Trigonometric Functions

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The unit circle
Definitions of trigonometric functions for a right triangle
Definitions of trigonometric functions for a unit circle
Exact values for trigonometric functions of most commonly used angles
Trigonometric functions of any angle θ' in terms of angle θ in quadrant I
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Reminder: Relationship Between Degrees and Radians

A radian is defined as an angle θ subtended at the center of a circle for which the arc length is equal to the radius of that circle (see Fig.1).

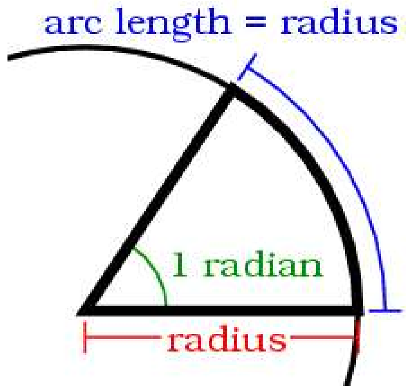

Fig.1. Definition of a radian.

The circumference of the circle is equal to $2\pi R$, where R is the radius of the circle. Consequently, $360° = 2\pi$ radians. Thus,

$$1 \text{ radian} = 360°/2\pi \approx 57.296°$$

$$\mathbf{1° = (2\pi/360) \text{ radians} \approx 0.01745 \text{ radians}}$$

The Unit Circle

In mathematics, a unit circle is defined as a circle with a radius of 1. Often, especially in applications to trigonometry, the unit circle is centered at the origin (0,0) in the coordinate plane. The equation of the unit circle in the coordinate plane is

$$x^2 + y^2 = 1.$$

As mentioned above, the unit circle is taken to be 360°, or 2π radians. We can divide the coordinate plane, and therefore, the unit circle, into 4 quadrants. The first quadrant is defined in terms of coordinates by $x>0$, $y>0$, or, in terms of angles, by $0°<\theta<90°$, or $0<\theta<\pi/2$. The second quadrant is defined by $x<0$, $y>0$, or $90°<\theta<180°$, or $\pi/2<\theta<\pi$. The third quadrant is defined by $x<0$, $y<0$, or $180°<\theta<270°$, or $\pi<\theta<3\pi/2$. Finally, the fourth quadrant is defined by $x>0$, $y<0$, or $270°<\theta<360°$, or $3\pi/2<\theta<2\pi$.

Trigonometric Functions

Definitions of Trigonometric Functions For a Right Triangle

A right triangle is a triangle with a right angle (90°) (See Fig.2).

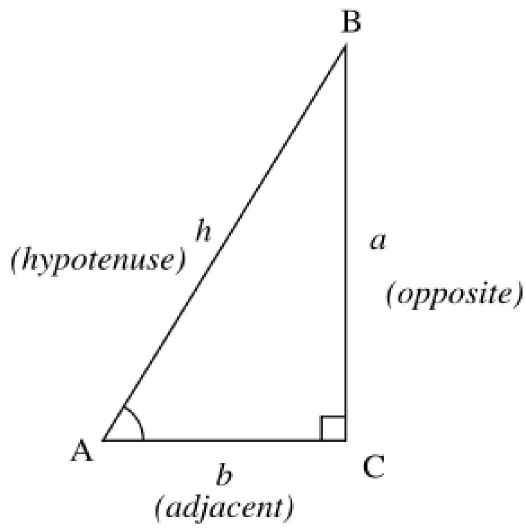

Fig.2. Right triangle.

For every angle θ in the triangle, there is the side of the triangle adjacent to it (from here on denoted as "adj"), the side opposite of it (from here on denoted as "opp"), and the hypotenuse (from here on denoted as "hyp"), which is the longest side of the triangle located opposite of the right angle. For angle θ, the trigonometric functions are defined as follows:

$$\text{sine of } \theta = \sin\theta = \frac{opp}{hyp}$$

$$\text{cosine of } \theta = \cos\theta = \frac{adj}{hyp}$$

$$\text{tangent of } \theta = \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{opp}{adj}$$

$$\text{cotangent of } \theta = \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta} = \frac{adj}{opp}$$

$$\text{secant of } \theta = \sec\theta = \frac{1}{\cos\theta} = \frac{hyp}{adj}$$

$$\text{cosecant of } \theta = \csc\theta = \frac{1}{\sin\theta} = \frac{hyp}{opp}$$

Definitions of Trigonometric Functions For a Unit Circle

In the unit circle, one can define the trigonometric functions cosine and sine as follows. If (x,y) is a point on the unit cirlce, and if the ray from the origin (0,0) to that point (x,y) makes an angle θ with the positive x-axis, (such that the counterclockwise direction is considered positive), then,

$$\cos\theta = x/1 = x$$
$$\sin\theta = y/1 = y$$

Then, each point (x,y) on the unit circle can be written as (cosθ, sinθ). Combined with the equation $x^2 + y^2 = 1$, the definitions above give the relationship $\sin^2\theta+\cos^2\theta=1$. In addition, other trigonometric functions can be defined in terms of x and y:

$$\tan\theta = \sin\theta/\cos\theta = y/x$$
$$\cot\theta = \cos\theta/\sin\theta = x/y$$
$$\sec\theta = 1/\cos\theta = 1/x$$
$$\csc\theta = 1/\sin\theta = 1/y$$

Fig.3 below shows a unit circle in the coordinate plane, together with some useful values of angle θ, and the points (x,y)=(cosθ, sinθ), that are most commonly used (also see table in the following section).

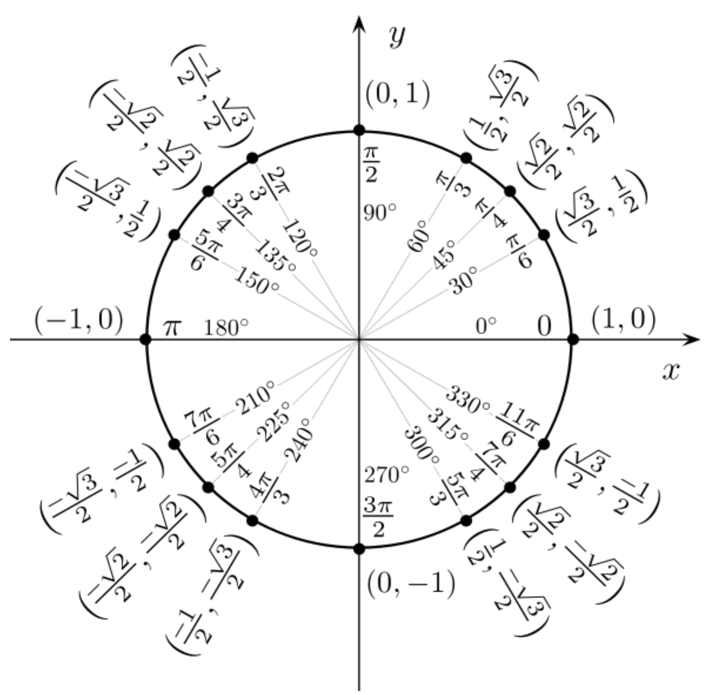

Fig.3. Most commonly used angles and points of the unit circle.

Note: For θ in quadrant I, sinθ>0, cosθ >0; for θ in quadrant II, sinθ>0, cosθ<0; for θ in quadrant III, sinθ<0, cosθ<0; and for θ in quadrant IV, sin θ

Exact Values for Trigonometric Functions of Most Commonly Used Angles

θ in degrees	**θ in radians**	**sinθ**	**cosθ**	**tanθ**
0	0	0	1	0
30	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
45	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90	$\frac{\pi}{2}$	1	0	undefined
180	π	0	-1	0
270	$\frac{3\pi}{2}$	-1	0	undefined
360	2π	0	1	0

Note: Exact values for other trigonometric functions (such as cotθ, secθ, and cscθ) as well as trigonometric functions of many other angles can be derived by using the following sections.

Trigonometric Functions of Any Angle θ' in Terms of Angle θ in Quadrant I

θ'	**sinθ'**	**cosθ'**	**tanθ'**	**θ'**	**sinθ'**	**cosθ'**	**tanθ'**
90°+θ π/2+θ	cosθ	−sinθ	-cotθ	90°−θ π/2-θ	cosθ	sinθ	cotθ
180°+θ π+θ	-sinθ	-cosθ	tanθ	180°−θ π−θ	sinθ	-cosθ	-tanθ
270°+θ 3π/2+θ	-cosθ	sinθ	-cotθ	270°−θ 3π/2-θ	-cosθ	-sinθ	cotθ
k(360°)+θ k(2π)+θ k=integer	sinθ	cosθ	tanθ	k(360°)−θ k(2π)−θ k=integer	-sinθ	cosθ	-tanθ

Trigonometric Functions of Negative Angles

$$\sin(-\theta) = -\sin\theta$$
$$\cos(-\theta) = \cos\theta$$
$$\tan(-\theta) = -\tan\theta$$

Some Useful Relationships Among Trigonometric Functions

$$\sin^2\theta + \cos^2\theta = 1$$
$$\sec^2\theta - \tan^2\theta = 1$$
$$\csc^2\theta - \cot^2\theta = 1$$

Double Angle Formulas

$$\sin 2\theta = 2 \sin\theta \cos\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1\text{-}2 \sin^2\theta = 2 \cos^2\theta \text{ -}1$$
$$\tan 2\theta = 2\frac{tan\theta}{1-\tan^2\theta}$$

Half Angle Formulas

Note: in the formulas in this section, the “+” sign is used in the quadrants where the respective trigonometric function is positive for angle $\theta/2$, and the “-” sign is used in the quadrants where the respective trigonometric function is negative for angle $\theta/2$.

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-cos\theta}{2}}$$
$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+cos\theta}{2}}$$
$$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-cos\theta}{1+\cos\theta}} = \frac{sin\theta}{1+cos\theta} = \frac{1-cos\theta}{sin\theta}$$

Angle Addition Formulas

Note: in this and the following section, letters A and B are used to denote the angles of interest, instead of the letter θ.

$$\sin(A \pm B) = sinA\,cosB \pm cosA\,sinB$$

$$\cos(A \pm B) = cosA\cos B \mp sinA\,sinB$$

$$\tan(A \pm B) = \frac{tanA \pm tanB}{1 \mp tanA\,tanB}$$

$$\text{cot(A±B)=} \frac{cotA\,cotB \mp 1}{cotB \pm cotA}$$

Sum, Difference and Product of Trigonometric Functions

$$\text{sinA + sinB =} \quad 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\text{sinA – sinB =} \quad 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$$

$$\text{cosA + cosB =} \quad 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\text{cosA – cosB =} \quad -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\text{sinA sinB =} \quad \frac{1}{2}[\cos(A-B)-\cos(A+B)]$$

$$\text{cosA cosB =} \quad \frac{1}{2}[\cos(A-B)+\cos(A+B)]$$

$$\text{sinA cosB =} \quad \frac{1}{2}[\sin(A-B)+\sin(A+B)]$$

Graphs of Trigonometric Functions (Fig.4, a-f)

Note: In each graph in Fig.4, the horizontal axis (x) is measured in radians.
Ref. Weisstein, Eric W. "Sine." "Cosine." "Tangent." "Cotangent." "Secant." "Cosecant." From MathWorld – A Wolfram Web Resource: http://mathworld.worlfram.com

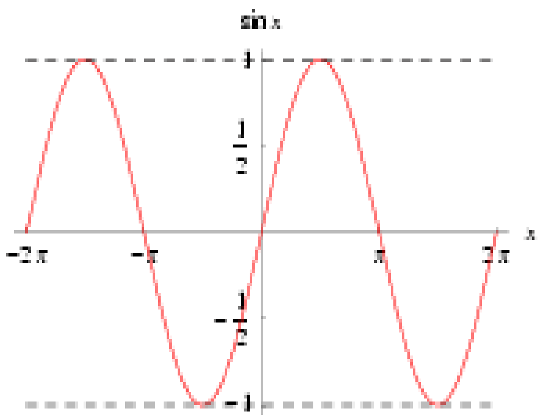

Fig.4a. Graph of sin(x).

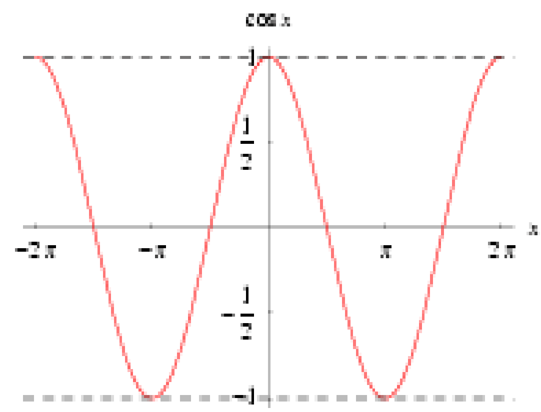

Fig.4b. Graph of cos(x).

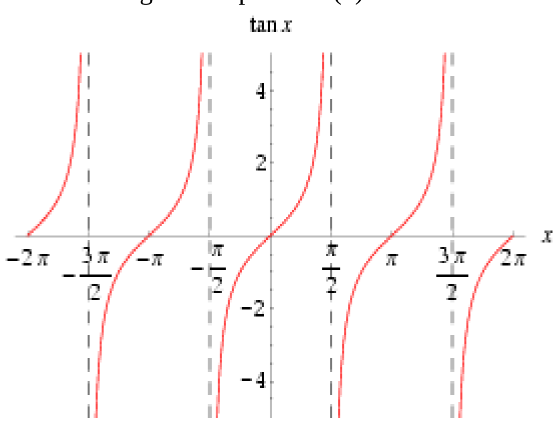

Fig.4c. Graph of tan(x).

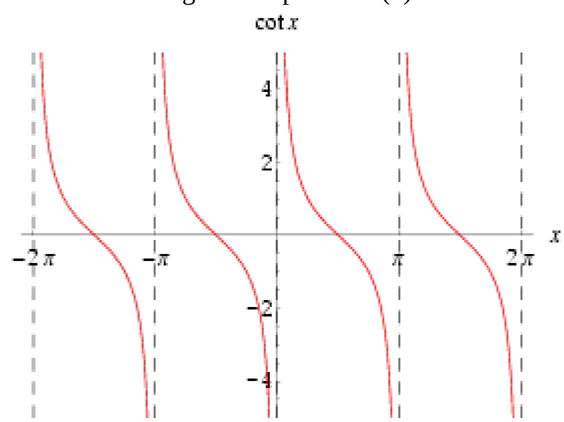

Fig.4d. Graph of cot(x).

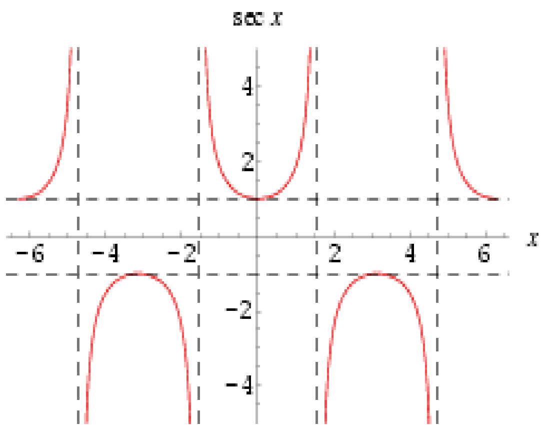

Fig.4e. Graph of sec(x).

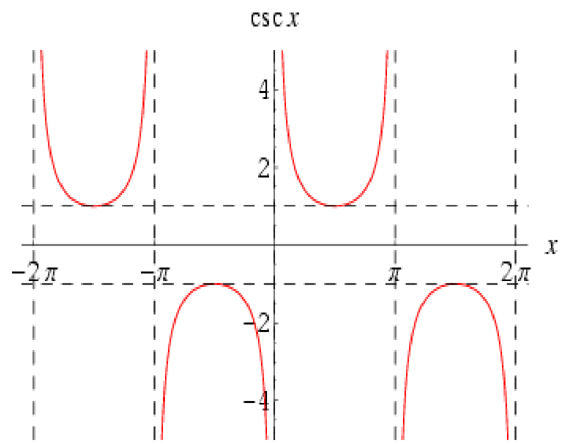

Fig.4f. Graph of csc(x).

Inverse Trigonometric Functions

Inverse Trigonometric Functions

If x=sin(y), then y=$\sin^{-1}$(x), i.e. s is the angle whose sine is y. In other words, x is the inverse sine of y. Another name for inverse sine is arcsine, and the notation used is y=arcsin(x). Similarly, we can define inverse cosine, inverse tangent, inverse cotangent, inverse secant and inverse cosecant. All of the inverse functions are many-valued functions of x (for each value of x, there are many corresponding values of y), which are collections of single-valued functions (for each value of x, there is only one corresponding value of y) called branches. For many purposes a particular branch is required. This is called the principal branch and the values for this branch are called principal values.

x = sin(y)	y = $\sin^{-1}$(x) = arcsin(x)
x = cos(y)	y = $\cos^{-1}$(x) = arccos(x)
x = tan(y)	y = $\tan^{-1}$(x) = arctan(x)
x = cot(y)	y = $\cot^{-1}$(x) = arccot(x)
x = sec(y)	y = $\sec^{-1}$(x) = arcsec(x)
x = csc(y)	y = $\csc^{-1}$(x) = arccsc(x)

Principal Values for Inverse Trigonometric Functions

Principal Values for $x \geqslant 0$	***Principal Values for*** $x < 0$
$0 \leqslant \sin^{-1}(x) \leqslant \pi/2$	$-\pi/2 \leqslant \sin^{-1}(x) < 0$
$0 \leqslant \cos^{-1}(x) \leqslant \pi/2$	$\pi/2 < \cos^{-1}(x) \leqslant \pi$
$0 \leqslant \tan^{-1}(x) < \pi/2$	$-\pi/2 < \tan^{-1}(x) < 0$
$0 < \cot^{-1}(x) \leqslant \pi/2$	$\pi/2 < \cot^{-1}(x) < \pi$
$0 \leqslant sec^{-1}(x) < \pi/2$	$\pi/2 < sec^{-1}(x) \leqslant \pi$
$0 < csc^{-1}(x) \leqslant \pi/2$	$-\pi/2 \leqslant csc^{-1}(x) < 0$

Relations Between Inverse Trigonometric Functions

Note: In all cases, it is assumed that principal values are used.

$$\sin^{-1}x + \cos^{-1}x = \pi/2$$
$$\tan^{-1}x + \cot^{-1}x = \pi/2$$
$$\sec^{-1}x + \csc^{-1}x = \pi/2$$
$$\csc^{-1}x = \sin^{-1}(1/x)$$
$$\sec^{-1}x = \cos^{-1}(1/x)$$
$$\cot^{-1}x = \tan^{-1}(1/x)$$
$$\sin^{-1}(-x) = -\sin^{-1}x$$
$$\cos^{-1}(-x) = \pi - \cos^{-1}x$$
$$\tan^{-1}(-x) = -\tan^{-1}x$$
$$\cot^{-1}(-x) = \pi - \cot^{-1}x$$
$$\sec^{-1}(-x) = \pi - \sec^{-1}x$$
$$\csc^{-1}(-x) = -\csc^{-1}x$$

Graphs of Inverse Trigonometric Functions (Fig.5, a-f)

Note: In each graph in Fig.5, the vertical axis (y) is measured in radians. Only portions of curves corresponding to principal values are shown.
Ref. Weisstein, Eric W. "Inverse Sine." "Inverse Cosine." "Inverse Tangent." "Inverse Cotangent." "Inverse Secant." "Inverse Cosecant." From MathWorld – A Wolfram Web Resource: http://mathworld.worlfram.com

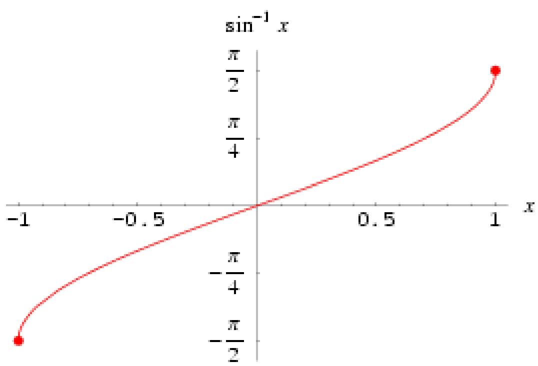

Fig.5a. Graph of $\sin^{-1}(x)$.

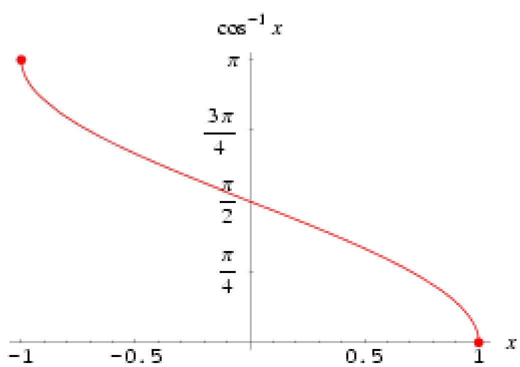

Fig.5b. Graph of $\cos^{-1}(x)$.

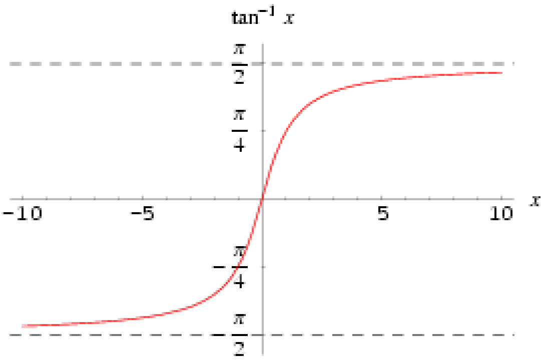

Fig.5c. Graph of tan^{-1}(x).

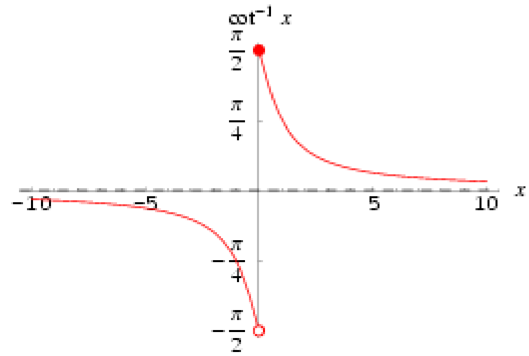

Fig.5d. Graph of cot^{-1}(x).

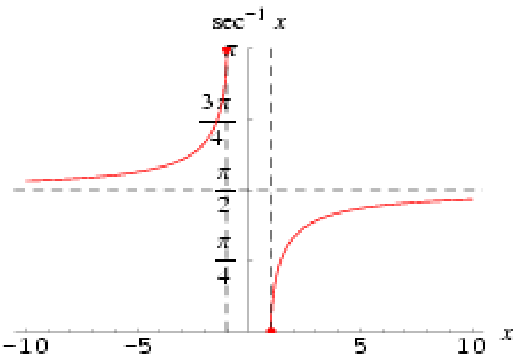

Fig.5e. Graph of sec^{-1}(x).

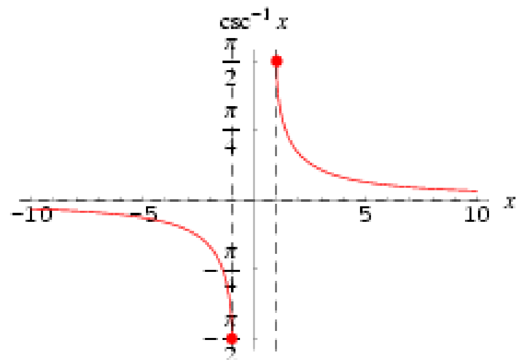

Fig.5f. Graph of csc^{-1}(x).

Using Trigonometric Functions

Resolving Vectors into Components:

The geometric way of adding vectors is not recommended whenever great accuracy is required or in three-dimensional problems. In such cases, it is better to make use of the projections of vectors along coordinate axis, also known as components of the vector. Any vector can be completely described by its components.

Consider an arbitrary vector **A** (from now on, the bold-cased letters are used to signify vectors, whereas the regular-font letter A signifies the length of the vector **A**) lying in the xy-plane and making an arbitrary angle θ with the positive x-axis, as shown in Fig.6:

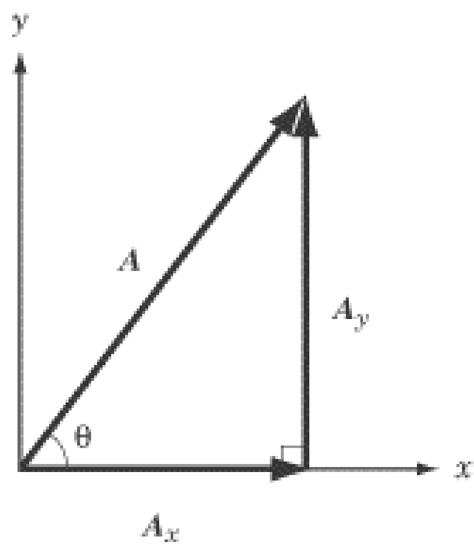

Fig.6. Arbitrary vector **A** in the xy-plane.

This vector **A** can be expressed as the sum of two other vectors, $\mathbf{A}_x$ and $\mathbf{A}_y$. From Fig.6, it can be seen that the three vectors form a right triangle and that $\mathbf{A}=\mathbf{A}_x+\mathbf{A}_y$. It is conventional to refer to the "components of vector **A**," written A_x and A_y (without the boldface notation). The component A_x represents the projection of **A** along the x-axis, and the component A_y represents the projection of **A** along the y-axis. These components can be positive or negative.

From Fig.6 and the definition of sine and cosine for a right triangle, we see that $\cos\theta=A_x/A$ and $\sin\theta=A_y/A$. Hence, the components of **A** are

$$A_x = A\cos\theta$$
$$A_y = A\sin\theta$$

These components from the two sides of a right triangle with hypotenuse of length A. Thus, it follows that the magnitude and direction of **A** are related to its components through the expressions

$$A = \sqrt{Ax^2 + Ay^2}$$
$$\theta = \tan^{-1}\left(\frac{Ay}{Ax}\right)$$

Note that the signs of the comonents A_x and A_y depend on the angle θ. For example, if θ=120°, then A_x is negative and A_y is positive.

When solving problems, one can specify a vector **A** either with its components Ax and Ay or with its magnitude and direction A and θ. Furthermore, one can express a vector A as

$$A = Ax\,\hat{x} + Ay\,\hat{y} = A(cos\theta\,\hat{x} + sin\theta\,\hat{y}) \quad ,$$

where $\hat{x}$ and $\hat{y}$ are unit vectors (length of one) in the direction of x- and y-axis, respectively.

Writing a Phase Shift of a Wave

Waves can have many different shape. One of the simplest to deal with and also one that is of a particular interest when it comes to simple harmonic motion is a sinusoidal wave. It is called "sinusoidal" because the shape that the wave takes either in space or in time is that of a sine curve (see Fig.7a,b):

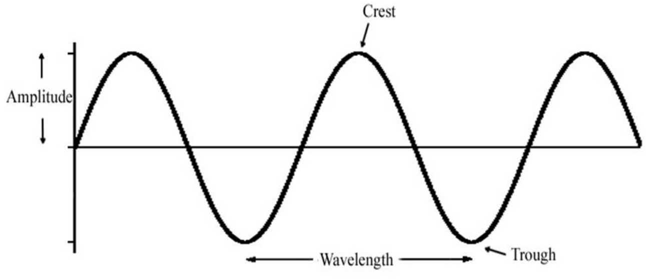

Fig.7.(a) A one-dimensional sinusoidal wave in space.

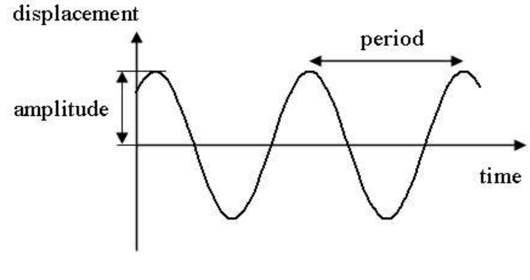

Fig.7b. A one-dimensional sinusoidal wave in time.

A function describing such a wave can be written as

$$y = A\sin\left(\frac{2\pi}{\lambda}x \pm \frac{2\pi}{T}t + \varphi\right) \quad ,$$

where A is the amplitude of the wave, λ is the wavelength of the wave, T is the period of the wave, and φ is the phase shift of the wave. The ± sign in front of the second term in the parentheses depends on whether the wave is moving to the right (-) or to the left (+). The entire expression in the parentheses can also be written as Φ(x,t) and is called the total phase of the wave, and is a function of position x and time t, in contrast to the phase shift φ, which is a constant. Then, the entire expression becomes y = A sin Φ(x,t).

Because the sine function is periodic with period 2π, sin(Φ+2πn)=sin(Φ) for any integer n.

φ is referred to as a phase-shift, because it represents a "shift" from zero phase. But a change in φ is also referred to as a phase-shift. Fig.8 shows two curves: the red one with zero phase, and the blue one with a non-zero phase.

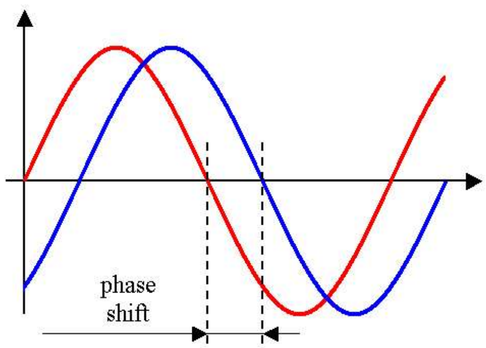

Fig.8. Phase shift.

Two oscillators that have the same frequency and same wavelength, will have a phase difference, if their phase shift φ is different. If that is the case, the oscillators are said to be out of phase with each other. The amount by which such oscillators are out of phase with each other can be expressed in degrees from 0° to 360°, or in radians from 0 to 2π (see Fig.9a,b).

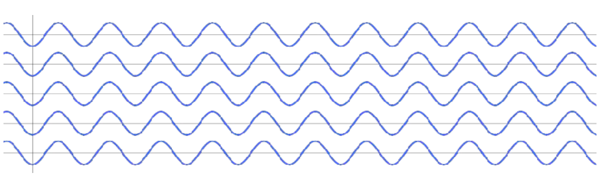

Fig.9a. Waves that are in phase.

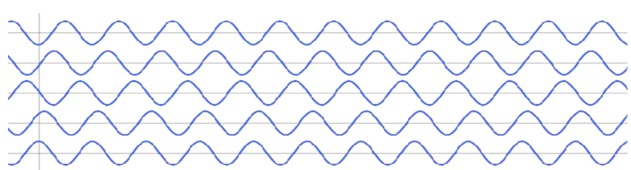

Fig.9b. Waves that are out of phase.

Derivatives of Trigonometric and Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin x)=\cos x$$

$$\frac{d}{dx}(\cos x)=-\sin x$$

$$\frac{d}{dx}(\tan x)=sec^2 x$$

$$\frac{d}{dx}(\cot x)=-csc^2 x$$

$$\frac{d}{dx}(sec\, x)=sec\, x \tan x$$

$$\frac{d}{dx}(csc\, x)=-csc\, x \cot x$$

$$\frac{d}{dx}(\sin^{-1} x)=\frac{1}{\sqrt{1-x^2}} \quad , \quad \frac{-\pi}{2}<\sin^{-1} x<\frac{\pi}{2}$$

$$\frac{d}{dx}(\cos^{-1} x)=\frac{-1}{\sqrt{1-x^2}} \quad , \quad 0<\cos^{-1} x<\pi$$

$$\frac{d}{dx}(\tan^{-1} x)=\frac{1}{1+x^2} \quad , \quad \frac{-\pi}{2}<\tan^{-1} x<\frac{\pi}{2}$$

$$\frac{d}{dx}(\cot^{-1} x)=\frac{-1}{1+x^2} \quad , \quad 0<\cot^{-1} x<\pi$$

$$\frac{d}{dx}(sec^{-1} x)=\frac{\pm 1}{x\sqrt{x^2-1}} \quad \text{, + if} \quad 0<sec^{-1} x<\frac{\pi}{2} \quad \text{, - if} \quad \frac{\pi}{2}<sec^{-1} x<\pi$$

$$\frac{d}{dx}(csc^{-1} x)=\frac{\mp 1}{x\sqrt{x^2-1}} \quad \text{, - if} \quad 0<csc^{-1} x<\frac{\pi}{2} \quad \text{, + if} \quad \frac{-\pi}{2}<csc^{-1} x<0$$