PHY–396 K. Final Exam. Due Thursday, December 10, 4:45 PM.

Please do not waste time and paper by copying the posted homework solutions or supplementary notes. If you need to use any homework result, simply reference the appropriate question or equation and go ahead. Likewise, don't re-derive anything I derived in class.

---

1. Consider a non-abelian $SO(3)$ gauge theory coupled to a triplet of *complex* scalar fields Φ_a,

$$\mathcal{L} = -\tfrac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + D_\mu \Phi^{*a} D^\mu \Phi^a - V, \tag{1}$$

$$V = m^2 \Phi^{*a}\Phi^a + \frac{\alpha}{2} (\Phi^{*a}\Phi^a)^2 + \frac{\beta}{2} \sum_a \left| \epsilon^{abc} \Phi^{*b} \Phi^c \right|^2, \tag{2}$$

where $a, b, c = 1, 2, 3$ and repeated indices are summed over.

(a) Identify all symmetries of this theory, global or local, discrete or continuous. For simplicity, skip the spacetime symmetries (Lorentz, translations, P, and T) and focus on the internal symmetries only.

(b) Show that for $\alpha, \beta > 0$ but $m^2 < 0$, the scalar potential (2) has a continuous family of degenerate minima, and that all those minima are related by symmetries to

$$\langle \Phi \rangle = \frac{v}{\sqrt{2}} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad v = \sqrt{\frac{-2m^2}{\alpha}}. \tag{3}$$

(c) Which symmetries are spontaneously broken by this vacuum expectation value (VEV)? Which symmetries remain unbroken? How many massless Goldstone bosons does this symmetry breaking calls for and what should be their quantum numbers? Which vector fields becomes massive by the Higgs mechanism and which remain massless?

(d) Expand the Lagrangian (1) in powers of $\Phi^a(x) - \langle \Phi^a \rangle$ and $A^a_\mu(x)$, write down the quadratic terms, and use them to determine the mass spectrum of the theory. Check that this spectrum agrees with predictions you have made in part (c).
Hint: split the complex fields $\Phi^a(x) - \langle \Phi^a \rangle$ into their real and imaginary parts.

(e) Generalize parts (a–c) to the $SO(N)$ gauge theory coupled to N complex scalar fields. Note: to generalize the third term in the potential (2), use

$$\sum_a \left|\epsilon^{abc}\Phi^{*b}\Phi^c\right|^2 = \left(\Phi^{*a}\Phi^a\right)^2 - \left(\Phi^{*a}\Phi^{*a}\right)\left(\Phi^b\Phi^b\right) \\ = 4\left(\operatorname{Re}\Phi^a \operatorname{Re}\Phi^a\right)\left(\operatorname{Im}\Phi^b \operatorname{Im}\Phi^b\right) - 4\left(\operatorname{Re}\Phi^a \operatorname{Im}\Phi^a\right)^2. \tag{4}$$

2. The rest of this exam is about Quantum Electro-Dynamics that has charged scalar particles $S^\mp$ besides the usual photons γ and electrons $e^\mp$. The Lagrangian of this theory is

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\Psi}(i\not{D} - m)\Psi + D_\mu\Phi^* D^\mu\Phi - M^2\Phi^*\Phi = \mathcal{L}^{\text{free}} + \mathcal{L}^{\text{interactions}}. \tag{5}$$

In the Feynman rules, the propagators and the external lines follow from the $\mathcal{L}^{\text{free}}$ while the vertices follow from the $\mathcal{L}^{\text{interactions}}$. Altogether, the Feynman rules for QED with both electrons and charged scalars are

Photonic propagator: A^μ  A^ν ($q \to$) $= \dfrac{-ig^{\mu\nu}}{q^2 + i0}$, (F.1)

Incoming photon:  $= e_\mu(k,\lambda)$, (F.2)

Outgoing photon:  $= e^*_\mu(k,\lambda)$, (F.3)

Electron propagator: $\overline{\Psi}$  Ψ ($q \to$) $= \dfrac{i}{\not{q} - m + i0}$, (F.4)

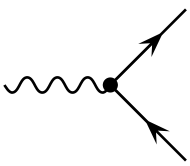
Incoming e^- or outgoing e^+:  $= u(p,s)$ or $v(p,s)$, (F.5)

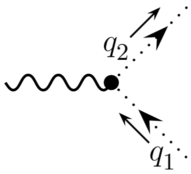
Outgoing e^- or incoming e^+:  $= \bar{u}(p,s)$ or $\bar{v}(p,s)$, (F.6)

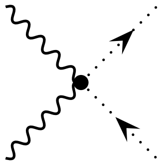
Scalar propagator: Φ^*  Φ ($q \to$) $= \dfrac{i}{q^2 - M^2 + i0}$, (F.7)

Incoming S^- or outgoing S^+:  $= 1$, (F.8)

Outgoing S^- or incoming S^+:  $= 1$, (F.9)

QED vertex $ee\gamma$:  $= +ie\gamma^\mu$, (F.10)

Scalar QED vertex $SS\gamma$:  $= +ie(q_1+q_2)^\mu$ (F.11)

Seagull vertex $SS\gamma\gamma$:  $= +2ie^2\,g^{\mu\nu}.$ (F.12)

Note: the dotted lines (F.7–9) for the charged scalars have arrows. Also note that in the $S^-S^+\gamma$ vertex (F.11), the directions of momenta q_1 and q_2 must agree with the arrows of the scalar lines; otherwise, the vertex becomes $+ie(q_1-q_2)^\mu$ or $+ie(-q_1+q_2)^\mu$ or $+ie(-q_1-q_2)^\mu$.

(a) The QED Feynman rules (F.1–6) and (F.10) were explained in class. Explain the remaining rules (F.7–9) (especially the arrows on the scalar lines) and (F.11–12) in terms of the Lagrangian (5).

Note: don't re-derive the Feynman rules as such, just explain why the scalar propagators, external lines, and vertices are as in eqs. (F.7–9) and (F.11–12).

(b) Given the Feynman rules, draw the tree diagram(s) for the scalar pair production $e^-e^+ \to S^-S^+$ and calculate the tree-level amplitude $\left\langle S^-, S^+\right| \mathcal{M} \left|e^-, e^+\right\rangle$.

Hint: Mind the arrow directions on the dotted lines of scalars.

(c) Average $|\mathcal{M}|^2$ over the incoming particles' spins and calculate the partial cross-section for the scalar pair production. Compare its angular dependence with that of the muon pair production we have studied in class. Also, calculate the total cross-section $\sigma_{\rm tot}(e^-e^+ \to S^-S^+)$ and compare its energy dependence to that of the $\sigma_{\rm tot}(e^-e^+ \to \mu^-\mu^+)$.

For simplicity, neglect the electron's mass m. But don't neglect the scalar's mass M.

3. Finally, consider the annihilation of the charged scalars into photons, $S^+S^- \to \gamma\gamma$.

   (a) Draw and evaluate **all** tree diagrams contributing to the $\langle\gamma\gamma|\,\mathcal{M}\,|S^+S^-\rangle$ amplitude. Make sure the amplitude respects the Bose symmetry between the two photons.

   (b) Write the tree amplitude as $\mathcal{M} = \mathcal{M}^{\mu\nu} \times \mathcal{E}^*_\mu(k_1,\lambda_1)\mathcal{E}^*_\nu(k_2,\lambda_2)$ and verify the Ward identities

$$k_1^\mu \times \mathcal{M}_{\mu\nu} \;=\; 0, \qquad k_2^\nu \times \mathcal{M}_{\mu\nu} \;=\; 0. \tag{6}$$

   Hint: If these identities seem to be broken, go back to part (a) and make sure you have not missed a diagram. If this does not help, check your signs.

   (c) Sum $|\mathcal{M}|^2$ over the outgoing photon polarizations and calculate the partial cross-section of the $S^+S^- \to \gamma\gamma$ annihilation.

- For simplicity, assume $E \gg M$ and neglect the scalar mass M in your calculations.

⋆ Extra credit if you do take M into account, and do it right. But beware: the kinematics is much messier for $M \neq 0$, and you might need a couple of hours to work through the algebra.