Notes on Elastic and Inelastic Collisions

In any collision of 2 bodies, their net momentum is conserved. That is, the net momentum vector of the bodies just after the collision is the same as it was just before the collision,

$$\vec{\mathbf{P}}_{\rm net} \;=\; m_1\vec{\mathbf{v}}_1' \;+\; m_2\vec{\mathbf{v}}_2' \;=\; m_1\vec{\mathbf{v}}_1 \;+\; m_2\vec{\mathbf{v}}_2 \tag{1}$$

So if we know the velocity vectors of both bodies before the collision and if we also know the velocity vector of one body after the collision, then using this formula we may find out the velocity vector of the other body after the collision.

But if we only know the initial velocities of the two bodies and we want to find out their velocities after the collision, we need to invoke additional physics. In particular, we need to know what happens to the net kinetic energy of the two bodies,

$$K_{\rm net} \;=\; \tfrac{1}{2}m_1v_1^2 \;+\; \tfrac{1}{2}m_2v_2^2\,. \tag{2}$$

It is convenient to reorganize this net kinetic energy into two terms, one due to the net momentum (1) of two particles and the other due to their *relative velocity* $\vec{\mathbf{v}}_{\rm rel} = \vec{\mathbf{v}}_1 - \vec{\mathbf{v}}_2$,

$$K_{\rm net} \;=\; \frac{\vec{\mathbf{P}}_{\rm net}^2}{2(m_1+m_2)} \;+\; \frac{m_1\,m_2}{2(m_1+m_2)}\times\vec{\mathbf{v}}_{\rm rel}^2\,. \tag{3}$$

- Proof: First, let's expand vector squares

$$\begin{aligned}\vec{\mathbf{P}}_{\rm net}^2 \;&=\; (m_1\vec{\mathbf{v}}_1+m_2\vec{\mathbf{v}}_2)^2 \;=\; m_1^2\times\vec{\mathbf{v}}_1^2 \;+\; m_2^2\times\vec{\mathbf{v}}_2^2 \;+\; 2m_1m_2\times\vec{\mathbf{v}}_1\cdot\vec{\mathbf{v}}_2\,,\\ \text{and }\vec{\mathbf{v}}_{\rm rel}^2 \;&=\; (\vec{\mathbf{v}}_1-\vec{\mathbf{v}}_2)^2 = \;\vec{\mathbf{v}}_2^2 \;+\; \vec{\mathbf{v}}_2^2 \;+\; 2\vec{\mathbf{v}}_1\cdot\vec{\mathbf{v}}_2\,.\end{aligned} \tag{4}$$

Next, let's combine

$$\begin{aligned}\vec{\mathbf{P}}_{\rm net}^2 \;+\; m_1m_2\times\vec{\mathbf{v}}_{\rm rel}^2 \;&=\; m_1^2\times\vec{\mathbf{v}}_1^2 \;+\; m_2^2\times\vec{\mathbf{v}}_2^2 \;+\; 2m_1m_2\times\vec{\mathbf{v}}_1\cdot\vec{\mathbf{v}}_2\\ &\qquad+\; m_1m_2\times\vec{\mathbf{v}}_1^2 \;+\; m_1m_2\times\vec{\mathbf{v}}_2^2 \;-\; 2m_1m_2\times\vec{\mathbf{v}}_1\cdot\vec{\mathbf{v}}_2\\ &=\; (m_1^2+m_1m_2)\times\vec{\mathbf{v}}_1^2 \;+\; (m_2^2+m_1m_2)\times\vec{\mathbf{v}}_2^2 \;+\; 0\times\vec{\mathbf{v}}_1\cdot\vec{\mathbf{v}}_2\\ &=\; 2(m_1+m_2)\times\left[\tfrac{1}{2}m_1\times\vec{\mathbf{v}}_1^2 + \tfrac{1}{2}m_2\times\vec{\mathbf{v}}_2^2\right].\end{aligned} \tag{5}$$

Finally, let's divide both sides of this long equation by by $2(m_1+m_2)$:

$$\frac{\vec{\mathbf{P}}_{\rm net}}{2(m_1+m_2)} \;+\; \frac{m_1\,m_2}{2(m_1+m_2)}\times\vec{\mathbf{v}}_{\rm rel}^2 \;=\; \tfrac{1}{2}m_1\times\vec{\mathbf{v}}_1^2 \;+\; \tfrac{1}{2}m_2\times\vec{\mathbf{v}}_2^2 \;=\; K_{\rm net}\,. \tag{6}$$

Quod erat demonstrandum.

The first term in eq. (3) is the kinetic energy due to motion of the center of mass of the two-body system (see second half of these notes for explanation),

$$K_{\rm cm} \;=\; \frac{\vec{\mathbf{P}}^2_{\rm net}}{2(m_1+m_2)} \;=\; \frac{m_1+m_2}{2}\times\vec{\mathbf{v}}^2_{\rm cm}\,. \tag{7}$$

This term is conserved in any two-body collision because the net momentum $\vec{\mathbf{P}}_{\rm net}$ is conserved.

The second term in eq. (3) is the kinetic energy due to *relative* motion of the two colliding bodies,

$$K_{\rm rel} \;=\; \frac{m_1\,m_2}{2(m_1+m_2)}\times\vec{\mathbf{v}}^2_{\rm rel} \;=\; \frac{1}{2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)^{-1}\times(\vec{\mathbf{v}}_1\,-\,\vec{\mathbf{v}}_2)^2\,. \tag{8}$$

What happens to this term during a collision depends on the elasticity of the colliding bodies:

- In an *elastic collision,* kinetic energy of the relative motion is converted into the elastic energies of two momentarily compressed bodies, and then is converted back into the kinetic energy, $K_{\rm rel}\to U_{\rm elastic}\to K_{\rm rel}$. Therefore, $K'_{\rm rel}$ immediately after the collision is the same as $K_{\rm rel}$ immediately before the collision, and consequently *the net kinetic energy of the two colliding bodies is conserved,*

$$K'_{\rm net} \;=\; K_{\rm net} \quad\Longleftrightarrow\quad \tfrac12 m_1\,\vec{\mathbf{v}}_1'^2 \;+\; \tfrac12 m_2\,\vec{\mathbf{v}}_2'^2 \;=\; \tfrac12 m_1\,\vec{\mathbf{v}}_1^2 \;+\; \tfrac12 m_2\,\vec{\mathbf{v}}_2^2\,. \tag{9}$$

  Also, in light of eq. (8), $K'_{\rm rel}=K_{\rm rel}$ implies *the same relative speed of the two bodies before and after the collision,*

$$\left|\vec{\mathbf{v}}'_{\rm rel}\right| \;=\; \left|\vec{\mathbf{v}}_{\rm rel}\right| \quad\Longleftrightarrow\quad \left|\vec{\mathbf{v}}'_1-\vec{\mathbf{v}}'_2\right| \;=\; \left|\vec{\mathbf{v}}_1-\vec{\mathbf{v}}_2\right|, \tag{10}$$

  although the direction of the relative velocity vector is different.

- In an *inelastic collision*, a part of the $K_{\rm rel}$ is converted into the elastic energy and then back into the kinetic energy, while the rest of the initial $K_{\rm rel}$ is converted into heat (or some other non-mechanical forms of energy). Therefore,

$$0 \ < \ K'_{\rm rel} \ < \ K_{\rm rel} \tag{11}$$

and hence

$$0 \ < \ \left|\vec{\mathbf{v}}'_1 - \vec{\mathbf{v}}'_2\right| \ < \ \left|\vec{\mathbf{v}}_1 - \vec{\mathbf{v}}_2\right| \tag{12}$$

and

$$K_{\rm cm} \ < \ K'_{\rm net} \ < \ K_{\rm net}\,. \tag{13}$$

- In a *totally inelastic collision*, all of kinetic energy of relative motion is converted into heat (or other non-mechanical energies), so *after the collision* $K'_{\rm rel} = 0$ *and there is no relative motion:*

$$\vec{\mathbf{v}}'_{\rm rel} \ = \ \vec{\mathbf{0}} \quad \Longleftrightarrow \quad \vec{\mathbf{v}}'_1 \ = \ \vec{\mathbf{v}}'_2 \tag{14}$$

and

$$K'_{\rm net} \ = \ K_{\rm cm} \ = \ \frac{\vec{\mathbf{P}}^2_{\rm net}}{2(m_1+m_2)} \ < \ K_{\rm net}\,. \tag{15}$$

Totally Inelastic Collisions

In a totally inelastic collision, the two colliding bodies stick together and move at the same velocity $\vec{\mathbf{v}}'_1 = \vec{\mathbf{v}}'_2 = \vec{\mathbf{v}}'$ after the collision. This common final velocity can be found from the momentum conservation equation (1):

$$\vec{\mathbf{P}}_{\rm net} \ = \ m_1\vec{\mathbf{v}}_1 \ + \ m_2\vec{\mathbf{v}}_2 \ = \ m_1\vec{\mathbf{v}}' \ + \ m_2\vec{\mathbf{v}}' \ = \ (m_1+m_2)\vec{\mathbf{v}}', \tag{16}$$

hence

$$\vec{\mathbf{v}}'_1 \ = \ \vec{\mathbf{v}}'_2 \ = \ \vec{\mathbf{v}}' \ = \ \frac{m_1}{m_1+m_2}\,\vec{\mathbf{v}}_1 \ + \ \frac{m_2}{m_1+m_2}\,\vec{\mathbf{v}}_2\,. \tag{17}$$

<u>Special case: fixed target.</u>

In particular, when only one particle moves before the collision, say $\vec{\mathbf{v}}_1 \neq 0$ but $\vec{\mathbf{v}}_2 = 0$, then after the collision they both move with velocity

$$\vec{\mathbf{v}}_1' = \vec{\mathbf{v}}_2' = \vec{\mathbf{v}}' = \frac{m_1}{m_1 + m_2}\,\vec{\mathbf{v}}_1\,. \tag{18}$$

The direction of this velocity is the same as the initial velocity $\vec{\mathbf{v}}_1$ but the speed is reduced by the factor $m_1/m_{\rm net}$,

$$v' = \frac{m_1}{m_1 + m_2} \times v_1\,. \tag{19}$$

General case: two moving particles.

If both bodies move before the collision, we should use eq. (17) as it is. Moreover, if the bodies move in different directions — like two cars colliding at an intersection — we must sum their momenta as vectors in order to obtain the final velocity. In components,

$$\begin{aligned} v_{1x}' &= v_{2x}' = \frac{m_1 v_{1x} + m_2 v_{2x}}{m_1 + m_2}\,, \\ v_{1y}' &= v_{2y}' = \frac{m_1 v_{1y} + m_2 v_{2y}}{m_1 + m_2}\,, \\ v_{1z}' &= v_{2z}' = \frac{m_1 v_{1z} + m_2 v_{2z}}{m_1 + m_2}\,. \end{aligned} \tag{20}$$

Head-on Elastic Collisions

In a perfectly elastic collision, the two bodies' velocities before and after the collision satisfy two constraints: eq. (10) stemming from kinetic energy conservation, and also

$$\vec{\mathbf{P}}_{\rm net}' = \vec{\mathbf{P}}_{\rm net} \quad \Longleftrightarrow \quad m_1\vec{\mathbf{v}}_1' + m_2\vec{\mathbf{v}}_2' = m_1\vec{\mathbf{v}}_1 + m_2\vec{\mathbf{v}}_2 \tag{21}$$

which is valid for any collision, elastic and otherwise.

Let's focus on head-on elastic collisions where both bodies move along the same straight line both before and after the collision. Such collisions are effectively one-dimensional, so we may dispense with vector notation and write eqs. (10) and (21) as

$$v_1' - v_2' = \pm(v_1 - v_2) \tag{22}$$

and

$$m_1 v_1' + m_2 v_2' = m_1 v_1 + m_2 v_2 \,. \tag{23}$$

Together, they give us two independent linear equations for two unknowns, v_1' and v_2', so there a unique solution. Or rather, there are two solutions, one for each sign in eq. (22): for the '+' sign, the solution is

$$v_1' = v_1 \,, \quad v_2' = v_2 \,, \tag{24}$$

which happens when there is no collision at all. When the particles do collide, their velocities have to change, so the sign in eq. (22) should be '−'. Multiplying both sides of this equation by m_2 and adding to eq. (23), we obtain

$$(m_2 + m_1) \times v_1' \;+\; (-m_2 + m_2 = 0) \times v_2' \;=\; (-m_2 + m_1) \times v_1 \;+\; (+m_2 + m_2) \times v_2 \tag{25}$$

and hence

$$v_1' = \frac{m_1 - m_2}{m_1 + m_2} \times v_1 + \frac{2m_2}{m_1 + m_2} \times v_2 \,. \tag{26}$$

Similarly,

$$v_2' = \frac{m_2 - m_1}{m_1 + m_2} \times v_2 + \frac{2m_1}{m_1 + m_2} \times v_1 \,. \tag{27}$$

Together, eqs. (26) and (27) give us the velocities of both bodies immediately after a perfectly elastic collision in terms of their velocities just before the collision.

<u>Special case: equal masses.</u>

When the two colliding bodies have equal masses, $m_1 = m_2$, eqs. (26) and (27) become much simpler:

$$v_1' = v_2 \quad \text{and} \quad v_2' = v_1. \tag{28}$$

In other words, *the two colliding bodies exchange their velocities.*

Special case: fixed target.

Another situation where eqs. (26) and (27) become simpler is when only one body moves before the collision, say $v_1 \neq 0$ but $v_2 = 0$. After a perfectly elastic collision, the second body moves away with velocity

$$v_2' \;=\; \frac{2m_1}{m_1 + m_2} \times v_1\,, \tag{29}$$

which is twice the velocity it would have obtained in an inelastic collision. In particular,

$$\text{for } m_2 \ll m_1\,, \quad v_2' \;\approx\; 2 \times v_1\,. \tag{30}$$

For example, if a small body initially at rest suffers a perfectly elastic collision with a truck, its velocity after the collision is twice the truck's velocity, and it does not matter how heavy is the truck as long as its much more massive than the body it hits.

As to the first body, its velocity after a perfectly elastic collision is

$$v_1' \;=\; \frac{m_1 - m_2}{m_1 + m_2} \times v_1\,. \tag{31}$$

If $m_1 > m_2$, it continues moving forward at a reduced speed, if $m_1 = m_2$, it stops moving, and if $m_1 < m_2$, it bounces back! In an extreme case of $m_1 \ll m_2$, *i.e.* hitting a target much heavier than itself, it bounces back with $v_1' = -v_1$: same speed in the opposite direction.

Glancing Elastic Collisions

In a glancing collision, the two bodies bounce off at some angles from their initial directions. The motion in such collisions is inherently two-dimensional or three-dimensional, and we absolutely have to treat all velocities as vectors. In other words, we are stuck with the vector form of eqs. (10) and (21), or the equivalent component equations:

$$\begin{aligned}
m_1 v_{1x}' \;+\; m_2 v_{2x}' \;&=\; m_1 v_{1x} \;+\; m_2 v_{2x}\,,\\
m_1 v_{1y}' \;+\; m_2 v_{2y}' \;&=\; m_1 v_{1y} \;+\; m_2 v_{2y}\,,\\
m_1 v_{1z}' \;+\; m_2 v_{2z}' \;&=\; m_1 v_{1z} \;+\; m_2 v_{2z}\,,\\
(v_{1x}' - v_{2x}')^2 \;+\; (v_{1y}' - v_{12}')^2 \;+\; (v_{1z}' - v_{2z}')^2 \;&=\; (v_{1x} - v_{2x})^2 \;+\; (v_{1y} - v_{2y})^2 \;+\; (v_{1z} - v_{2z})^2.
\end{aligned} \tag{32}$$

(The first three equations here spell out eq. (21) in components; the last equation is the square of eq. (10).) Clearly, this is a much more complicated equation system than just two linear eqs. (22) and (23) for the head-on elastic collisions.

Before we even try to solve eqs. (32), let's count the unknowns and the equations. In three dimensions, the two unknown velocity vectors $\vec{\mathbf{v}}'_1$ and $\vec{\mathbf{v}}'_2$ after the collision amount to 6 unknown components — v'_{1x}, v'_{1y}, v'_{1z}, v'_{2x}, v'_{2y}, and v'_{2z} — but there are only 4 equations (32). Consequently, there is no unique solution for all the unknowns but a continuous two-parameter family of solutions. Thus, the initial velocities of the two bodies do not completely determine their final velocities; to find them, we need to know the shapes of the two colliding bodies and how exactly do they collide. Alternatively, if we know two independent properties of the final velocities, then we can determine the other four. For example, if we know the direction of one final velocity (which calls for two angles in 3D), then we can determine its magnitude, and bot direction and magnitude of the other velocity.

Likewise, in two dimensions we have 3 equations

$$\begin{aligned} m_1 v'_{1x} \;+\; m_2 v'_{2x} \;&=\; m_1 v_{1x} \;+\; m_2 v_{2x}\,, \\ m_1 v'_{1y} \;+\; m_2 v'_{2y} \;&=\; m_1 v_{1y} \;+\; m_2 v_{2y}\,, \\ (v'_{1x} - v'_{2x})^2 \;+\; (v'_{1y} - v'_{12})^2 \;&=\; (v_{1x} - v_{2x})^2 \;+\; (v_{1y} - v_{2y})^2, \end{aligned} \tag{33}$$

for 4 velocity components v'_{1x}, v'_{1y}, v'_{2x}, and v'_{2y}, so we are one equation short. Again, the initial velocities do not completely determine the final velocities, and we need one more data point to find the outcome of the collision. For example, if we know either direction or speed of one body after the collision, then we may solve eqs. (33) for the remaining direction(s) and speed(s).

But I would not do it here because the algebra is too messy for this 309 K class.

Notes on the Center of Mass and its Motion

Let's start with a system of two point-like particles. The center of mass of this system lies between the particles on the straight line connecting them. Specifically, for particles of masses m_1 and m_2 at a distance L from each other, the center of mass is

$$\begin{aligned} \text{at distance } L_1 \;&=\; \frac{m_2}{m_1 + m_2} \times L \text{ from the first particle} \\ \text{and at distance } L_2 \;&=\; \frac{m_1}{m_1 + m_2} \times L \text{ from the second particle.} \end{aligned} \tag{34}$$

Equivalently, let the x axis run through both particles; then the center of mass of the two-particle system is at

$$x_{\rm cm} = \frac{m_1}{m_1 + m_2} \times x_1 + \frac{m_2}{m_1 + m_2} \times x_2 \,. \tag{35}$$

For an N particle system, we have a similar vector formula for the radius-vector of the center of mass

$$\vec{\mathbf{R}}_{\rm cm} = \sum_{i=1}^{N} \frac{m_i}{M_{\rm tot}} \times \vec{\mathbf{r}}_i \tag{36}$$

where $M_{\rm tot} = m_1 + m_2 + \cdots + m_N$ is the total mass of all the particles. In components, the center of mass is at

$$\begin{aligned} X_{\rm cm} &= \sum_{i=1}^{N} \frac{m_i}{M_{\rm tot}} \times x_i \,, \\ Y_{\rm cm} &= \sum_{i=1}^{N} \frac{m_i}{M_{\rm tot}} \times y_i \,, \\ Z_{\rm cm} &= \sum_{i=1}^{N} \frac{m_i}{M_{\rm tot}} \times z_i \,. \end{aligned} \tag{37}$$

Finding the center of mass of a macroscopic body such as a human body or a piece of machinery is more difficult. Formally, we can treat such a body as a system of $N \sim 10^{27}$ point-like atoms, then find the center of mass according to eq. (36) or eqs. (37). This is absolutely correct, but alas totally impractical. Alternatively, we can treat the body in question as continuous and replace the discrete sums in eqs. (37) with volume integrals:

$$X_{\rm cm} = \frac{1}{M_{\rm tot}} \iiint\limits_{\rm body} dx\, dy\, dz\, \rho(x, y, z) \times x \tag{38}$$

and similar formulae for the Y and Z coordinates of the center of mass. This method is actually used in science and engineering, but it's way too complicated for the 309 K class, so I am not going to explain it any further.

Instead, I will give two rules which often let you find the center of mass without doing any integrals.

1. The symmetry rule, for bodies of uniform density $\rho(x, y, z) = \text{const}$.
   If a body has a uniform density and a symmetric geometry, then the center of mass lies on the axis or plane of that symmetry. If there are several symmetry axis or planes, the center of mass lies at their intersection. In particular, if there is a geometric center, the center of mass is at that center. Here are a few examples:
   - A solid ball of uniform density or a spherical shell of uniform density and thickness has its CM at the geometric center.
   - A cube of uniform density has CM at the geometric center, at equal distance from all sides.
   - The CM of a rectangular slab of uniform density lies at the intersection of all three mid-planes of the slab, half-way between each pair of opposite sides.
   - The CM of a solid cylinder of uniform density lies on the cylinder's axis, half-way between the opposite flat sides. Ditto for a disk, or for a cylindrical shell of a uniform density and thickness.
2. The split-into-parts rule.
   Suppose a body is made of several parts, and we know how to locate the centers of mass for each individual part. Then the center of mass of the whole body is at

$$\vec{\mathbf{R}}_{\text{cm}}(\text{whole}) \;=\; \frac{1}{M_{\text{tot}}} \sum_{i}^{\text{parts}} M(i) \times \vec{\mathbf{R}}_{cm}(i) \tag{39}$$

   In other words, we can replace each part with a point particle of the same mass located at the part's center of mass, and then apply eq. (36) for the resulting collection of particles.

   Example: To find the CM of the Earth–Moon system, we may treat Earth as a point particle of mass $M_E = 5.97 \cdot 10^{24}$ kg located at the Earth CM (which is at the Earth geometric center by the symmetry rule) and the Moon as another point particle of mass $M_M = 7.35 \cdot 10^{22}$ kg at the Moon's CM (which is at the Moon's geometric center). The distance between these two 'particles' is $L = 384,000$ km, so by eq. (34), the CM of the Earth–Moon system lies at the distance

$$L_E \;=\; \frac{M_M}{M_E + M_M} \times L \;=\; 4660 \text{ km} \tag{40}$$

from the Earth's center in the direction of the Moon. Note that the distance (40) is smaller than the Earth's radius $R_E = 6380$km, so the system's CM lies 1720 km below the ground.

Note: the parts of a body or a system do not have to be separate from each other. As long as you can mentally split the body into several parts and you can find the CM of each part, you may use this rule. You do not need to actually cut the body!

Motion of the Center of Mass

Now consider a system of several *moving* particles or bodies. The motion of the system's center of mass is described by a time-dependent version of eq. (36):

$$\vec{\mathbf{R}}_{\mathrm{cm}}(t) \;=\; \sum_{i=1}^{N} \frac{m_i}{M_{\mathrm{tot}}} \times \vec{\mathbf{r}}_i(t) \tag{41}$$

Consequently, the CM's displacement is related to individual particles' displacements as

$$\Delta\vec{\mathbf{R}}_{\mathrm{cm}} \;=\; \sum_{i=1}^{N} \frac{m_i}{M_{\mathrm{tot}}} \times \Delta\vec{\mathbf{r}}_i \tag{42}$$

This formula works just as well for a system of macroscopic bodies (or body parts) rather than point-like particles. And if a body or a part does not rotate or change its shape, we don't need to know where exactly is its CM located, we may simply use the overall displacement of that body or part.

The velocity of the system's CM follows by dividing the displacement (42) by the time interval Δt and taking the limit $\Delta t \to 0$. Thus,

$$\begin{aligned}
\vec{\mathbf{v}}_{\mathrm{cm}} &= \lim_{\Delta t\to 0} \frac{\Delta\vec{\mathbf{R}}_{\mathrm{cm}}}{\Delta t} \\
&= \lim_{\Delta t\to 0} \frac{1}{\Delta t} \sum_{i=1}^{N} \frac{m_i}{M_{\mathrm{tot}}} \times \Delta\vec{\mathbf{r}}_i \\
&= \sum_{i=1}^{N} \frac{m_i}{M_{\mathrm{tot}}} \times \lim_{\Delta t\to 0} \frac{\Delta\vec{\mathbf{r}}_i}{\Delta t} \\
&= \sum_{i=1}^{N} \frac{m_i}{M_{\mathrm{tot}}} \times \vec{\mathbf{v}}_i
\end{aligned} \tag{43}$$

Moreover,

$$\sum_{i=1}^{N} \frac{m_i}{M_{\mathrm{tot}}} \times \vec{\mathbf{v}}_i \;=\; \frac{1}{M_{\mathrm{tot}}} \sum_{i=1}^{N} m_i \times \vec{\mathbf{v}}_i \;=\; \frac{1}{M_{\mathrm{tot}}} \sum_{i=1}^{N} \vec{\mathbf{P}}_i \;=\; \frac{1}{M_{\mathrm{tot}}} \times \vec{\mathbf{P}}_{\mathrm{net}}\,,$$

hence the CM's velocity is related to the net momentum of the system as simply

$$\vec{\mathbf{v}}_{\mathrm{cm}} = \frac{\vec{\mathbf{P}}_{\mathrm{net}}}{M_{\mathrm{tot}}} \quad\Longleftrightarrow\quad \vec{\mathbf{P}}_{\mathrm{net}} = M_{\mathrm{tot}} \times \vec{\mathbf{v}}_{\mathrm{cm}}\,. \tag{44}$$

The net momentum has a nice property that it does not care for any internal forces between different parts of the system. Only the external forces affect the net momentum according to

$$\Delta\vec{\mathbf{P}}_{\mathrm{net}} = \text{net impulse of external forces only.} \tag{45}$$

For a very short period of time, the impulse of a force is simply $\vec{\mathbf{F}} \times \Delta t$ and therefore

$$\Delta\vec{\mathbf{P}}_{\mathrm{net}} = \Delta t \times \vec{\mathbf{F}}^{\mathrm{net}}_{\mathrm{ext}}\,. \tag{46}$$

Combining this formula with eq. (44) and dividing by Δt we find

$$M_{\mathrm{tot}} \times \frac{\vec{\mathbf{v}}_{\mathrm{cm}}}{\Delta t} = \frac{\Delta\vec{\mathbf{P}}_{\mathrm{net}}}{\Delta t} = \vec{\mathbf{F}}^{\mathrm{net}}_{\mathrm{ext}}\,, \tag{47}$$

and therefore

$$M_{\text{tot}} \times \vec{\mathbf{a}}_{\text{cm}} \;=\; \vec{\mathbf{F}}^{\text{net}}_{\text{ext}}\,. \tag{48}$$

In other words, *the center of mass moves as a particle of mass* M_{tot} *subject to the external forces acting on the system, but it does not care about the internal forces.* In particular, when the external forces balance each other, the CM either stays at rest or moves along a straight line at constant speed. The forces between different parts of the system affect the *relative* motion of those parts, but they don't affect the CM motion.

Likewise, when the only external force is gravity, the CM moves like a projectile in free fall. For example, when you take a high jump, your arms, legs, head, butt, *etc.*, move in rather complicated ways, but your center of mass follows a simple parabola.