INTERFERENCE, BEATS, AND STANDING WAVES

Harmonic Interference in General

Consider a superposition of two signals of some kind — displacements of a rope, pressures in a sound wave, voltages in an electric circuit, whatever — that change harmonically with time with the same frequency f,

$$y_{\rm net} = A_1 \times \sin(2\pi f t + \phi_1) + A_2 \times \sin(2\pi f t + \phi_2). \tag{1}$$

This superposition looks complicated, but if the two signals have similar amplitudes $A_1 = A_2 = A$, we may use a trigonometric identity

$$\sin\alpha + \sin\beta = 2\cos\frac{\alpha-\beta}{2} \times \sin\frac{\alpha+\beta}{2} \tag{2}$$

to re-write the two-signal superposition (1) as a single harmonic signal

$$y = 2A\cos\frac{\phi_1-\phi_2}{2} \times \sin(2\pi f t + \phi_{\rm avg}) \tag{3}$$

with amplitude

$$\mathcal{A} = 2A \times \left|\cos\frac{\phi_1-\phi_2}{2}\right|. \tag{4}$$

Note that this amplitude depends on the phase difference $\Delta\phi = \phi_1 - \phi_2$ between the two original signals.

The power of a signal is proportional to the square of its amplitude — $P = CA^2$ for some constant coefficient C. Thus, each of the two original signals has power $P_{\rm single} = CA^2$ while the combined signal has power

$$P_{\rm comb} = C\mathcal{A}^2 = C\times 4A^2\cos^2\frac{\Delta\phi}{2} = 2CA^2\times\bigl(1+\cos(\Delta\phi)\bigr) = 2P_{\rm single} + 2P_{\rm single}\times\cos(\Delta\phi). \tag{5}$$

The first term on the right hand side is simply the net power of two single signals, while the second phase-dependent term is due to interference. For $\Delta\phi = 0, \pm2\pi, \pm4\pi, \pm6\pi, \ldots$, $\cos(\Delta\phi) = +1$ and the interference is *constructive:* the combined signal has more power than the two single signals put together. On the other hand, for $\Delta\phi = \pm\pi, \pm3\pi, \pm5\pi, \ldots$, $\cos(\Delta\phi) = -1$ and the interference is *destructive:* the two signals cancel each other.

When the two original signals have different amplitudes $A_1 \neq A_2$ and hence different powers P_1 and P_2, the mathematics of interference is more complicated, but the net result for the combined power is similar to eq. (5):

$$P_{\rm comb} \;=\; P_1 \;+\; P_2 \;+\; 2\sqrt{P_1 P_2} \times \cos(\Delta\phi). \tag{6}$$

In particular, the interference of the two signals depends on the phase difference $\Delta\phi = \phi_1 - \phi_2$ in exactly the same way: For $\Delta\phi = 0, \pm 2\pi, \pm 4\pi, \pm 6\pi, \ldots$ the interference is constructive and the combined signal has more power than $P_1 + P_2$, while for $\Delta\phi = \pm\pi, \pm 3\pi, \pm 5\pi, \ldots$ the interference is destructive and the combined signal has less power than $P_1 + P_2$.

Interference Between two Speakers

Now consider interference between sound waves produced by two speakers playing the same harmonic signal. A single speaker would produce a spherical harmonic wave expanding in all directions of space: the pressure deviation $\delta\mathcal{P}$ depends on location (x, y, z) in 3D space and time t as

$$\delta\mathcal{P}(x,y,z,t) \;=\; \frac{A}{r} \times \sin\left(2\pi f \times \left(t \,-\, \frac{r}{u}\right)\right) \;=\; \frac{A}{r} \times \sin\left(2\pi f t \,-\, \frac{2\pi r}{\lambda}\right) \tag{7}$$

where $r(x, y, z)$ is the distance between the speaker and the point (x, y, z), u is the speed of sound, f is the frequency, and $\lambda = u/f$ is the wavelength. The phase of this acoustic signal at point (x, y, z)

$$\phi \;=\; 2\pi f \times \frac{-r}{u} \;=\; -\frac{2\pi r}{\lambda} \tag{8}$$

depends on time r/u the sound needs to travel from the speaker to that point.

When two speakers powered by the same electric signal are playing at the same time, the pressure deviations add up:

$$\delta\mathcal{P}_{\rm net}(x,y,z,t) \;=\; \frac{A}{r_1} \times \sin\left(2\pi f t \,-\, \frac{2\pi r_1}{\lambda}\right) \;+\; \frac{A}{r_2} \times \sin\left(2\pi f t \,-\, \frac{2\pi r_2}{\lambda}\right) \tag{9}$$

where r_1 and r_2 are the distances from the point (x, y, z) to the two speakers. In general, these two distance are different, which leads to interference between between two signals with different phases:

$$\phi_1 \;=\; -\frac{2\pi r_1}{\lambda} \quad \text{and} \quad \phi_2 \;==\; -\frac{2\pi r_2}{\lambda}\,. \tag{10}$$

The phase difference is

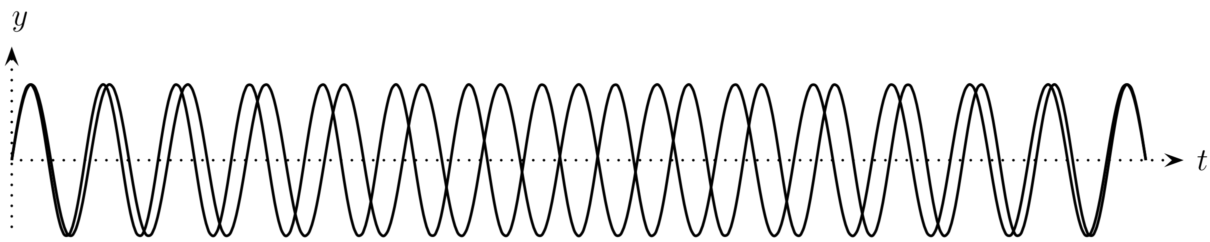
$$\Delta\phi \;=\; 2\pi \times \frac{r_2 - r_1}{\lambda}\,, \tag{11}$$

so *the interference is constructive when* $r_2 - r_1 = 0, \pm\lambda, \pm 2\lambda, \pm 3\lambda, \ldots$ *and destructive when* $r_2 - r_1 = \pm\frac{1}{2}\lambda, \pm\frac{3}{2}\lambda, \pm\frac{5}{2}\lambda, \ldots$

Beats

Now consider two harmonic signals with slightly different frequencies,

$$y_1(t) \;=\; A \times \sin(2\pi f_1 t + \phi_1) \quad \text{and} \quad y_2(t) \;=\; A \times \sin(2\pi f_2 t + \phi_2) \tag{12}$$



If the frequency difference $f_1 - f_2$ is relatively small, we can treat it as a time-dependent phase difference. Indeed, let's apply trigonometric identity (2) to the superposition

$$y_{\rm net}(t) \;=\; y_1(t) \;+\; y_2(t) \;=\; A\Big(\sin(2\pi f_1 t + \phi_1) \;+\; \sin(2\pi f_2 t + \phi_2)\Big). \tag{13}$$

For $\alpha = 2\pi f_1 t + \phi_1$ and $\beta = 2\pi f_2 t + \phi_2$, we have

$$\frac{\alpha+\beta}{2} \;=\; \pi(f_1 + f_2) \times t \;+\; \tfrac{1}{2}(\phi_1 + \phi_2)\,, \quad \frac{\alpha-\beta}{2} \;=\; \pi(f_1 - f_2) \times t \;+\; \tfrac{1}{2}(\phi_1 - \phi_2),$$

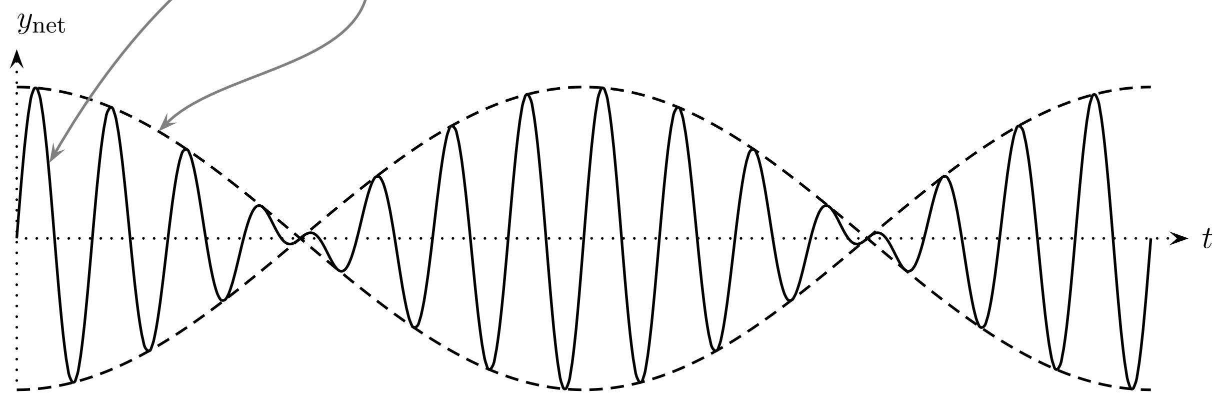
and hence

$$y_{\rm net}(t) \;=\; 2A\cos\Big(\pi(f_1 - f_2) \times t \;+\; \tfrac{1}{2}(\phi_1 - \phi_2)\Big) \times \sin\Big(\pi(f_1 + f_2) \times t \;+\; \tfrac{1}{2}(\phi_1 + \phi_2)\Big). \tag{14}$$

If the frequency difference $\Delta f = f_1 - f_2$ is much smaller than the average frequency $f_{\rm avg} = \frac{1}{2}(f_1 + f_2)$, we can treat the second factor here as a harmonic signal with frequency $f_{\rm avg}$

while the first factor acts as a *time dependent amplitude:*

$$\begin{aligned} y(t) &= \mathcal{A}(t) \times \sin(2\pi f_{\text{avg}} \times t + \phi_{\text{avg}}), \\ \text{where } \mathcal{A}(t) &= 2A \cos\Bigl(\pi \Delta f \times t + \tfrac{1}{2}(\phi_1 - \phi_2)\Bigr). \end{aligned} \tag{15}$$



The amplitude depends on time because the phase difference between the two signals grows with time as $2\pi\Delta f \times t$ + const. Consequently, the interference between the two signals alternates between constructive and destructive, so the signal power goes up and down. This is knows as the *beats.*

Indeed, the power of the combined signal works similarly to eq. (5), but with a time-dependent phase difference:

$$P_{\text{net}} = 2P_{\text{single}} + 2P_{\text{single}} \times \cos(2\pi\Delta f \times t + \text{const}), \tag{16}$$

or more generally, for two signals of different powers

$$P_{\text{net}} = P_1 + P_2 + 2\sqrt{P_1 P_2} \times \cos(2\pi\Delta f \times t + \text{const}). \tag{17}$$

In any case, the interference term here oscillates with *beat frequency*

$$f_{\text{beat}} = |\Delta f| = |f_1 - f_2|. \tag{18}$$

For example, if the two signals have frequencies $f_1 = 1234$ Hz and $f_2 = 1233$ Hz that differ by just 1 Hertz, their superposition beats once every second.

Standing waves

Let's excite a harmonic traveling wave on a string,

$$y(x,t) \;=\; A\,\sin\left(-\frac{2\pi}{\lambda}\times(x-ut)\right) \;=\; A\,\sin\left(2\pi ft\;-\;\frac{2\pi x}{\lambda}\right). \tag{19}$$

When this wave reaches the end of the string, it is reflected and travels back. Then its is reflected again from the other end, *etc.*, *etc.* Eventually, we end up with a superposition of two waves traveling in opposite directions, one to the right and the other to the left,

$$y(x,t) \;=\; A_R\times\sin\left(2\pi ft\;-\;\frac{2\pi x}{\lambda}\right)\;+\;A_L\times\sin\left(2\pi ft\;+\;\frac{2\pi x}{\lambda}\right). \tag{20}$$

Suppose both traveling waves have equal amplitudes $A_L = A_R = A$. In this case we may simplify the net wave using the trigonometric identity (2) for

$$\alpha \;=\; 2\pi ft\;-\;\frac{2\pi x}{\lambda}\,,\quad \beta \;=\; 2\pi ft\;+\;\frac{2\pi x}{\lambda}\,, \tag{21}$$

and consequently

$$\frac{\alpha+\beta}{2} \;=\; 2\pi ft \quad\text{and}\quad \frac{\alpha-\beta}{2} \;=\; -\frac{2\pi x}{\lambda}\,. \tag{22}$$

Therefore, the combined wave can be written as

$$y(x,t) \;=\; 2A\,\cos\left(\frac{2\pi x}{\lambda}\right)\times\sin(2\pi ft). \tag{23}$$

This function of x and t is a product of a sine wave of t and a cosine wave of x, so it describes a *standing wave.*

Note the power of superposition: By superposing two traveling waves moving in opposite direction, we got a standing wave! Likewise, it is possible to superpose two standing waves and get a traveling wave, but I will not do it here.

In the standing wave (23), every point x of the string oscillates with with exactly the same phase (modulo the overall sign). In particular, every point x crosses $y = 0$ at exactly the same time, so the whole string becomes momentarily flat. On the other hand, the amplitude of oscillations

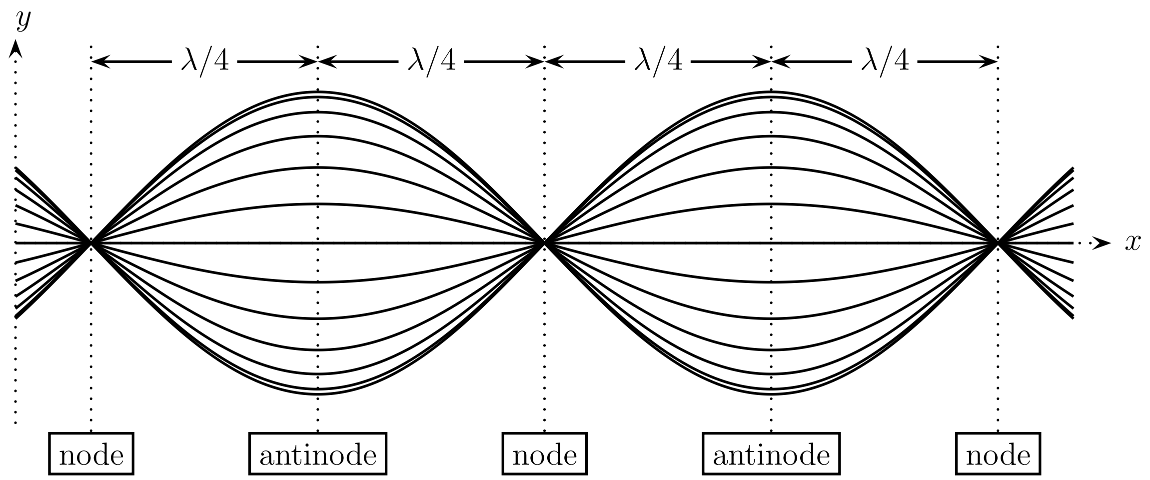
$$\mathcal{A}(x) \;=\; 2A\,\cos\frac{2\pi x}{\lambda} \tag{24}$$

depends on position x. Some points x where $\mathcal{A}(x) = 0$ do not oscillate at all; they are called *nodes* of the standing waves. For the wave (23) the nodes are located at zeros of the cosine $\cos(2\pi x/\lambda)$, thus

$$x_{\text{node}} \;=\; \pm\tfrac{1}{4}\lambda, \pm\tfrac{3}{4}\lambda, \pm\tfrac{5}{4}\lambda, \ldots \tag{25}$$

For other standing waves all the nodes may be shifted to the right or to the left, but the distances between the nodes are always the same: *half-wavelength $\lambda/2$ between each node and the next one.*

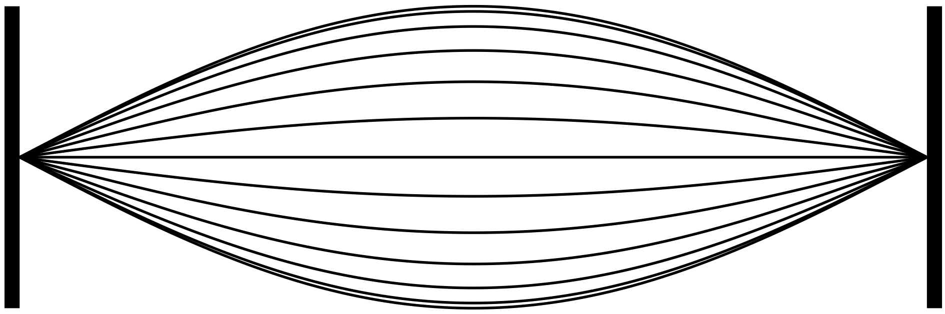
The points of a standing wave where the oscillation amplitude $\mathcal{A}(x)$ is maximal are called the *antinodes*. Unlike the nodes located at zeros of the cosine $\cos(2\pi x/\lambda)$, the antinodes are points where this cosine becomes $+1$ or -1. For the wave (23), the antinodes are at $x = 0, \pm\frac{1}{2}\lambda, \pm\lambda, \pm\frac{3}{2}\lambda, \ldots$ but other standing wave can have all the antinodes shifted left or right. But in any standing wave, *the antinodes are located at mid-points between the nodes.*



Resonances and Harmonics

Suppose a string is tied up — or otherwise fixed in place — at both ends. For example, the playing part of the guitar string is fixed to the deck at both ends. If you want to set up a standing wave on such a string, it must have a node at each fixed end. There could also be additional nodes in the middle of the string, but they are optional.

Suppose a standing wave has just two nodes at the ends of the string but no other nodes.



The distance between the two nodes of a standing wave is $\lambda/2$ — half of the wavelength. But if the nodes happen to be at the two ends of the string, the distance between them must be equal to the string's length L, which calls for

$$L = \frac{\lambda}{2}. \tag{26}$$

In other words, if a standing wave on a string has just the two required nodes at the string's ends and no other nodes, then this standing wave must have wavelength

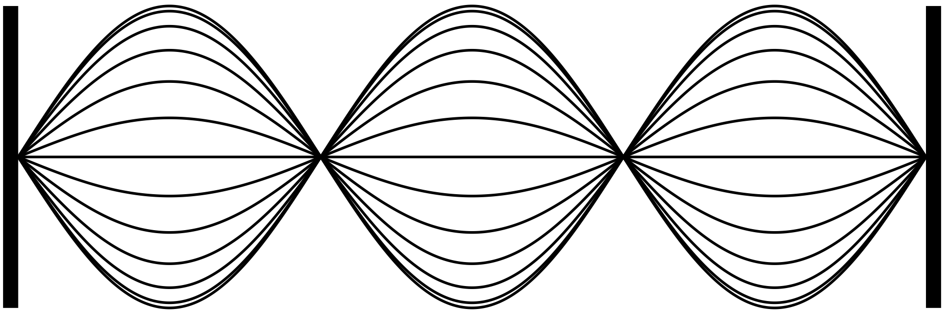
$$\lambda = 2L. \tag{27}$$

Consequently, it must have a specific frequency

$$f = \frac{u}{\lambda} = \frac{u}{2L} \tag{28}$$

where u is the speed of waves on the string.

Now suppose a standing wave has $n+1 = 3, 4, 5, \ldots$ nodes: one node at each end, plus $n-1$ additional nodes in the middle.



$n = 3$

The nodes are at equal distances $\lambda/2$ from each other, so they must divide the string's length L into n inter-node intervals, with an antinode — a point of maximal amplitude — in the middle of each interval. The length of each interval is $\lambda/2$, so the net length of the string must be equal to n such intervals,

$$L = n \times \frac{\lambda}{2}, \quad n = 2, 3, 4, \ldots. \tag{29}$$

Thus, a standing wave with a node at each end of the string and $n-1$ nodes in the middle must have wavelength

$$\lambda_n = \frac{2L}{n} \tag{30}$$

Consequently, the frequency of this wave must be

$$f_n = \frac{u}{\lambda_n} = \frac{u}{2L/n} = n \times \frac{u}{2L} \tag{31}$$

For $n = 1$ we have a wave with no nodes in the middle. Its wavelength and frequency are given by eqs. (26) and (28) — which are indeed special cases of eqs. (30) and (31).

Altogether, there is a discrete series of allowed standing waves on a string; they are called *modes*. Each such mode has a specific frequency (31), and there is a simple relation between them:

$$f_n = n \times f_1 . \tag{32}$$

This relation is very important for music because harmony of sounds depends on their frequencies being integer multiples of a common frequency. This is true for all modes of any particular string: their frequencies — called the *harmonics* — are integer multiples of the *fundamental frequency*

$$f_1 = \frac{u}{2L}. \tag{28}$$

The fundamental frequency itself is the lowest harmonic (also called the fundamental harmonic), the second harmonic is $f_2 = 2f_1$, the third harmonic is $f_3 = 3f_1$, *etc.*, *etc.*

Strictly speaking, a *harmonic* of a string is a frequency with with it can oscillate, while the standing wave oscillating with that frequency is a *mode*. However, people often use the word *harmonic* to mean the standing wave itself — *i.e.*, the mode — rather than just its frequency. This terminology is incorrect, but it's rather common. The textbook for this class uses it, so I might use it myself in class or in homeworks.

Many musical instruments use vibrating strings. Striking, plucking, or bowing a string creates some kind of a wave in it; usually this wave is not a single mode but a superposition of many modes. Consequently, the sound created by the string is a superposition of many harmonics. Indeed, such multi-harmonic sounds are much more pleasing to the human ear than a single-frequency sine wave. But since all the harmonics are multiples of the fundamental frequency f_1 of the string, the whole multi-harmonic sound wave is periodic with period $T = 1/f_1$. Consequently, the fundamental frequency of a string controls the *pitch* of the sound it produces. In musical notations, note names correspond to pitches, *i.e.*, fundamental frequencies; for example, the "middle A" (the A note of the middle octave) has fundamental frequency 440 Hz.

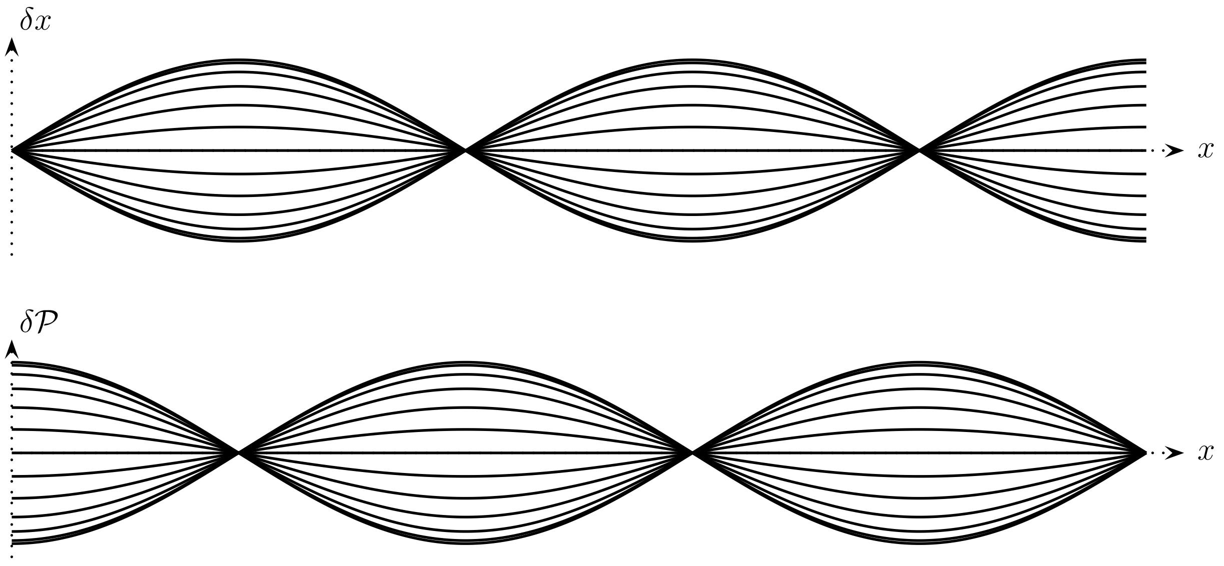
In music, the *timbre* of a sound is just as important as its pitch. Physically, the timbre corresponds to the way the sound power is distributed between different harmonics. In a string instrument, the timbre depends on how exactly is the string plucked, struck, or bowed. The point of the string that's plucked, struck or bowed is also important. For example, the usual way of plucking a guitar string produces sound with most power in the first three harmonics f_1, f_2, f_3, but plucking a string closer to the bridge would give more power to the higher harmonics f_4, f_5, f_6, f_7.

You don't have to pluck, strike, or bow a string to make it vibrate. If there is some vibration in the deck, or even sound in the air, it would act on the string and make a wave in it. For a generic frequency of the external vibration or sound, the string wave it creates is rather weak. However, if the frequency of the external vibration or sound matches one of the string's harmonics, then the string's response would be much stronger. This is called the *resonance.*

Many other physical system can resonate in response to a perturbation with just the right frequency. For example, if a pendulum is pushed and pulled with a weak alternating force whose frequency happens to match the frequency with which the free pendulum would swing, this little perturbation can make the pendulum swing with a very large amplitude. But while the pendulum would resonate at only one frequency, the string has a whole series of harmonics, and it can resonate to any of them.

Standing Waves in Air Columns

A sound wave is described by two related quantities, the displacement $\delta x(x,t)$ of the air forward or backward, and the pressure deviation from the average $\delta P(x,t) = P(x,t) - P_{\rm avg}$. In a standing wave, both of these quantities oscillate with the same frequency and wavelength, but they have different nodes (points where the amplitude vanishes) and antinodes (points where the amplitude is maximal). In fact, *the nodes of pressure are antinodes of displacement while the nodes of displacement are antinodes of pressure.*



Therefore, the distance between two nearby nodes of pressure is $\lambda/2$, the distance between two nearby nodes of displacement if also $\lambda/2$, but the distance between a pressure node and the nearest displacement node is only half of that, namely $\lambda/4$. Also, pressure and displacement nodes alternate every $\lambda/4$: pressure node, displacement node, pressure node, displacement node, pressure node, displacement node, *etc.*., *etc.*.

When a standing wave is established in air inside a pipe, the wave must have nodes at both ends of the pipe. However, the type of a node — a pressure node or a displacement node — depends on whether the pipe end in question is closed or open to the atmosphere. A closed end of the pipe must be a node of displacement since the air cannot be displaced through a closed end. On the other hand, an open end of the pipe has constant pressure — equal to the pressure outside the pipe — so it must be a node of pressure rather than displacement. Consequently, the allowed standing waves in a pipe depend not only on the pipe's length L but also on whether its ends are both closed, both open, or one closed and one open.

Let's start with a pipe with two open ends. A standing wave in this pipe must have pressure nodes at both ends. If there are no additional *pressure* nodes in the middle of the pipe, the distance between the two nodes at pipe's ends is half-wavelength, which calls for $L = \frac{1}{2}\lambda$. This gives us the fundamental harmonic of the pipe: its wavelength (in air) is

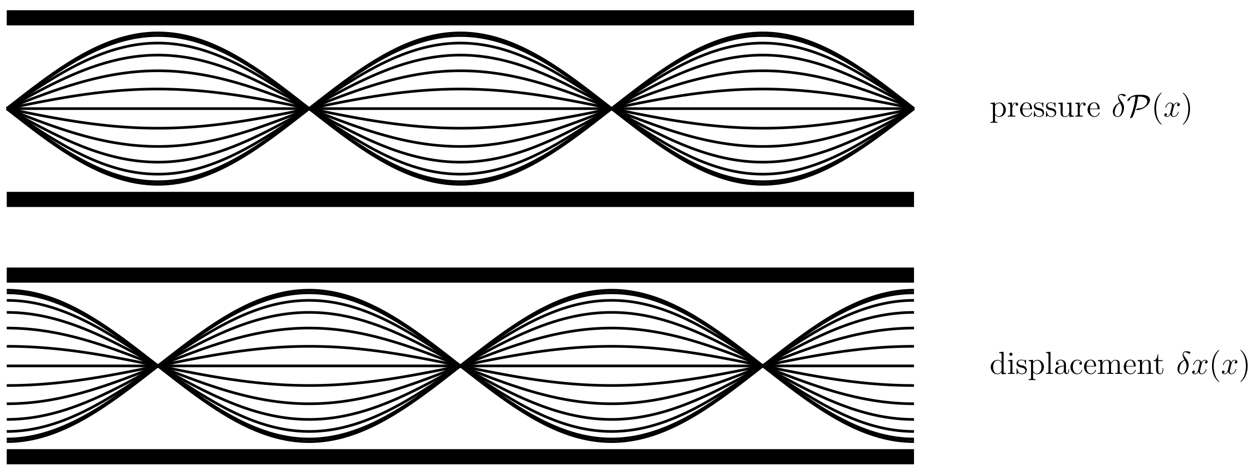
$$\lambda_1^{\text{open-open}} \;=\; 2L \tag{33}$$

and frequency

$$f_1^{\text{open-open}} \;=\; \frac{u}{\lambda_1^{\text{open-open}}} \;=\; \frac{u}{2L} \tag{34}$$

where u is the speed of sound in the air inside the pipe.

The higher modes in a pipe with two open ends have $n+1$ pressure nodes: one node at each end of the pipe, plus $n-1$ nodes in the middle. Here is the diagram of the $n=3$ mode:



Focusing on the pressure wave, we see the pipe divided into n intervals between the pressure node; since each interval has length $\lambda/2$, this calls for

$$L = n \times \frac{\lambda}{2}. \tag{35}$$

Consequently, the $n^{\underline{\text{the}}}$ harmonic of the pipe with 2 open ends has wavelength (in air)

$$\lambda_n^{\text{open}-\text{open}} = \frac{2L}{n} \tag{36}$$

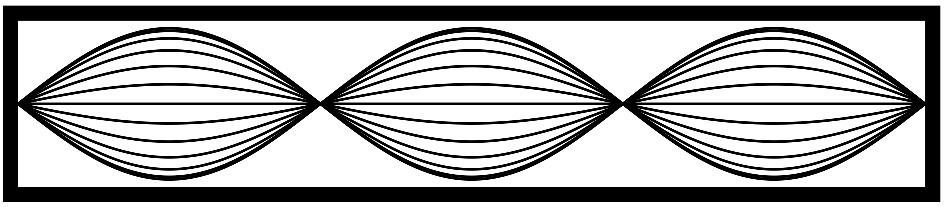
and frequency

$$f_n^{\text{open}-\text{open}} = \frac{u}{\lambda_n^{\text{open}-\text{open}}} = \frac{u}{2L/n} = n \times \frac{u}{2L}. \tag{37}$$

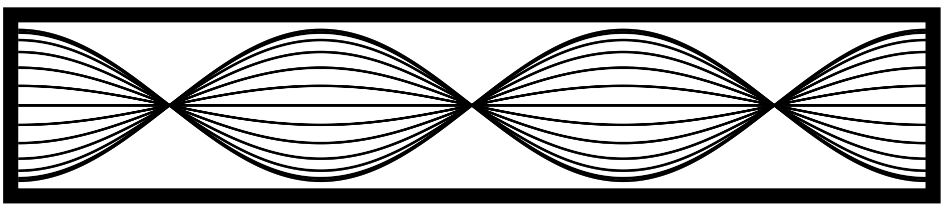
Similar to the string, the harmonics of an open-open pipe are integer multiples of the fundamental frequency,

$$f_n^{\text{open}-\text{open}} = n \times f_{\text{fundamental}}^{\text{open}-\text{open}}, \quad f_{\text{fundamental}}^{\text{open}-\text{open}} = \frac{u_{\text{sound}}}{2L}. \tag{38}$$

A pipe with two closed ends works similarly to the open-open case, only the pressure and the displacement nodes of a standing wave trade places: Both closed ends must be nodes of displacement rather than pressure. The fundamental mode of the standing wave has only two displacement nodes — one at each end of the pipe — while the higher modes have additional displacement nodes in the middle of the pipe. Here is the diagram for the $n = 3$ mode:



displacement $\delta x(x)$



pressure $\delta\mathcal{P}(x)$

Focusing on the displacement wave, we see the pipe divided into n intervals between the pressure node; since each interval has length $\lambda/2$, this calls for

$$L \;=\; n \times \frac{\lambda}{2}\,. \tag{39}$$

Consequently, the $n^{\underline{\text{the}}}$ harmonic of the pipe with 2 closed ends has wavelength (in air)

$$\lambda_1^{\text{closed}-\text{closed}} \;=\; \frac{2L}{n} \tag{40}$$

and frequency

$$f_n^{\text{closed}-\text{closed}} \;=\; \frac{u}{\lambda_n^{\text{closed}-\text{closed}}} \;=\; \frac{u}{2L/n} \;=\; n \times \frac{u}{2L}\,. \tag{41}$$

Note that the fundamental frequency

$$f_{\text{fundamental}}^{\text{closed}-\text{closed}} \;=\; \frac{u_{\text{sound}}}{2L} \tag{42}$$

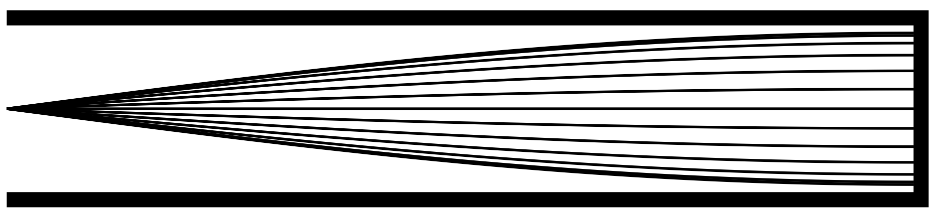
and all the harmonics

$$f_n^{\text{closed}-\text{closed}} \;=\; n \times f_{\text{fundamental}}^{\text{closed}-\text{closed}}\,, \quad n = 1, 2, 3, 4, \ldots$$

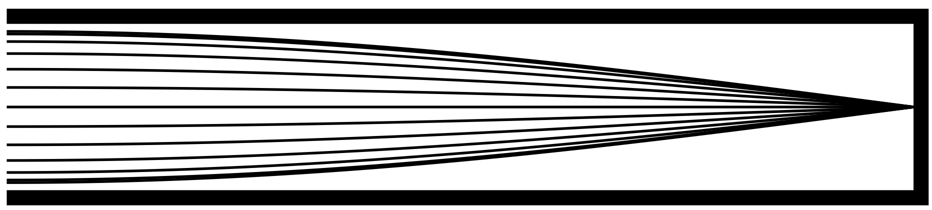
of a pipe with two closed ends are exactly the same as for a pipe with two open ends.

But a pipe with different ends — once closed and one open — has more complicated standing sound waves. At the open end of the pipe, the wave must have a node of pressure, while at the closed end the wave must have a node of displacement. Consequently, we cannot focus on just the pressure wave or just the displacement wave but must consider both waves at once.

Let's start with the fundamental mode of the open-closed pipe.



pressure $\delta\mathcal{P}(x)$



displacement $\delta x(x)$

This mode does not have any nodes at all — pressure or displacement — in the middle of the pipe; there is only a pressure node at the open end and a displacement node at the closed end. The distance between such nearby nodes of opposite kinds is $\lambda/4$ (unlike the $\lambda/2$ distance between nearby nodes of the same kind), and since they sit at the two ends of a pipe of length L, we must have

$$L = \frac{\lambda}{4}. \tag{43}$$

Consequently, the fundamental harmonic of the open-closed pipe has wavelength

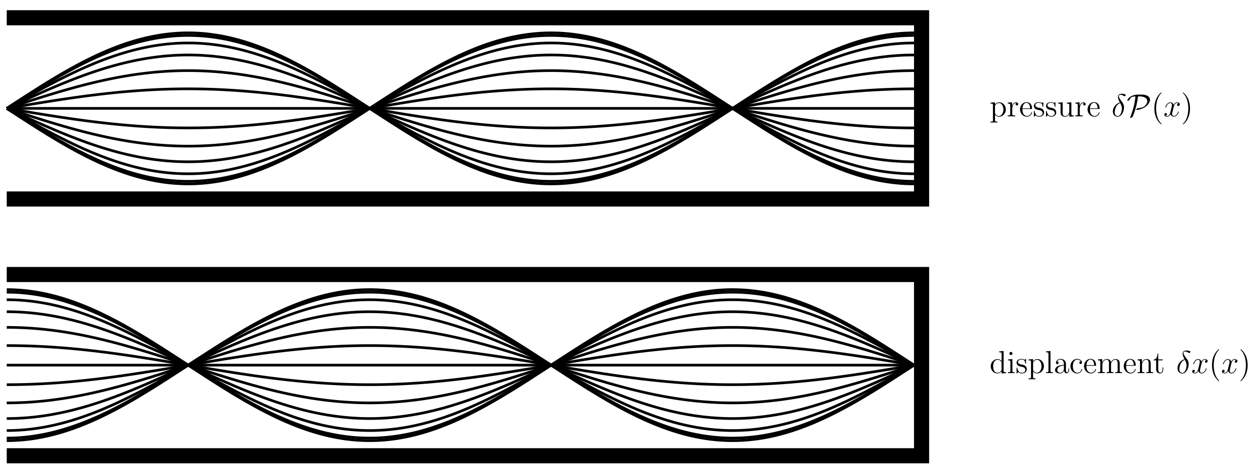
$$\lambda^{\text{open-closed}}_{\text{fundamental}} = 4L \tag{44}$$

and frequency

$$f^{\text{open-closed}}_{\text{fundamental}} = \frac{u_{\text{sound}}}{4L}. \tag{45}$$

Note that a pipe of length L with different ends (one open, one closed) has the same fundamental frequency — *i.e.*, the pitch — as a pipe of double length $2L$ with similar ends (both open or both closed). For example, a one-meter-long open-closed pipe has the same pitch 85 Hz as a two-meter-long pipe with both ends open or both closed. However, the higher harmonics of those pipes would be quite different.

Indeed, higher modes of standing wave in an open-closed pipe have both pressure and displacement modes in the middle of the pipe. Since the pressure and the displacement nodes alternate, the $n^{\underline{\text{th}}}$ mode has n pressure nodes (including one node at the open end and $n-1$ nodes in the middle) and also n displacement nodes (including one node at the closed end and $n-1$ nodes in the middle). Here is the diagram for $n=3$:



Altogether there are $2n$ nodes separated by $\lambda/4$ distances between neighboring nodes. Consequently, the total length L of the pipe divides into $2n-1$ intervals of length $\lambda/4$ each. This calls for

$$(2n-1)\times\frac{\lambda}{4} \;=\; L, \tag{46}$$

hence the $n^{\underline{\rm th}}$ harmonic of the open-closed pipe has wavelength

$$\lambda_n^{\rm open-closed} \;=\; \frac{4L}{2n-1} \tag{47}$$

and therefore frequency

$$f_n^{\rm open-closed} \;=\; \frac{u_{\rm sound}}{4L/(2n-1)} \;=\; (2n-1)\times\frac{u_{\rm sound}}{4L}\,. \tag{48}$$

In terms of the fundamental frequency

$$f_{\rm fundamental}^{\rm open-closed} \;=\; \frac{u_{\rm sound}}{4L}\,, \tag{49}$$

the higher harmonics are its *odd* multiplets

$$f_n^{\rm open-closed} \;=\; f_{\rm fundamental}^{\rm open-closed}\times(2n-1). \tag{50}$$

Thus, the sound created by an organ pipe with different ends (one open and one closed) would have only the odd multiples of the fundamental frequency: the f_1 itself, $3\times f_1$, $5\times f_1$, $7\times f_1$, *etc.*, *etc.* In contrast, the sound of a pipe with both ends open (or both closed) has both odd and even multiplets of the fundamental frequency: f_1, $2\times f_1$, $3\times f_1$, $4\times f_1$, $5\times f_1$, *etc.*, *etc.*.