

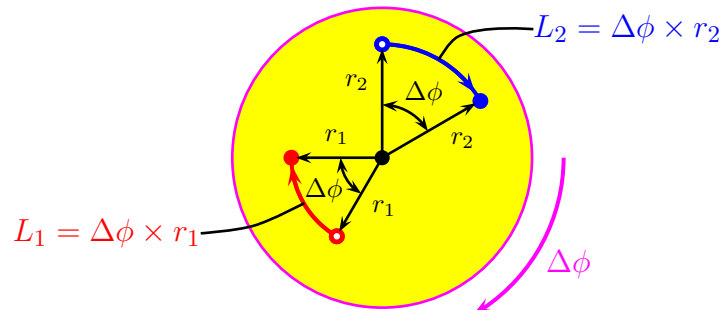
Rotation Kinematics, Moment of Inertia, and Torque

Mathematically, rotation of a rigid body about a fixed axis is analogous to a linear motion in one dimension. Although the physical quantities involved in rotation are quite distinct from their counterparts for the linear motion, the formulae look very similar and may be manipulated in similar ways. Here is the correspondence table:

Linear Motion	Angular Motion
Linear displacement Δx	Angular displacement $\Delta\varphi$
Linear velocity $v = \frac{\Delta x}{\Delta t}$	Angular velocity $\omega = \frac{\Delta\varphi}{\Delta t}$
Linear acceleration $a = \frac{\Delta v}{\Delta t}$	Angular acceleration $\alpha = \frac{\Delta\omega}{\Delta t}$
Motion with constant acceleration: $v(t) = v_0 + at, \quad x(t) = x_0 + v_0t + \frac{1}{2}at^2$	Rotation with constant angular acceleration: $\omega(t) = \omega_0 + \alpha t, \quad \varphi(t) = \varphi_0 + \omega_0t + \frac{1}{2}\alpha t^2$
Mass m	Moment of inertia I
Kinetic energy $K = \frac{1}{2}mv^2$	Kinetic energy $K = \frac{1}{2}I\omega^2$
Force F	Torque τ
Equation of Motion $F = ma$	Equation of Motion $\tau = I\alpha$
Work $W = F\Delta x$	Work $W = \tau\Delta\varphi$
Linear momentum $P = mv$	Angular Momentum $L = I\omega$

Linear Motion of the Parts of a Rotating Body

When a rigid body rotates around a fixed axis, each part of the body moves in a circle, or in a circular arc:



Each part moves in an arc of different radius r — equal to the distance of that part to the axis — but they all span the same angle equal to the angular displacement $\Delta\phi$ of the body. The distance traveled by each part is

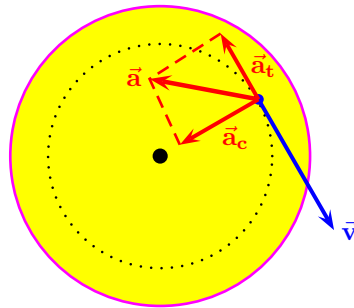
$$L = \Delta\phi \times r \quad \text{for } \Delta\phi \text{ in radians.} \quad (1)$$

Dividing this distance by the time interval Δt , we find the linear speed of the part:

$$v = \frac{L}{\Delta t} = \frac{\Delta\phi}{\Delta t} \times r = \omega \times r \quad (2)$$

where ω is the *angular velocity* of the rotation. Again, ω should be measured in radians per second.

The velocity vector \vec{v} of the part is tangent to the circle centered on the axis as shown below:



The acceleration vector of the part has two components:

$$\vec{a} = \vec{a}_t + \vec{a}_c \quad (3)$$

the tangential acceleration and the centripetal acceleration. The tangential acceleration has magnitude

$$a_t = r \times \alpha \quad \text{for } \alpha \text{ in radian/s}^2 \quad (4)$$

and direction parallel to the velocity vector \vec{v} . That is, it has the same direction as \vec{v} if the rotational speed increases or opposite direction to \vec{v} if the rotational speed decreases.

The centripetal acceleration has magnitude

$$a_c = \frac{v^2}{r} = \frac{(\omega \times r)^2}{r} = \omega^2 \times r \quad \text{for } \omega \text{ in radian/s} \quad (5)$$

— regardless of the angular acceleration α — and its direction is towards the axis of rotation. Since the tangential and the centripetal accelerations are always perpendicular to each other, the net acceleration vector \vec{a} has magnitude

$$a = \sqrt{a_t^2 + a_c^2} = r \times \sqrt{\alpha^2 + \omega^4}. \quad (6)$$

Kinetic Energy of Rotation

The net kinetic energy of a rotating body is simply the sum of kinetic energies of all its parts. Thus,

$$\begin{aligned} K &= \sum_i^{\text{parts}} \frac{1}{2} m_i \times v_i^2 \\ &= \sum_i^{\text{parts}} \frac{1}{2} m_i \times (r_i \times \omega)^2 \\ &= \frac{\omega^2}{2} \times \sum_i^{\text{parts}} m_i \times r_i^2 \\ &= \frac{\omega^2}{2} \times I \end{aligned} \quad (7)$$

where

$$I = \sum_i^{\text{parts}} m_i \times r_i^2 \quad (8)$$

is the **moment of inertia** of the body — the rotational analogue of the mass.

The moment of inertia of a body depends on its mass, size, and shape, and also on a particular axis around which the body is rotated. Indeed, in eq. (8), r_i is the distance of part

i from the axis of rotation. For example, for the rotation about the z axis of the 3D body,

$$r_i^2 = x_i^2 + y_i^2 + \cancel{x_i^2}, \quad (9)$$

so the moment of inertia around this axis would be

$$I^{\text{wrt } z \text{ axis}} = \sum_i^{\text{parts}} m_i \times (x_i^2 + y_i^2 + \cancel{x_i^2}). \quad (10)$$

But if the same body rotates around the x axis, the moment of rotation would be given by a different sum

$$I^{\text{wrt } x \text{ axis}} = \sum_i^{\text{parts}} m_i \times (y_i^2 + z_i^2 + \cancel{x_i^2}). \quad (11)$$

Also, the location of the axis is just as important as its direction.

For a continuous body, the sum in eq. (8) for the moment of inertia becomes a volume integral, but calculating such integrals is way beyond the scope of this class. Instead, let me give you a few examples of moments of inertia for bodies of particularly simple shapes:

- Solid rod of length L and uniform density; axis \perp to the rod. For axis through the end of the rod, $I = \frac{1}{3}ML^2$. For axis through the middle of the rod, $I = \frac{1}{12}ML^2$.
- A thin cylindrical shell of radius R and uniform density. For axis=cylinder's axis, $I = MR^2$. Same formula for a thin hoop and an axis through the center, \perp to the hoops' plane. But for axis=diameter of the hoop, $I = \frac{1}{2}MR^2$.
- A solid cylinder or disk of radius R and uniform density. For axis=cylinder's axis, $I = \frac{1}{2}MR^2$.
- A thin spherical shell of radius R and uniform density and thickness. For any axis through the sphere's center, $I = \frac{2}{3}MR^2$.
- A solid ball of radius R and uniform density. For any axis through the sphere's center, $I = \frac{2}{5}MR^2$.
- Any body whose mass is concentrated in several points (or rather parts of relatively small sizes) — use eq. (8).

Torque

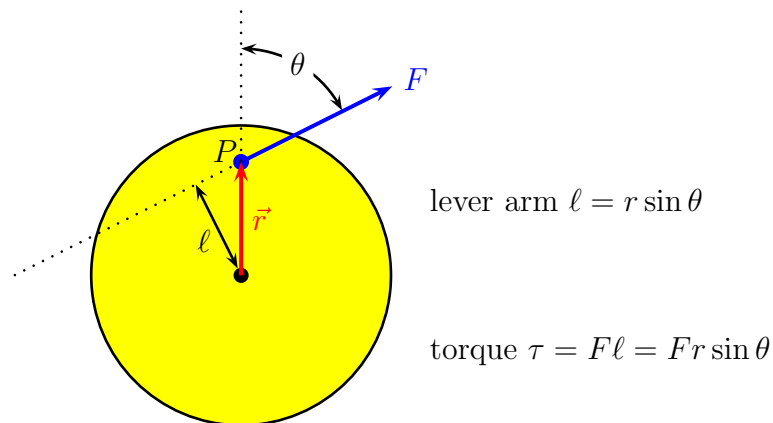
The rotational analogue of the Newton's Second Law $ma = F$ is

$$I\alpha = \tau \quad (12)$$

where τ is the **torque** — the rotational analogue of the force. The torque of a force $\vec{\mathbf{F}}$ depends on the force's magnitude and direction, and also on location of the point where the force is applied. It is defined as a product of a force (acting on a rotating body) and its **lever arm**,

$$\tau = F \times \ell, \quad (13)$$

where the lever arm ℓ is the distance between the axis of rotation and the line of force — line through the point P where the force acts in the direction of the force vector $\vec{\mathbf{F}}$. In a 2D plane perpendicular to the axis of rotation,



In components, force $\vec{\mathbf{F}} = (F_x, F_y, F_z)$ acting at point $\vec{\mathbf{r}} = (x, y, z)$ has torque (around the z axis)

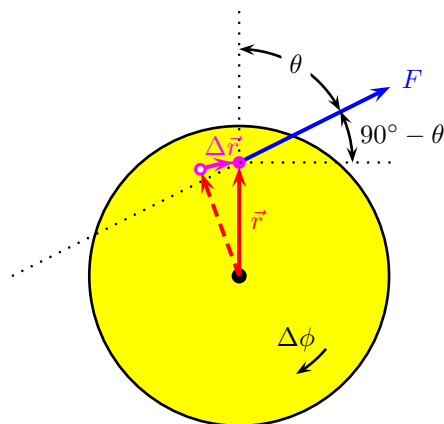
$$\tau = yF_x - xF_y. \quad (14)$$

Note that the F_z component of the force — parallel to the axis — does not contribute to the torque.

The work of a torque on a rotating body is simply

$$W = \tau \times \Delta\phi \quad \text{for } \Delta\phi \text{ in radians.} \quad (15)$$

This is the rotational analogy of the work $W = F \times \Delta x$ of a linear force. To see how this works, consider a rotation through a small angle $\Delta\phi$ as shown below:



angle between $\Delta\vec{r}$ and $\vec{F} = 90^\circ - \theta$,

$$|\Delta\vec{r}| = r \times \Delta\phi,$$

$$W = |\Delta\vec{r}| \times |\vec{F}| \times \cos(90^\circ - \theta)$$

$$= (r \times \Delta\phi) \times F \times \sin\theta$$

$$= (r \times F \times \sin\theta) \times \Delta\phi$$

$$= \tau \times \Delta\phi.$$

(16)

When a body does not have a fixed axis of rotation but can turn in all three dimensions, we need to generalize the torque to a vector $\vec{\tau}$. The torque of a force \vec{F} acting at point \vec{r} is the *vector product*

$$\vec{\tau} = \vec{r} \times \vec{F}. \quad (17)$$

In components,

$$\tau_x = yF_z - zF_y,$$

$$\tau_y = zF_x - xF_z, \quad (18)$$

$$\tau_z = xF_y - yF_x.$$

The magnitude of the torque vector is

$$|\vec{\tau}| = |\vec{r}| \times |\vec{F}| \times \sin(\text{angle between } \vec{r} \text{ and } \vec{F}) \quad (19)$$

and its direction is \perp to both the force and the radius vector,

$$\vec{\tau} \perp \vec{F}, \quad \vec{\tau} \perp \vec{r}. \quad (20)$$

But in this class, I will mostly focus on torques around a fixed axis.