Bogolyubov Transform

Given some kind of annihilation and creation operators $\hat{a}_{\mathbf{k}}$ and $\hat{a}^{\dagger}_{\mathbf{k}}$ which satisfy the bosonic commutation relations

$$[\hat{a}_{\mathbf{k}},\hat{a}_{\mathbf{k}'}] \;=\; [\hat{a}^{\dagger}_{\mathbf{k}},\hat{a}^{\dagger}_{\mathbf{k}'}] \;=\; 0, \qquad [\hat{a}_{\mathbf{k}},\hat{a}^{\dagger}_{\mathbf{k}'}] \;=\; \delta_{\mathbf{k},\mathbf{k}'}\,, \tag{1}$$

we may define new operators $\hat{b}_{\mathbf{k}}$ and $\hat{b}^{\dagger}_{\mathbf{k}}$ according to

$$\hat{b}_{\mathbf{k}} \;=\; \cosh(t_{\mathbf{k}})\hat{a}_{\mathbf{k}} \;+\; \sinh(t_{\mathbf{k}})\hat{a}^{\dagger}_{-\mathbf{k}}\,, \quad \hat{b}^{\dagger}_{\mathbf{k}} \;=\; \cosh(t_{\mathbf{k}})\hat{a}^{\dagger}_{\mathbf{k}} \;+\; \sinh(t_{\mathbf{k}})\hat{a}_{-\mathbf{k}} \tag{2}$$

for some arbitrary real parameters $t_{\mathbf{k}} = t_{-\mathbf{k}}$. These new operators $\hat{b}_{\mathbf{k}}$ and $\hat{b}^{\dagger}_{\mathbf{k}}$ satisfy the same the same bosonic commutation relations as the $\hat{a}_{\mathbf{k}}$ and the $\hat{a}^{\dagger}_{\mathbf{k}}$:

$$[\hat{b}_{\mathbf{k}},\hat{b}_{\mathbf{k}'}] \;=\; [\hat{b}^{\dagger}_{\mathbf{k}},\hat{b}^{\dagger}_{\mathbf{k}'}] \;=\; 0, \qquad [\hat{b}_{\mathbf{k}},\hat{b}^{\dagger}_{\mathbf{k}'}] \;=\; \delta_{\mathbf{k},\mathbf{k}'}\,. \tag{3}$$

The Bogolyubov transform — replacing the 'original' creation and annihilation operators $\hat{a}^{\dagger}_{\mathbf{k}}$ and $\hat{a}_{\mathbf{k}}$ with the 'transformed' operators $\hat{b}^{\dagger}_{\mathbf{k}}$ and $\hat{b}_{\mathbf{k}}$ — is useful for diagonalizing quadratic Hamiltonians of the form

$$\hat{H} \;=\; \sum_{\mathbf{k}} A_{\mathbf{k}}\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}} \;+\; \tfrac{1}{2}\sum_{\mathbf{k}} B_{\mathbf{k}}\left(\hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}} \;+\; \hat{a}^{\dagger}_{\mathbf{k}}\hat{a}^{\dagger}_{-\mathbf{k}}\right) \tag{4}$$

where for all momenta $\mathbf{k}$, $A_{\mathbf{k}} = A_{-\mathbf{k}}$, $B_{\mathbf{k}} = B_{-\mathbf{k}}$, and $A_{\mathbf{k}} > |B_{\mathbf{k}}|$. Indeed, for a suitable choice of the $t_{\mathbf{k}}$ parameters,

$$\hat{H} \;=\; \sum_{\mathbf{k}} \omega_{\mathbf{k}}\hat{b}^{\dagger}_{\mathbf{k}}\hat{b}_{\mathbf{k}} \;+\; \text{const} \qquad \text{where } \omega_{\mathbf{k}} \;=\; \sqrt{A^2_{\mathbf{k}} - B^2_{\mathbf{k}}}\,. \tag{5}$$

Moreover, $\hat{b}^{\dagger}_{\mathbf{k}}\hat{b}_{\mathbf{k}} - \hat{b}^{\dagger}_{-\mathbf{k}}\hat{b}_{-\mathbf{k}} = \hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}} - \hat{a}^{\dagger}_{-\mathbf{k}}\hat{a}_{-\mathbf{k}}$ and consequently

$$\hat{\mathbf{P}} \;\equiv\; \sum_{\mathbf{k}} \mathbf{k}\times\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}} \;=\; \sum_{\mathbf{k}} \mathbf{k}\times\hat{b}^{\dagger}_{\mathbf{k}}\hat{b}_{\mathbf{k}}\,. \tag{6}$$

Proof of (3):

Combining definitions (2) with commutation relations (1), we immediately calculate

$$[\hat{b}_{\mathbf{k}}, \hat{b}_{\mathbf{k}'}] \;=\; \cosh(t_{\mathbf{k}})\sinh(t_{\mathbf{k}'})\delta_{\mathbf{k},-\mathbf{k}'} \;-\; \sinh(t_{\mathbf{k}})\cosh(t_{\mathbf{k}'})\delta_{-\mathbf{k},\mathbf{k}'} \;=\; 0 \tag{7}$$

where the second equality follows from $t_{\mathbf{k}} = t_{\mathbf{k}'}$ for $\mathbf{k} = -\mathbf{k}'$. Likewise, $[\hat{b}^\dagger_{\mathbf{k}}, \hat{b}^\dagger_{\mathbf{k}'}] = 0$. Finally,

$$\begin{aligned}[\hat{b}_{\mathbf{k}}, \hat{b}^\dagger_{\mathbf{k}'}] \;&=\; \cosh(t_{\mathbf{k}})\cosh(t_{\mathbf{k}'})\delta_{\mathbf{k},\mathbf{k}'} \;-\; \sinh(t_{-\mathbf{k}})\sinh(t_{-\mathbf{k}'})\delta_{-\mathbf{k},-\mathbf{k}'} \\ &=\; \delta_{\mathbf{k},\mathbf{k}'}\Big(\cosh^2(t_{\mathbf{k}}) - \sinh^2(t_{\mathbf{k}}) = 1\Big).\end{aligned} \tag{8}$$

In other words, the $\hat{b}_{\mathbf{k}}$ and $\hat{b}^\dagger_{\mathbf{k}}$ operators satisfy the same bosonic commutations relations

$$[\hat{b}_{\mathbf{k}}, \hat{b}_{\mathbf{k}'}] \;=\; 0, \quad [\hat{b}^\dagger_{\mathbf{k}}, \hat{b}^\dagger_{\mathbf{k}'}] \;=\; 0, \quad [\hat{b}_{\mathbf{k}}, \hat{b}^\dagger_{\mathbf{k}'}] \;=\; \delta_{\mathbf{k},\mathbf{k}'}\,. \tag{9}$$

as the original $\hat{a}_{\mathbf{k}}$ and $\hat{a}^\dagger_{\mathbf{k}}$ operators. $\mathcal{Q.E.D.}$

Proof of (5):

Applying eqs. (2) twice, we immediately obtain

$$\hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}} \;=\; \cosh^2(t_{\mathbf{k}})\,\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \cosh(t_{\mathbf{k}})\sinh(t_{\mathbf{k}})\,(\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}} + \hat{a}_{-\mathbf{k}}\hat{a}_{\mathbf{k}}) + \sinh^2(t_{\mathbf{k}})\,(\hat{a}^\dagger_{-\mathbf{k}}\hat{a}_{-\mathbf{k}}+1). \tag{10}$$

Next, we use $t_{-\mathbf{k}} = t_{\mathbf{k}}$ to combine

$$\begin{aligned}\hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}} \;+\; \hat{b}^\dagger_{-\mathbf{k}}\hat{b}_{-\mathbf{k}} \;=\;& \Big(\cosh^2(t_{\mathbf{k}}) + \sinh^2(t_{\mathbf{k}}) = \cosh(2t_{\mathbf{k}})\Big) \times (\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \hat{a}^\dagger_{-\mathbf{k}}\hat{a}_{-\mathbf{k}}) \\ &+\; \Big(2\cosh(t_{\mathbf{k}})\sinh(t_{\mathbf{k}}) = \sinh(2t_{\mathbf{k}})\Big) \times (\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}} \;+\; \hat{a}_{-\mathbf{k}}\hat{a}_{\mathbf{k}}) \;+\; \text{const.}\end{aligned} \tag{11}$$

Finally, for $\omega_{-\mathbf{k}} \equiv \omega_{\mathbf{k}}$ we have

$$\begin{aligned}\sum_{\mathbf{k}} \omega_{\mathbf{k}}\hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}} \;&=\; \frac{1}{2}\sum_{\mathbf{k}} \omega_{\mathbf{k}}(\hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}} + \hat{b}^\dagger_{-\mathbf{k}}\hat{b}_{-\mathbf{k}}) \\ &= \sum_{\mathbf{k}} \omega_{\mathbf{k}}\cosh(2t_{\mathbf{k}})\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} \;+\; \tfrac{1}{2}\sum_{\mathbf{k}} \omega_{\mathbf{k}}\sinh(2t_{\mathbf{k}})\left(\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}} \;+\; \hat{a}_{-\mathbf{k}}\hat{a}_{\mathbf{k}}\right) \;+\; \text{const.}\end{aligned} \tag{12}$$

Consequently, the Hamiltonian (4)can be “diagonalized” in terms of the transformed creation

/ annihilation operators (2), provided we can find $\omega_{\mathbf{k}} = \omega_{-\mathbf{k}}$ and $t_{\mathbf{k}} = t_{-\mathbf{k}}$ such that

$$\omega_{\mathbf{k}} \cosh(2t_{\mathbf{k}}) \;=\; A_{\mathbf{k}} \quad \text{and} \quad \omega_{\mathbf{k}} \sinh(2t_{\mathbf{k}}) \;=\; B_{\mathbf{k}}\,. \tag{13}$$

These equations are easy to solve, and the solution exists as long as $A_{\mathbf{k}} = A_{-\mathbf{k}}$, $B_{\mathbf{k}} = B_{-\mathbf{k}}$, and $A_{\mathbf{k}} > |B_{\mathbf{k}}|$, namely

$$t_{\mathbf{k}} \;=\; \frac{1}{2}\,\text{artanh}\,\frac{B_{\mathbf{k}}}{A_{\mathbf{k}}} \quad \text{and} \quad \omega_{\mathbf{k}} \;=\; \sqrt{A_{\mathbf{k}}^2 - B_{\mathbf{k}}^2}\,. \tag{14}$$

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Proof of (6):
Using eq. (10) and $t_{-\mathbf{k}} = t_{\mathbf{k}}$, we immediately see that

$$\hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}} \;-\; \hat{b}^\dagger_{-\mathbf{k}}\hat{b}_{-\mathbf{k}} \;=\; \Big(\cosh^2(t_{\mathbf{k}}) - \sinh^2(t_{\mathbf{k}}) = 1\Big) \times (\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} \;-\; \hat{a}^\dagger_{-\mathbf{k}}\hat{a}_{-\mathbf{k}}). \tag{15}$$

Consequently,

$$\begin{aligned} \hat{\mathbf{P}} \;&=\; \sum_{\mathbf{k}} \mathbf{k} \times \hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} \;=\; \sum_{\mathbf{k}} (-\mathbf{k}) \times \hat{a}^\dagger_{-\mathbf{k}}\hat{a}_{-\mathbf{k}} \\ &=\; \frac{1}{2}\sum_{\mathbf{k}} \mathbf{k} \times (\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} \;-\; \hat{a}^\dagger_{-\mathbf{k}}\hat{a}_{-\mathbf{k}}) \\ &=\; \frac{1}{2}\sum_{\mathbf{k}} \mathbf{k} \times (\hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}} \;-\; \hat{b}^\dagger_{-\mathbf{k}}\hat{b}_{-\mathbf{k}}) \\ &=\; \sum_{\mathbf{k}} \mathbf{k} \times \hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}}\,. \end{aligned} \tag{16}$$

$\mathcal{Q.E.D.}$