors are massive, *cf.* part (a). In the B theory, the quark masses come from the VEVs of the Φ fields, so what we need is $\det\bigl(\langle\Phi\rangle\bigr) \neq 0$. By analogy with part (a), we get

$$\lambda\Phi \times M^B \;\overset{\text{c.r.}}{=}\; M^B \times \lambda\Phi \;\overset{\text{c.r.}}{=}\; -\frac{S_B}{16\pi^2} \times \mathbf{1}_{N_f \times N_f}\,, \tag{8.b}$$

$$S_B^{N_c^B} \;\overset{\text{c.r.}}{=}\; \Lambda_B^{3N_c^B - N_f} \times \det(-\lambda\Phi). \tag{8.c}$$

Now let's solve the equations (8.a–c). Taking the determinants for both sides of eqs. (8.a) and (8.b) gives us

$$\begin{aligned} \det(M^B) &\overset{\text{c.r.}}{=} (\mu/\lambda)^{N_f} \times \det(m), \\ \det(-\lambda\Phi) \times \det(M^B) &\overset{\text{c.r.}}{=} \left(\frac{S_B}{16\pi^2}\right)^{N_f}, \end{aligned} \tag{S.22}$$

and hence

$$S_B^{N_f} \overset{\text{c.r.}}{=} (16\pi^2\mu/\lambda)^{N_f} \times \det(m) \times \det(-\lambda\Phi). \tag{S.23}$$

Comparing this formula to eq. (8.c), we obtain

$$S_B^{N_f - N_c^B} \overset{\text{c.r.}}{=} (16\pi^2\mu/\lambda)^{N_f} \times \det(m) \times \Lambda_B^{N_f - 3N_c^B} \tag{S.24}$$

and hence

$$\langle S_B \rangle = \left(\frac{(16\pi^2\mu/\lambda)^{N_f} \times \det(m)}{\Lambda_B^{3N_c^B - N_f}}\right)^{1/(N_f - N_c^B)}. \tag{S.25}$$

As to the $\langle\Phi\rangle$ matrix, combining eqs. (8.b) and (8.a) gives us

$$\langle\Phi\rangle = -\frac{\langle S_B \rangle}{16\pi^2} \times (\lambda M^B)^{-1} = -\frac{\langle S_B \rangle}{16\pi^2} \times (\mu m)^{-1} \tag{S.26}$$

and hence

$$\mu\Phi = -\frac{\langle S_B \rangle}{16\pi^2} \times m^{-1} = -m^{-1} \times \left(\frac{(16\pi^2)^{N_c^B} \mu^{N_f} \times \det(m)}{\lambda^{N_f} \Lambda_B^{3N_c^B - N_f}}\right)^{1/(N_f - N_c^B)}. \tag{S.27}$$

Note that this formula tells us that in order to get $\mu\Phi = +\mathcal{M} = +S_A \times m^{-1}$ we need $S_B = -S_A$. That's the reason for the minus sign in the first eq. (9).

Problem **3**(c):

Let's compare eqs. (S.17) and (S.25) for the gaugino condensates. Taking the $N_c^A = N_f - N_c^B$ powers of both sides of each equation, we get

$$\begin{aligned} S_A^{N_c^A} &= \det(m) \times \Lambda_A^{3N_c^A - N_f}, \\ S_B^{N_c^A} &= \det(m) \times \frac{(16\pi^2\mu/\lambda)^{N_f}}{\Lambda_B^{3N_c^B - N_f}}, \end{aligned} \tag{S.28}$$

Both sides have the same dependence on the quark mass matrix m, while the different Λ dependencies can be corrected by a suitable value of the μ parameter. Specifically, in order to get $S_B = -S_A$ we need

$$\frac{(16\pi^2\mu/\lambda)^{N_f}}{\Lambda_B^{3N_c^B - N_f}} = (-1)^{N_c^A} \Lambda_A^{3N_c^A - N_f} \tag{S.29}$$

and hence

$$\mu = \lambda \Big((-1)^{N_c^A} \Lambda_A^{3N_c^A - N_f} \Lambda_B^{3N_c^B - N_f} \Big)^{1/N_f}. \tag{5}$$

And once we obtain $S_B = -S_A$, equations (S.25) and (S.17) immediately lead to $\mu\Phi = +\mathcal{M}$. *Quod erat demonstrandum.*

Problem **3**(d):

In the A theory, $N_f - \mathrm{rank}(m)$ quark flavors remain massless. If the number of massless flavors is non-zero but less than N_c^A, the effective potential for the massless squarks behaves as $|\langle Q\rangle|^{\text{negative}}$ and does not have any minima. Instead, the squark vevs run away to infinity, and the A theory does not have a stable vacuum.

In the B theory, the dual O'Raifeartaigh terms lead to the squark VEVS and the Higgs mechanism. In a supersymmetric vacuum, the matrix of B-meson VEVs $\langle M^B \rangle$ should have the same rank as the mass matrix m — *cf.* eq. (8.a). But classically, this rank cannot exceed $\mathrm{rank}(q) = N_c^B$, while the quantum corrections either vanish altogether when $\langle S_B\rangle = 0$ or else lead to $\mathrm{rank}(\langle M_B\rangle) = N_f$ when $\langle S_B\rangle \neq 0$, *cf.* eq. (8.b). Thus, if $\mathrm{rank}(\langle M_B\rangle) < N_f$ then $\mathrm{rank}(\langle M_B\rangle) \geq N_c^B$.

In terms of the m matrix, this means that if $N_c^B < \mathrm{rank}(m) < N_f$, then eqs. (8.a) for the B-meson VEVs becomes impossible to satify. Since these equations must hold for any SUSY vacuum, it follows that the B theory does not have SUSY vacua — it either breaks supersymmetry or does not have any stable vacua at all.

In terms of A theory, the bad range of rank(m) for the B theory becomes

$$0 \ < \ N_f \ - \ \mathrm{rank}(m) \ < \ N_f \ - \ N_c^B \ = \ N_c^A\,, \tag{S.30}$$

i.e., the number of massless flavors is non-zero but less than N_c^A. And this is precisely the bad range for the A theory itself.

<u>Problem **3**(e)</u>:

In the A theory, integrating out the heavy quarks leaves us with the effective theory AE which is pure SQCD with

$$N_c^{AE} \ = \ N_c^A \quad \text{and} \quad N_f^{AE} \ = \ N_f \ - \ \mathrm{rank}(m) \ \geq N_c^{AE}. \tag{S.31}$$

In the B theory, the dual O'Raifeartaigh terms lead to the Higgs mechanism. In light of eq. (8.a), the number of broken B-colors is rank(m), so for $\mathrm{rank}(m) < N_f - N_c^A - 1 = N_c^B - 1$ part of the gauge symmetry survives unbroken. Thus, the low-energy limit of the Higgsed B theory is an SQCD-like theory BE with

$$N_c^{BE} \ = \ N_c^B \ - \ rank(m) \quad \text{colors.} \tag{S.32}$$

In SQCD, the supersymmetric Higgs mechanism eats a quark flavor for each Higgsed-down color, so the low-energy BE theory has

$$N_f^{BE} \ = \ N_f \ - \ \mathrm{rank}(m) \quad \text{flavors.} \tag{S.33}$$

Comparing eqs. (S.32) and (S.33) to (S.31), we immediately see that

$$N_f^{BE} \ = \ N_f^{AE} \ = \ N_c^{AE} \ + \ N_c^{BE}, \tag{S.34}$$

as appropriate for the Seiberg-dual theories.

Now let's check the color-singlet fields of both effective theories. The AE theory does not have any color singlets — it's a pure SQCD — but the BE theory has singlets coming from two sources: (1) the quarks and antiquarks broken colors left un-eaten by the Higgs mechanism, and (2) the Φ fields of the original B theory. However, many of these fields become massive due to their Yukawa couplings to the squark and antisquark VEVs.

To sort out the mass terms, let's work in the basis where the quark VEVs are diagonal,

$$\left\langle q_f^c \right\rangle \;=\; \left\langle \tilde{q}_{c,f} \right\rangle \;=\; \delta_{cf} \times \sqrt{\frac{\mu}{\lambda} m_f} \quad \text{for } c = f = 1, \ldots, \text{rank}(m) \text{ only.} \tag{S.35}$$

Consequently, the Yukawa couplings

$$W^B \;\supset\; \lambda \sum_{c,f,f'} \Phi^{f,f'} q_f^c \tilde{q}_{c,f'} \tag{S.36}$$

give raise to the mass terms

$$W \supset \sum_{i=1}^{\text{rank}(m)} \sum_{f=1}^{N_f} \sqrt{\lambda \mu m_i} \Big(q_f^i \times \Phi^{f,i} \;+\; \Phi^{i,f} \times \tilde{q}_{i,f} \Big). \tag{S.37}$$

In particular, all the $\Phi^{f,f'}$ singlets with $f \leq \text{rank}(m)$ or $f' \leq \text{rank}(m)$ (or both) become massive. On the other hand, the $\phi^{f,f'}$ with $f, f' > \text{rank}(m)$ remain massless and remain in the low-energy BE theory; their flavor quantum numbers correspond to a single $(\mathbf{N_f^{BE}}, \mathbf{N_f^{BE}})$ multiplet.

As to the quarks and antiquarks of the B theory, the mass terms (S.37) include masses for all the q_f^c and $\tilde{q}_{c,f}$ with broken colors $c = 1, \ldots, \text{rank}(m)$ and unbroken flavors $f = \text{rank}(m) + 1, \ldots N_f$. For broken colors and broken flavors $c, f = 1, \ldots, \text{rank}(m)$, one combination of q_f^c and $\tilde{q}_{c,f}$ becomes massive via (S.37), while the other combination is eaten by the supersymmetric Higgs mechanism. Finally, the quarks and antiquarks with broken flavors and unbroken colors are also eaten by the Higgs mechanism. Thus, the only massless survivors are q_f^c and $\tilde{q}_{c,f}$ with unbroken colors $c = \text{rank}(m) + 1, \ldots, N_c^B$ and also unbroken flavors $f = \text{rank}(m) + 1, \ldots N_f$. These massless quarks and antiquarks provide the N_c^{BE} colors and N_f^{BE} flavors of the low-energy BE theory.

Altogether, the only singlets of the BE theory are the $\Phi^{f,f'}$ with unbroken flavors, which is precisely what we need for the Seiberg dual of the AE theory.

<u>*Nota bene:*</u> The above argument does not depend on the conformal window. It works equally well for $N_f^{AE} > \frac{3}{2}N_c^{AE}$ — in which case the AE and BE theories are both superconformal — and for $N_f^{AE} \leq \frac{3}{2}N_c^{AE}$, which makes the AE theory confining while the BE theory is IR-free. In the latter case, we have the second kind of the Seiberg duality in which the AE and the BE theories describe the different energy ranges of the the same physical theory.

<u>Problem **3**(f):</u>
For the A theory with $N_f - \text{rank}(m) = N_c^A$ or $N_c^A + 1$, integrating out the massive quarks leaves us with an effective SQCD with $N_f = N_c$ or $N_f = N_c + 1$. In both cases the effective theory is confining and has a continuous family of supersymmetric vacua parametrized by the mesonic and baryonic VEVs. In the deep IR limit, the effective AE theory becomes a non-linear sigma model of the $\mathcal{M}$, $\mathcal{B}$, and $\widetilde{\mathcal{B}}$ superfields.

In the B theory, the O'Raifeartaigh terms of rank $= N_f - N_c^A = N_c^B$ or $N_c^B - 1$ lead to complete Higgsing of the $SU(N_c^B)$ gauge symmetry, so the low-energy BE theory does not have any vector multiplets at all. As to the quark, antiquark, and singlet fields, we may analyze their masses in the same manner as we did in part (e). In particular, for $\text{rank}(m) = N_c^B - 1$, we find that the massless chiral superfields comprise

$$\text{one color and} N_f^{BE} = N_f^{AE} \text{ flavors} \tag{S.38}$$

or quarks and antiquarks, plus $(N_f^{BE})^2$ singlet fields $\Phi^{f,f'}$. The quantum numbers of these fields with respect to the unbroken flavor symmetry $SU(N_f^{BE})_L \times SU(N_f^{BE})_r \times U(1)_V$

$$q \in (\overline{\mathbf{N_f^{BE}}}, \mathbf{1}, V = +1), \quad \tilde{q} \in (\mathbf{1}, \overline{\mathbf{N_f^{BE}}}, V = -1), \quad \Phi \in (\mathbf{N_f^{BE}}, \mathbf{N_f^{BE}}, V = 0). \tag{S.39}$$

match the quantum numbers of the baryon, antibaryon, and meson fields of the low-energy limit of the A theory, which suggests the following low-energy Seiberg duality:

$$\mathcal{B}_f \leftrightarrow q_f\,, \quad \widetilde{\mathcal{B}}_f \leftrightarrow \tilde{q}_f\,, \quad \mathcal{M}^{f,f'} \leftrightarrow \Phi^{f,f'}. \tag{S.40}$$

Now consider the moduli spaces of the two low-energy theories, or at least the complex structures of the moduli spaces which follow from the chiral ring equations. In homework set#9, we saw that the off-shell chiral ring equations of SQCD with $N_f = N_c + 1$ — such as A theory after integrating out the massive flavors — may be emulated in the effective low-energy theory by the on-shell equations stemming from an effective superpotential

$$W_{\text{eff}}^{AE} \;=\; C \times \Big(\mathcal{B}_f \mathcal{M}^{f,f'} \widetilde{\mathcal{B}}_{f'} \;-\; \det(\mathcal{M}) \Big) \tag{S.41}$$

where

$$C \;=\; \frac{O(1)\ \text{numeric constant}}{\left(\Lambda^{3N_c - N_f}\right)_{AE}}\,. \tag{S.42}$$

In terms of the dual BE theory, the superpotential (S.41) translates to

$$W_{\text{eff}}^{BE} \;=\; C\mu\rho^2 \times q_f \Phi^{f,f'} \tilde{q}_{f'} \;-\; C\mu^{N_f^{BE}} \times \det(\Phi) \tag{S.43}$$

where μ and ρ are dimensionful constants relating the non-canonical fields of the AE and BE theories,

$$\mathcal{M} \;=\; \mu\Phi, \quad \mathcal{B} \;=\; \rho q, \quad \widetilde{\mathcal{B}} \;=\; \rho\tilde{q}. \tag{S.44}$$

The first term in the superpotential (S.43) has the form of tree-level Yukawa couplings

$$W_{\text{tree}}^{BE} \;=\; \lambda \sum_{f,f'} \Phi^{f,f'} q_f \tilde{q}_{f'} \tag{S.45}$$

inherited by the BE from the higher-energy B theory, provided we identify $C\mu\rho^2 = \lambda$. Since we do not have any other constraints on the ρ factor, we may use this identification as the equation for the ρ. If we take μ as in eq. (5) for the low-energy AE and BE theories,

$$\mu \;=\; \lambda \times \left(\Lambda_{AE}^{2N-1} \times \Lambda_B^{2-N} \right)^{1/(N+1)} \times \text{a numeric constant} \tag{S.46}$$

where $N = N_c^a = N_c^{AE}$ (so that $N_f^{AE} = N_f^{BE} = N + 1$), Λ_{AE} is the IR-strong scale of the

AE theory, and Λ_B is the UV Landau pole of the BE theory, then $C\mu\rho^2 = \lambda$ requires

$$\rho \;=\; \Lambda_A^{N-1} \times \left(\frac{\Lambda_B}{\Lambda_A}\right)^{(N-2)/2(N+1)} \times \text{a numeric constant.} \tag{S.47}$$

As to the second term in eq. (S.43), eq. (S.46) for the μ leads to

$$C\mu^{N+1} \;=\; \lambda^{N+1} \times \Lambda_{BE}^{2-N} \times \text{a numeric constant,} \tag{S.48}$$

but independent of the Λ_A. Consequently, the second term in the effective superpotential of the BE theory seems to be a nonperturbative effect due to instantons of the broken gauge symmetries of the higher-energy B theory. Indeed, writing the second term as

$$\begin{aligned} W_{\text{np}}^{BE} \;&=\; \text{numeric constant} \times \Lambda_{BE}^{N-2} \times \det(\lambda\Phi) \\ &=\; \text{numeric constant} \times \frac{\left(\Lambda^{3N_c-N_f}\right)_B}{\text{minor}(M^B)} \times \det(\lambda\Phi) \end{aligned} \tag{S.49}$$

we see the dependence on the Λ_B which is appropriate for a one-instanton effect.

Finally, consider the case of $N_f - \text{rank}(m) = N_c^A$. In this case, integrating the massless flavors form the A theory leads to an effective theory with $N_c = N_f$. Again, the IR limit of this theory is the non-linear sigma model of the mesons and baryons, but this time instead of an effective superpotential (S.41), the AE theory has a constraint

$$\det(\mathcal{M}) \;-\; \mathcal{B} \times \widetilde{\mathcal{B}} \;=\; \Lambda_{AB}^{2N}. \tag{S.50}$$

As to the B theory, integrating out all fields which become massive due to the Higgs mechanism or due to Yukawa couplings, we are left with N^2 $\Phi f, f'$ fields plus a single combination $\hat{q}$ of all the quark and the antiquark fields. As usual, we may identify the Φ fields as dual to the meson fields $\mathcal{M}$ of the AE theory, but the dual of the $\hat{q}$ field is obscure.

The moduli space of the constraint (S.50) has a complex line

$$\mathcal{B} \times \widetilde{\mathcal{B}} \;=\; -\Lambda^{2N} \tag{S.51}$$

where the $SU(N)_L \times SU(N)_R$ chiral symmetry remains unbroken. *In the vicinity of this line*, we may identify $\hat{q}$ with some baryonic modulus along this line, for example

$$\hat{q} \;\longleftrightarrow\; \log \frac{\widetilde{\mathcal{B}}}{\mathcal{B}}\,. \tag{S.52}$$

However, far away from this line — and especially at the points where $\mathcal{B} = 0$, or $\widetilde{\mathcal{B}} = 0$, or both — this identification becomes problematic.