Dirac Matrices and Lorentz Spinors

Background: In 3D, the spinor $j = \frac{1}{2}$ representation of the Spin(3) rotation group is constructed from the Pauli matrices σ^x, σ^y, and σ^z, which obey both commutation and anticommutation relations

$$[\sigma^i, \sigma^j] \;=\; 2i\epsilon^{ijk}\sigma^k \quad \text{and} \quad \{\sigma^i, \sigma^j\} \;=\; 2\delta^{ij} \times \mathbf{1}_{2\times 2}\,. \tag{1}$$

Consequently, the spin matrices

$$\mathbf{S} \;=\; -\tfrac{i}{4}\boldsymbol{\sigma} \times \boldsymbol{\sigma} \;=\; \tfrac{1}{2}\boldsymbol{\sigma} \tag{2}$$

commute with each other like angular momenta, $[S^i, S^j] = i\epsilon^{ijk}S^k$, so they represent the generators of the rotation group. Moreover, under finite rotations $R(\phi, \mathbf{n})$ represented by

$$M(R) \;=\; \exp\bigl(-i\phi\mathbf{n}\cdot\mathbf{S}\bigr), \tag{3}$$

the spin matrices transform into each other as components of a 3–vector,

$$M^{-1}(R)S^iM(R) \;=\; R^{ij}S^j. \tag{4}$$

In this note, I shall generalize this construction to the *Dirac spinor* representation of the Lorentz symmetry Spin$(3, 1)$.

Dirac Matrices γ^μ generalize the anti-commutation properties of the Pauli matrices σ^i to $3 + 1$ Minkowski dimensions:

$$\gamma^\mu\gamma^\nu \;+\; \gamma^\nu\gamma^\mu \;=\; 2g^{\mu\nu} \times \mathbf{1}_{4\times 4}\,. \tag{5}$$

The γ^μ are 4×4 matrices, but there are several different conventions for their specific form In my class I shall follow the same convention as the Peskin & Schroeder textbook, namely the Weyl convention where in 2×2 block notations

$$\gamma^0 \;=\; \begin{pmatrix} 0 & \mathbf{1}_{2\times 2} \\ \mathbf{1}_{2\times 2} & 0 \end{pmatrix}, \qquad \vec{\gamma} \;=\; \begin{pmatrix} 0 & +\vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}. \tag{6}$$

Note that the γ^0 matrix is hermitian while the γ^1, γ^2, and γ^3 matrices are anti-hermitian. Apart from that, the specific forms of the matrices are not important, the Physics follows from the anti-commutation relations (5).

Lorentz spin matrices generalize $\mathbf{S} = -\frac{i}{4}\boldsymbol{\sigma} \times \boldsymbol{\sigma}$ rather than $\mathbf{S} = \frac{1}{2}\boldsymbol{\sigma}$. In 4D, the vector product becomes the antisymmetric tensor product, so we define

$$S^{\mu\nu} \;=\; -S^{\nu\mu} \;\stackrel{\text{def}}{=}\; \tfrac{i}{4}[\gamma^\mu, \gamma^\nu]. \tag{7}$$

Thanks to the anti-commutation relations (5) for the γ^μ matrices, the $S^{\mu\nu}$ obey the commutation relations of the Lorentz generators $\hat{J}^{\mu\nu} = -\hat{J}^{\nu\mu}$. Moreover, the commutation relations of the spin matrices $S^{\mu\nu}$ with the Dirac matrices γ^μ are similar to the commutation relations of the $\hat{J}^{\mu\nu}$ with a Lorentz vector such as $\hat{P}^\mu$.

Lemma:

$$[\gamma^\lambda, S^{\mu\nu}] \;=\; ig^{\lambda\mu}\gamma^\nu \;-\; ig^{\lambda\nu}\gamma^\mu. \tag{8}$$

Proof: Combining the definition (7) of the spin matrices as commutators with the anti-commutation relations (5), we have

$$\gamma^\mu\gamma^\nu \;=\; \tfrac{1}{2}\{\gamma^\mu, \gamma^\nu\} \;+\; \tfrac{1}{2}[\gamma^\mu, \gamma^\nu] \;=\; g^{\mu\nu} \times \mathbf{1}_{4\times 4} \;-\; 2iS^{\mu\nu}. \tag{9}$$

Since the unit matrix commutes with everything, we have

$$[X, S^{\mu\nu}] \;=\; \tfrac{i}{2}[X, \gamma^\mu\gamma^\nu] \qquad \text{for any matrix } X, \tag{10}$$

and the commutator on the RHS may often be obtained from the Leibniz rules for the commutators or anticommutators:

$$\begin{aligned} [A, BC] \;&=\; [A, B]C \;+\; B[A, C] \;=\; \{A, B\}C \;-\; B\{A, C\}, \\ \{A, BC\} \;&=\; [A, B]C \;+\; B\{A, C\} \;=\; \{A, B\}C \;-\; B[A, C]. \end{aligned} \tag{11}$$

In particular,

$$[\gamma^\lambda, \gamma^\mu\gamma^\nu] \;=\; \{\gamma^\lambda, \gamma^\mu\}\gamma^\nu \;-\; \gamma^\mu\{\gamma^\lambda, \gamma^\nu\} \;=\; 2g^{\lambda\mu}\gamma^\nu \;-\; 2g^{\lambda\nu}\gamma^\mu \tag{12}$$

and hence

$$[\gamma^\lambda, S^{\mu\nu}] \;=\; \tfrac{i}{2}[\gamma^\lambda, \gamma^\mu\gamma^\nu] \;=\; ig^{\lambda\mu}\gamma^\nu \;-\; ig^{\lambda\nu}\gamma^\mu. \tag{13}$$

Quod erat demonstrandum.

Theorem: The $S^{\mu\nu}$ matrices commute with each other like Lorentz generators,

$$\left[S^{\kappa\lambda}, S^{\mu\nu}\right] \;=\; ig^{\lambda\mu}S^{\kappa\nu} \;-\; ig^{\kappa\nu}S^{\mu\lambda} \;-\; ig^{\lambda\nu}S^{\kappa\mu} \;+\; ig^{\kappa\mu}S^{\nu\lambda}. \tag{14}$$

<u>Proof</u>: Again, we use the Leibniz rule and eq. (9):

$$\begin{aligned}
\left[\gamma^\kappa\gamma^\lambda, S^{\mu\nu}\right] \;&=\; \gamma^\kappa\left[\gamma^\lambda, S^{\mu\nu}\right] \;+\; \left[\gamma^\kappa, S^{\mu\nu}\right]\gamma^\lambda \\
&=\; \gamma^\kappa\big(ig^{\lambda\mu}\gamma^\nu \;-\; ig^{\lambda\nu}\gamma^\mu\big) \;+\; \big(ig^{\kappa\mu}\gamma^\nu \;-\; ig^{\kappa\nu}\gamma^\mu\big)\gamma^\lambda \\
&=\; ig^{\lambda\mu}\gamma^\kappa\gamma^\nu \;-\; ig^{\kappa\nu}\gamma^\mu\gamma^\lambda \;-\; ig^{\lambda\nu}\gamma^\kappa\gamma^\mu \;+\; ig^{\kappa\mu}\gamma^\nu\gamma^\lambda \\
&=\; ig^{\lambda\mu}\big(g^{\kappa\nu} - 2iS^{\kappa\nu}\big) \;-\; ig^{\kappa\nu}\big(g^{\lambda\mu} + 2iS^{\lambda\mu}\big) \\
&\qquad -\; ig^{\lambda\nu}\big(g^{\kappa\mu} - 2iS^{\kappa\mu}\big) \;+\; ig^{\kappa\mu}\big(g^{\lambda\nu} + 2iS^{\lambda\nu}\big) \\
&=\; 2g^{\lambda\mu}S^{\kappa\nu} \;-\; 2g^{\kappa\nu}S^{\lambda\mu} \;-\; 2g^{\lambda\nu}S^{\kappa\mu} \;+\; 2g^{\kappa\mu}S^{\lambda\nu},
\end{aligned} \tag{15}$$

and hence

$$\left[S^{\kappa\lambda}, S^{\mu\nu}\right] \;=\; \tfrac{i}{2}\left[\gamma^\kappa\gamma^\lambda, S^{\mu\nu}\right] \;=\; ig^{\lambda\mu}S^{\kappa\nu} \;-\; ig^{\kappa\nu}S^{\mu\lambda} \;-\; ig^{\lambda\nu}S^{\kappa\mu} \;+\; ig^{\kappa\mu}S^{\nu\lambda}. \tag{16}$$

Quod erat demonstrandum.

In light of this theorem, the $S^{\mu\nu}$ matrices *represent* the Lorentz generators $\hat{J}^{\mu\nu}$ in a 4-component spinor multiplet.

Finite Lorentz transforms:

Any continuous Lorentz transform — a rotation, or a boost, or a product of a boost and a rotation — obtains from exponentiating an infinitesimal symmetry

$$X'^\mu \;=\; X^\mu \;+\; \epsilon^{\mu\nu}X_\nu \tag{17}$$

where the infinitesimal $\epsilon^{\mu\nu}$ matrix is antisymmetric when both indices are raised (or both lowered), $\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}$. Thus, the $L^\mu_{\ \nu}$ matrix of any continuous Lorentz transform is a matrix exponential

$$L^\mu_{\ \nu} \;=\; \exp(\Theta)^\mu_{\ \nu} \;\equiv\; \delta^\mu_\nu \;+\; \Theta^\mu_{\ \nu} \;+\; \tfrac{1}{2}\Theta^\mu_{\ \lambda}\Theta^\lambda_{\ \nu} \;+\; \tfrac{1}{6}\Theta^\mu_{\ \lambda}\Theta^\lambda_{\ \kappa}\Theta^\kappa_{\ \nu} \;+\; \cdots \tag{18}$$

of some matrix Θ that becomes antisymmetric when both of its indices are raised or lowered, $\Theta^{\mu\nu} = -\Theta^{\nu\mu}$. Note however that in the matrix exponential (18), the first index of Θ is raised while the second index is lowered, so the antisymmetry condition becomes $(g\Theta)^\top = -(g\Theta)$ instead of $\Theta^\top = -\Theta$.

The Dirac spinor representation of the finite Lorentz transform (18) is the 4×4 matrix

$$M_D(L) \;=\; \exp\bigl(-\tfrac{i}{2}\,\Theta_{\alpha\beta}S^{\alpha\beta}\bigr). \tag{19}$$

The group law for such matrices

$$\forall L_1, L_2 \in \mathrm{SO}^+(3,1), \quad M_D(L_2L_1) \;=\; M_D(L_2)M_D(L_1) \tag{20}$$

follows automatically from the $S^{\mu\nu}$ satisfying the commutation relations (14) of the Lorentz generators, so I am not going to prove it. Instead, let me show that when the Dirac matrices γ^μ are sandwiched between the $M_D(L)$ and its inverse, they transform into each other as components of a Lorentz 4–vector,

$$M_D^{-1}(L)\gamma^\mu M_D(L) \;=\; L^\mu_{\ \nu}\gamma^\nu. \tag{21}$$

This formula makes the Dirac equation transform covariantly under the Lorentz transforms.

Proof: In light of the exponential form (19) of the matrix $M_D(L)$ representing a finite Lorentz transform in the Dirac spinor multiplet, let's use the multiple commutator formula (AKA the Hadamard Lemma): for any 2 matrices F and H,

$$\exp(-F)H\exp(+F) \;=\; H \;+\; \bigl[H,F\bigr] \;+\; \tfrac{1}{2}\bigl[\bigl[H,F\bigr],F\bigr] \;+\; \tfrac{1}{6}\bigl[\bigl[\bigl[H,F\bigr],F\bigr],F\bigr] \;+\; \cdots. \tag{22}$$

In particular, let $H = \gamma^\mu$ while $F = -\frac{i}{2}\,\Theta_{\alpha\beta}S^{\alpha\beta}$ so that $M_D(L) = \exp(+F)$ and $M_D^{-1}(L) = \exp(-F)$. Consequently,

$$M_D^{-1}(L)\gamma^\mu M_D(L) \;=\; \gamma^\mu \;+\; \bigl[\gamma^\mu,F\bigr] \;+\; \tfrac{1}{2}\bigl[\bigl[\gamma^\mu,F\bigr],F\bigr] \;+\; \tfrac{1}{6}\bigl[\bigl[\bigl[\gamma^\mu,F\bigr],F\bigr],F\bigr] \;+\; \cdots \tag{23}$$

where all the multiple commutators turn out to be linear combinations of the Dirac matrices.

Indeed, the single commutator here is

$$\left[\gamma^\mu, F\right] \;=\; -\tfrac{i}{2}\Theta_{\alpha\beta}\left[\gamma^\mu, S^{\alpha\beta}\right] \;=\; \tfrac{1}{2}\Theta_{\alpha\beta}\big(g^{\mu\alpha}\gamma^\beta \;-\; g^{\mu\beta}\gamma^\alpha\big) \;=\; \Theta_{\alpha\beta}\, g^{\mu\alpha}\gamma^\beta \;=\; \Theta^\mu{}_\lambda\gamma^\lambda, \tag{24}$$

while the multiple commutators follow by iterating this formula:

$$\big[\left[\gamma^\mu, F\right], F\big] \;=\; \Theta^\mu{}_\lambda\left[\gamma^\lambda, F\right] \;=\; \Theta^\mu{}_\lambda\Theta^\lambda{}_\nu\gamma^\nu, \qquad \Big[\big[\left[\gamma^\mu, F\right], F\big], F\Big] \;=\; \Theta^\mu{}_\lambda\Theta^\lambda{}_\rho\Theta^\rho{}_\nu\gamma^\nu, \ldots. \tag{25}$$

Combining all these commutators as in eq. (23), we obtain

$$\begin{aligned}
M_D^{-1}\gamma^\mu M_D \;&=\; \gamma^\mu \;+\; \left[\gamma^\mu, F\right] \;+\; \tfrac{1}{2}\big[\left[\gamma^\mu, F\right], F\big] \;+\; \tfrac{1}{6}\Big[\big[\left[\gamma^\mu, F\right], F\big], F\Big] \;+\; \cdots \\
&=\; \gamma^\mu \;+\; \Theta^\mu{}_\nu\gamma^\nu \;+\; \tfrac{1}{2}\,\Theta^\mu{}_\lambda\Theta^\lambda{}_\nu\gamma^\nu \;+\; \tfrac{1}{6}\,\Theta^\mu{}_\lambda\Theta^\lambda{}_\rho\Theta^\rho{}_\nu\gamma^\nu \;+\; \cdots \\
&=\; \Big(\delta^\mu_\nu \;+\; \Theta^\mu{}_\nu \;+\; \tfrac{1}{2}\Theta^\mu{}_\lambda\Theta^\lambda{}_\nu \;+\; \tfrac{1}{6}\Theta^\mu{}_\lambda\Theta^\lambda{}_\rho\Theta^\rho{}_\nu \;+\; \cdots\Big)\gamma^\nu \\
&\equiv\; L^\mu{}_\nu\gamma^\nu.
\end{aligned} \tag{26}$$

Quod erat demonstrandum.

Dirac Equation and Dirac Spinor Fields

History:

Originally, the Klein–Gordon equation was thought to be the relativistic version of the Schrödinger equation — that is, an equation for the wave function $\psi(\mathbf{x}, t)$ for one relativistic particle. But pretty soon this interpretation run into trouble with bad probabilities (negative, or > 1) when a particle travels through high potential barriers or deep potential wells. There were also troubles with relativistic causality, and a few other things.

Paul Adrien Maurice Dirac had thought that the source of all those troubles was the ugly form of relativistic Hamiltonian $\hat{H} = \sqrt{\hat{\mathbf{p}}^2 + m^2}$ in the coordinate basis, and that he could solve all the problems with the Klein-Gordon equation by rewriting the Hamiltonian as a first-order differential operator

$$\hat{H} \;=\; \hat{\mathbf{p}}\cdot\vec{\alpha} \;+\; m\beta \quad\Longrightarrow\quad \text{Dirac equation}\quad i\,\frac{\partial\psi}{\partial t} \;=\; -i\vec{\alpha}\cdot\nabla\psi \;+\; m\beta\psi \tag{27}$$

where $\alpha_1, \alpha_2, \alpha_3, \beta$ are matrices acting on a multi-component wave function. Specifically, all four of these matrices are Hermitian, square to 1, and anticommute with each other,

$$\{\alpha_i, \alpha_j\} \;=\; 2\delta_{ij}\,, \quad \{\alpha_i, \beta\} \;=\; 0, \quad \beta^2 \;=\; 1. \tag{28}$$

Consequently

$$\left(\vec{\alpha}\cdot\hat{\mathbf{p}}\right)^2 \;=\; \alpha_i\alpha_j \times \hat{p}_i\hat{p}_j \;=\; \tfrac{1}{2}\{\alpha_i, \alpha_j\} \times \hat{p}_i\hat{p}_j \;=\; \delta_{ij} \times \hat{p}_i\hat{p}_j \;=\; \hat{\mathbf{p}}^2, \tag{29}$$

and therefore

$$\hat{H}^2_{\text{Dirac}} \;=\; \left(\vec{\alpha}\cdot\hat{\mathbf{p}} + \beta m\right)^2 \;=\; \left(\vec{\alpha}\cdot\hat{\mathbf{p}}\right)^2 \;+\; \{\alpha_i, \beta\} \times \hat{p}_i m \;+\; \beta^2 \times m^2 \;=\; \hat{\mathbf{p^2}} \;+\; 0 \;+\; m^2. \tag{30}$$

This, the Dirac Hamiltonian squares to $\hat{\mathbf{p}}^2 + m^2$, as it should for the relativistic particle.

The Dirac equation (27) turned out to be a much better description of a relativistic electron (which has spin $= \frac{1}{2}$) than the Klein–Gordon equation. However, it did not resolve the troubles with relativistic causality or bad probabilities for electrons going through big potential differences $e\Delta\Phi > 2m_e c^2$. Those problems are not solvable in the context of a relativistic single-particle quantum mechanics but only in quantum field theory.

Modern point of view:

Today, we interpret the Dirac equation as the equation of motion for a Dirac spinor field $\Psi(x)$, comprising 4 complex component fields $\Psi_\alpha(x)$ arranged in a column vector

$$\Psi(x) \;=\; \begin{pmatrix} \Psi_1(x) \\ \Psi_2(x) \\ \Psi_3(x) \\ \Psi_4(x) \end{pmatrix}, \tag{31}$$

and transforming under the continuous Lorentz symmetries $x'^\mu = L^\mu{}_\nu x^\nu$ according to

$$\Psi'(x') \;=\; M_D(L)\Psi(x). \tag{32}$$

The classical Euler–Lagrange equation of motion for the spinor field is the Dirac equation

$$i\,\frac{\partial}{\partial t}\Psi \;+\; i\vec{\alpha}\cdot\nabla\Psi \;-\; m\beta\Psi \;=\; 0. \tag{33}$$

To recast this equation in a Lorentz-covariant form, let

$$\beta \;=\; \gamma^0, \quad \alpha^i = \;\gamma^0\gamma^i\,; \tag{34}$$

it is easy to see that if the γ^μ matrices obey the anticommutation relations (5) then the $\vec{\alpha}$ and β matrices obey the relations (28) and vice verse. Now let's multiply the whole LHS of the Dirac equation (33) by the $\beta = \gamma^0$:

$$0 \;=\; \gamma^0\Big(i\partial_0 \,+\, i\gamma^0\vec{\gamma}\cdot\nabla \,-\, m\gamma^0\Big)\Psi(x) \;=\; \Big(i\gamma^0\partial_0 \,+\, i\gamma^i\partial_i \,-\, m\big)\Psi(x), \tag{35}$$

and hence

$$\big(i\gamma^\mu\partial_\mu \,-\, m\big)\Psi(x) \;=\; 0. \tag{36}$$

As expected from $\hat{H}^2_{\rm Dirac} = \hat{\mathbf{p}}^2 + m^2$, the Dirac equation for the spinor field implies the Klein–Gordon equation for each component $\Psi_\alpha(x)$. Indeed, if $\Psi(x)$ obey the Dirac equation, then obviously

$$\big(-i\gamma^\nu\partial_\nu \,-\, m\big) \times \big(i\gamma^\mu\partial_\mu \,-\, m\big)\Psi(x) \;=\; 0, \tag{37}$$

but the differential operator on the LHS is equal to the Klein–Gordon $m^2 + \partial^2$ times a unit matrix:

$$\big(-i\gamma^\nu\partial_\nu - m\big)\big(i\gamma^\mu\partial_\mu - m\big) \;=\; m^2 \,+\, \gamma^\nu\gamma^\mu\partial_\nu\partial_\mu = \; m^2 \,+\, \tfrac{1}{2}\{\gamma^\mu,\gamma^\nu\}\partial_\nu\partial_\mu \;=\; m^2 \,+\, g^{\mu\nu}\partial_\nu\partial_\mu\,. \tag{38}$$

The Dirac equation (36) **transforms covariantly under the Lorentz symmetries** — its LHS transforms exactly like the spinor field itself.

Proof: Note that since the Lorentz symmetries involve the x^μ coordinates as well as the spinor field components, the LHS of the Dirac equation becomes

$$\big(i\gamma^\mu\partial'_\mu \,-\, m\big)\Psi'(x') \tag{39}$$

where

$$\partial'_\mu \;\equiv\; \frac{\partial}{\partial x'^\mu} \;=\; \frac{\partial x^\nu}{\partial x'^\mu}\times\frac{\partial}{\partial x^\nu} \;=\; \big(L^{-1}\big)_\mu^{\;\nu}\times\partial_\nu\,. \tag{40}$$

Consequently,

$$\partial'_\mu \Psi'(x') \;=\; \left(L^{-1}\right)_\mu^{\ \nu} \times M_D(L)\,\partial_\nu \Psi(x) \tag{41}$$

and hence

$$\gamma^\mu \partial'_\mu \Psi'(x') \;=\; \left(L^{-1}\right)_\mu^{\ \nu} \times \gamma^\mu M_D(L)\,\partial_\nu \Psi(x). \tag{42}$$

But according to eq. (23),

$$\begin{aligned} M_D^{-1}(L)\gamma^\mu M_D(L) \;=\; L^\mu_{\ \nu}\gamma^\nu \quad &\Longrightarrow \quad \gamma^\mu M_D(L) \;=\; L^\mu_{\ \nu} \times M_D(L)\gamma^\nu \\ &\Longrightarrow \quad \left(L^{-1}\right)_\mu^{\ \nu} \times \gamma^\mu M_D(L) \;=\; M_D(L)\gamma^\nu, \end{aligned} \tag{43}$$

so

$$\gamma^\mu \partial'_\mu \Psi'(x') \;=\; M_D(L) \times \gamma^\nu \partial_\nu \Psi(x). \tag{44}$$

Altogether,

$$\bigl(i\gamma^\mu\partial_\mu \,-\, m\bigr)\Psi(x) \;\xrightarrow[\text{Lorentz}]{}\; \bigl(i\gamma^\mu\partial'_\mu \,-\, m\bigr)\Psi'(x') \;=\; M_D(L) \times \bigl(i\gamma^\mu\partial_\mu \,-\, m\bigr)\Psi(x), \tag{45}$$

which proves the covariance of the Dirac equation. *Quod erat demonstrandum.*

Dirac Lagrangian

The Dirac equation is a first-order differential equation, so to obtain it as an Euler–Lagrange equation, we need a Lagrangian which is linear rather than quadratic in the spinor field's derivatives. Thus, we want

$$\mathcal{L} \;=\; \overline{\Psi} \times \bigl(i\gamma^\mu\partial_\mu \,-\, m\bigr)\Psi \tag{46}$$

where $\overline{\Psi}(x)$ is some kind of a conjugate field to the $\Psi(x)$. Since Ψ is a complex field, we treat it as a linearly-independent from the $\overline{\Psi}$, so the Euler–Lagrange equation for the $\overline{\Psi}$

$$0 \;=\; \frac{\partial\mathcal{L}}{\partial\overline{\Psi}} \;-\; \partial_\mu \frac{\partial\mathcal{L}}{\partial\partial_\mu\overline{\Psi}} \;=\; \bigl(i\gamma^\nu\partial_\nu \,-\, m\bigr)\Psi \;-\; \partial_\mu\bigl(0\bigr) \tag{47}$$

immediately gives us the Dirac equation for the $\Psi(x)$ field.

To keep the action $S = \int d^4x \mathcal{L}$ Lorentz-invariant, the Lagrangian (46) should transform as a Lorentz scalar, $\mathcal{L}'(x') = \mathcal{L}(x)$. In light of eq. (19) for the $\Psi(x)$ field and covariance (45) of the Dirac equation, the conjugate field $\overline{\Psi}(x)$ should transform according to

$$\overline{\Psi}'(x') \;=\; \overline{\Psi}(x) \times M_D^{-1}(L) \quad \Longrightarrow \quad \mathcal{L}'(x') \;=\; \mathcal{L}(x). \tag{48}$$

Note that the $M_D(L)$ matrix is generally not unitary, so the inverse matrix $M_D^{-1}(L)$ in eq. (48) is different from the hermitian conjugate $M_D^\dagger(L)$. Consequently, the conjugate field $\overline{\Psi}(x)$ cannot be identified with the hermitian conjugate field $\Psi^\dagger(x)$, since the latter transforms to

$$\Psi'^\dagger(x') \;=\; \Psi^\dagger(x) \times M_D^\dagger(L) \;\neq \Psi^\dagger(x) \times M_D^{-1}(L). \tag{49}$$

Instead of the hermitian conjugate, we are going to use the Dirac conjugate spinor, see below.

Dirac conjugates:

Let Ψ be a 4-component Dirac spinor and Γ be any 4×4 matrix; we *define* their Dirac conjugates according to

$$\overline{\Psi} \;=\; \Psi^\dagger \times \gamma^0, \quad \overline{\Gamma} \;=\; \gamma^0 \times \Gamma^\dagger \times \gamma^0. \tag{50}$$

Thanks to $\gamma^0\gamma^0 \;=\; 1$, the Dirac conjugates behave similarly to hermitian conjugates or transposed matrices:

- For a a product of 2 matrices, $\overline{(\Gamma_1 \times \Gamma_2)} = \overline{\Gamma}_2 \times \overline{\Gamma}_1$.
- Likewise, for a matrix and a spinor, $\overline{(\Gamma \times \Psi)} = \overline{\Psi} \times \overline{\Gamma}$.
- The Dirac conjugate of a complex number is its complex conjugate, $\overline{(c \times \mathbf{1})} \;=\; c^* \times \mathbf{1}$.
- For any two spinors Ψ_1 and Ψ_2 and any matrix Γ, $\overline{\Psi}_1\overline{\Gamma}\Psi_2 \;=\; \left(\overline{\Psi}_2\Gamma\Psi_1\right)^*$.
  - The Dirac spinor fields are fermionic, so they anticommute with each other, even in the classical limit. Nevertheless, $\left(\Psi_\alpha^\dagger\Psi_\beta\right)^\dagger \;=\; +\Psi_\beta^\dagger\Psi_\alpha$, and therefore for any matrix Γ, $\overline{\Psi}_1\overline{\Gamma}\Psi_2 \;=\; +\left(\overline{\Psi}_2\Gamma\Psi_1\right)^*$.

The point of the Dirac conjugation (50) is that it works similarly for all four Dirac matrices γ^μ,

$$\overline{\gamma^\mu} \;=\; +\gamma^\mu. \tag{51}$$

Proof: For $\mu = 0$, the γ^0 is hermitian, hence

$$\overline{\gamma^0} \;=\; \gamma^0(\gamma^0)^\dagger\gamma^0 \;=\; +\gamma^0\gamma^0\gamma^0 \;=\; +\gamma^0. \tag{52}$$

For $\mu = i = 1, 2, 3$, the γ^i are anti-hermitian and also anticommute with the γ^0, hence

$$\overline{\gamma^i} \;=\; \gamma^0(\gamma^i)^\dagger\gamma^0 \;=\; -\gamma^0\gamma^i\gamma^0 \;=\; +\gamma^0\gamma^0\gamma^i \;=\; +\gamma^i. \tag{53}$$

Corollary: *The Dirac conjugate of the matrix*

$$M_D(L) \;=\; \exp\bigl(-\tfrac{i}{2}\Theta_{\mu\nu}S^{\mu\nu}\bigr) \tag{19}$$

representing any continuous Lorentz symmetry $L = \exp(\Theta)$ *is the inverse matrix*

$$\overline{M}_D(L) \;=\; M_D^{-1}(L) \;=\; \exp\bigl(+\tfrac{i}{2}\Theta_{\mu\nu}S^{\mu\nu}\bigr). \tag{54}$$

Proof: Let

$$X \;=\; -\tfrac{i}{2}\Theta_{\mu\nu}S^{\mu\nu} \;=\; +\tfrac{1}{8}\Theta_{\mu\nu}[\gamma^\mu,\gamma^\nu] \;=\; +\tfrac{1}{4}\Theta_{\mu\nu}\gamma^\mu\gamma^\nu \tag{55}$$

for some real antisymmetric Lorentz parameters $\Theta_{\mu\nu} = -\Theta_{\nu\mu}$. The Dirac conjugate of the X matrix is

$$\overline{X} \;=\; \overline{\tfrac{1}{4}\Theta_{\mu\nu}\gamma^\mu\gamma^\nu} \;=\; \tfrac{1}{4}\Theta^*_{\mu\nu}\overline{\gamma}^\nu\overline{\gamma}^\mu \;=\; \tfrac{1}{4}\Theta_{\mu\nu}\gamma^\nu\gamma^\mu \;=\; \tfrac{1}{4}\Theta_{\nu\mu}\gamma^\mu\gamma^\nu \;=\; -\tfrac{1}{4}\Theta_{\mu\nu}\gamma^\mu\gamma^\nu \;=\; -X. \tag{56}$$

Consequently,

$$\overline{X^2} \;=\; \overline{X}\times\overline{X} \;=\; +X^2, \quad \overline{X^3} \;=\; \overline{X\times X^2} \;=\; \overline{X^2}\times\overline{X} \;=\; -X^3, \quad \ldots, \quad \overline{X^n} \;=\; (-X)^n, \tag{57}$$

and hence

$$\overline{\exp(X)} \;=\; \sum_n \frac{1}{n!}\,\overline{X^n} \;=\; \sum_n \frac{1}{n!}\,(-X)^n \;=\; \exp(-X). \tag{58}$$

In light of eq. (55), this means

$$\overline{\exp\bigl(-\tfrac{i}{2}\Theta_{\mu\nu}S^{\mu\nu}\bigr)} \;=\; \exp\bigl(+\tfrac{i}{2}\Theta_{\mu\nu}S^{\mu\nu}\bigr), \tag{59}$$

that is,

$$\overline{M}_D(L) \;=\; M_D^{-1}(L). \tag{60}$$

Quod erat demonstrandum.

Back to the Dirac Lagrangian:

Thanks to the theorem (60), the conjugate field $\overline{\Psi}(x)$ in the Lagrangian (46) is simply the Dirac conjugate of the Dirac spinor field $\Psi(x)$,

$$\overline{\Psi}(x) \;=\; \Psi^\dagger(x)\times\gamma^0, \tag{61}$$

which transforms under Lorentz symmetries as

$$\overline{\Psi}'(x') \;=\; \overline{\Psi'(x')} \;=\; \overline{M_D(L)\times\Psi(x)} \;=\; \overline{\Psi}(x)\times\overline{M}_D(x) \;=\; \overline{\Psi}(x)\times M_D^{-1}(L). \tag{62}$$

Consequently, the Dirac Lagrangian

$$\mathcal{L} \;=\; \overline{\Psi}\times\bigl(i\gamma^\mu\partial_\mu\,-\,m\bigr)\Psi \;=\; \Psi^\dagger\gamma^0\times\bigl(i\gamma^\mu\partial_\mu\,-\,m\bigr)\Psi \tag{46}$$

is a Lorentz scalar and the action is Lorentz invariant.

Hamiltonian for the Dirac Field

Canonical quantization of the Dirac spinor field $\Psi(x)$ — just like any other field — begins with classical Hamiltonian formalism. Let's start with the canonical conjugate fields,

$$\Pi_\alpha = \frac{\partial\mathcal{L}}{\partial(\partial_0\Psi_\alpha)} = \left(i\overline{\Psi}\gamma^0\right)_\alpha = i\Psi^\dagger_\alpha \tag{63}$$

— the canonical conjugate to the Dirac spinor field $\Psi(x)$ is simply its hermitian conjugate $\Psi^\dagger(x)$. This is similar to what we had for the non-relativistic field, and it happens for the same reason — the Lagrangian which is linear in the time derivative.

In the non-relativistic field theory, the conjugacy relation (63) in the classical theory lead to the equal-time commutation relations in the quantum theory,

$$\left[\hat\psi(\mathbf{x},t),\hat\psi(\mathbf{y},t)\right] = 0, \quad \left[\hat\psi^\dagger(\mathbf{x},t),\hat\psi^\dagger(\mathbf{y},t)\right] = 0, \quad \left[\hat\psi(\mathbf{x},t),\hat\psi^\dagger(\mathbf{y},t)\right] = \delta^{(3)}(\mathbf{x}-\mathbf{y}). \tag{64}$$

However, the Dirac spinor field describes spin $=\frac{1}{2}$ particles — like electrons, protons, or neutrons — which are fermions rather than bosons. So instead of the commutations relations (64), the spinor fields obey *equal-time anti-commutation relations*

$$\begin{aligned}\left\{\hat\Psi_\alpha(\mathbf{x},t),\hat\Psi_\beta(\mathbf{y},t)\right\} &= 0,\\ \left\{\hat\Psi^\dagger_\alpha(\mathbf{x},t),\hat\Psi^\dagger_\beta(\mathbf{y},t)\right\} &= 0,\\ \left\{\hat\Psi_\alpha(\mathbf{x},t),\hat\Psi^\dagger_\beta(\mathbf{y},t)\right\} &= \delta_{\alpha\beta}\delta^{(3)}(\mathbf{x}-\mathbf{y}).\end{aligned} \tag{65}$$

Next, the classical Hamiltonian obtains as

$$\begin{aligned} H &= \int d^3\mathbf{x}\,\mathcal{H}(\mathbf{x}),\\ \mathcal{H} &= i\Psi^\dagger\partial_0\Psi - \mathcal{L}\\ &= i\Psi^\dagger\partial_0\Psi - \Psi^\dagger\gamma^0\left(i\gamma^0\partial_0 + i\vec\gamma\cdot\nabla - m\right)\Psi\\ &= \Psi^\dagger\left(-i\gamma^0\vec\gamma\cdot\nabla + \gamma^0 m\right)\Psi\end{aligned} \tag{66}$$

where the terms involving the time derivative ∂_0 cancel out. Consequently, the Hamiltonian operator of the quantum field theory is

$$\hat{H} \;=\; \int d^3\mathbf{x}\, \hat{\Psi}^\dagger(\mathbf{x}) \big(-i\gamma^0\vec{\gamma}\cdot\nabla \;+\; \gamma^0 m\big)\, \hat{\Psi}(\mathbf{x}). \tag{67}$$

Note that the derivative operator $(-i\gamma^0\vec{\gamma}\cdot\nabla+\gamma^0 m)$ in this formula is precisely the 1-particle Dirac Hamiltonian (27). This is very similar to what we had for the quantum non-relativistic fields,

$$\hat{H} \;=\; \int d^3\mathbf{x}\, \hat{\psi}^\dagger(\mathbf{x}) \left(\frac{-1}{2M}\,\nabla^2 \;+\; V(\mathbf{x})\right) \hat{\psi}(\mathbf{x}), \tag{68}$$

except for a different differential operator, Schrödinger instead of Dirac.

In the Heisenberg picture, the quantum Dirac field obeys the Dirac equation. To see how this works, we start with the Heisenberg equation

$$i\,\frac{\partial}{\partial t}\,\hat{\Psi}_\alpha(\mathbf{x},t) \;=\; \big[\hat{\Psi}_\alpha(\mathbf{x},t),\hat{H}\big] \;=\; \int d^3\mathbf{y}\,\big[\hat{\Psi}_\alpha(\mathbf{x},t),\hat{\mathcal{H}}(\mathbf{y},t)\big], \tag{69}$$

and then evaluate the last commutator using the anti-commutation relations (65) and the Leibniz rules (11). Indeed, let's use the Leibniz rule

$$[A,BC] \;=\; \{A,B\}C \;-\; B\{A,C\} \tag{70}$$

for

$$\begin{aligned} A \;&=\; \hat{\Psi}_\alpha(\mathbf{x},t), \\ B \;&=\; \hat{\Psi}^\dagger_\beta(\mathbf{y},t), \\ C \;&=\; \big(-i\gamma^0\vec{\gamma}\cdot\nabla \;+\; \gamma^0 m\big)_{\beta\gamma}\hat{\Psi}_\gamma(\mathbf{y},t), \end{aligned} \tag{71}$$

so that $BC=\hat{\mathcal{H}}(\mathbf{y},t)$. For the A,B,C at hand,

$$\{A,B\} \;=\; \delta_{\alpha\beta}\delta^{(3)}(\mathbf{x}-\mathbf{y}) \tag{72}$$

while

$$\{A,C\} \;=\; \big(-i\gamma^0\vec{\gamma}\cdot\nabla_y \;+\; \gamma^0 m\big)_{\beta\gamma}\{\hat{\Psi}_\alpha(\mathbf{x},t),\hat{\Psi}_\gamma(\mathbf{y},t)\} \;=\; (\text{diff.op.})\times 0 \;=\; 0. \tag{73}$$

Consequently

$$
\begin{aligned}
\left[\hat{\Psi}_\alpha(\mathbf{x},t), \hat{\mathcal{H}}(\mathbf{y},t)\right] \;&\equiv\; [A, BC] \\
&=\; \{A, B\} \times C \;-\; B \times \{A, C\} \\
&=\; \delta_{\alpha\beta}\delta^{(3)}(\mathbf{x}-\mathbf{y}) \times \left(-i\gamma^0\vec{\gamma}\cdot\nabla \,+\, \gamma^0 m\right)_{\beta\gamma}\hat{\Psi}_\gamma(\mathbf{y},t) \\
&\qquad\qquad -\; 0,
\end{aligned}
\tag{74}
$$

hence

$$
\begin{aligned}
\left[\hat{\Psi}_\alpha(\mathbf{x},t), \hat{H}\right] \;&=\; \int d^3\mathbf{y}\,\delta^{(3)}(\mathbf{x}-\mathbf{y}) \times \left(-i\gamma^0\vec{\gamma}\cdot\nabla \,+\, \gamma^0 m\right)_{\alpha\gamma}\hat{\Psi}_\gamma(\mathbf{y},t) \\
&=\; \left(-i\gamma^0\vec{\gamma}\cdot\nabla \,+\, \gamma^0 m\right)_{\alpha\gamma}\hat{\Psi}_\gamma(\mathbf{x},t),
\end{aligned}
\tag{75}
$$

and therefore

$$
i\partial_0\hat{\Psi}(\mathbf{x},t) \;=\; \left(-i\gamma^0\vec{\gamma}\cdot\nabla \,+\, \gamma^0 m\right)\hat{\Psi}(\mathbf{x},t). \tag{76}
$$

Or if you prefer,

$$
\left(i\gamma^\mu\partial_\mu \,-\, m\right)\hat{\Psi}(x) \;=\; 0. \tag{77}
$$