Glashow–Weinberg–Salam Theory

Glashow–Weinberg–Salam theory is a unified theory of weak and electromagnetic interactions. At its core is the $SU(2)_W \times U(1)_Y$ gauge theory spontaneously broken down to the $U(1)_{\rm EM}$. Out of 4 gauge fields W^a_μ $(a = 1, 2, 3)$ and B_μ, one linear combination remains massless and gives rise to the electromagnetism, while 3 other linear combinations become massive and give rise to the weak interactions.

The key to the spontaneous breakdown of the electroweak gauge symmetry is the doublet of complex fields H_α $(\alpha = 1, 2)$ called *the Higgs fields*. The $SU(2)_W \times U(1)_Y$ quantum numbers of these fields are $(\mathbf{2}, +\frac{1}{2})$; that is, they form a doublet of the $SU(2)_W$ and have the $U(1)_Y$ *hypercharge* $y = +\frac{1}{2}$. Thus,

$$D_\mu H_\alpha(x) \;=\; \partial_\mu H_\alpha + \frac{ig_2}{2}\,\tau^a_{\alpha\beta}\,W^a_\mu(x) H_\beta(x) \;+\; \frac{ig_1}{2}\,B_\mu H_\alpha \tag{1}$$

where g_2 is the $SU(2)_W$ gauge coupling and g_1 is the $U(1)_Y$ gauge coupling.

The gauge fields W^a_μ and B_μ and the Higgs fields H_α are the only bosonic fields of the GWS theory. There are also 24 fermionic fields describing the quarks and the leptons — I have a separate set of notes about them — but let's take care of the bosons first. The bosonic part of the theory's Lagrangian is

$$\mathcal{L} \;=\; -\tfrac{1}{4} W^a_{\mu\nu} W^{a\mu\nu} \;-\; \tfrac{1}{4} B_{\mu\nu} B^{\mu\nu} \;+\; D_\mu H^\dagger D^\mu H \;-\; \frac{\lambda}{2}\left(H^\dagger H \;-\; \frac{v^2}{2}\right)^2 \;+\; \text{fermionic terms} \tag{2}$$

where

$$\begin{aligned} B_{\mu\nu} \;&=\; \partial_\mu B_\nu \;-\; \partial_\nu B_\mu\,, \\ W^a_{\mu\nu} \;&=\; \partial_\mu W^a_\nu \;-\; \partial_\nu W^a_\mu \;-\; g_2 \epsilon^{abc} W^b_\mu W^c_\nu\,, \\ H \;&=\; \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}, \quad H^\dagger \;=\; \big(H_1^*, H_2^*\big), \end{aligned} \tag{3}$$

and $D_\mu H$, $D_\mu H^\dagger$ are row/column vector forms of $D_\mu H_\alpha$ and $D_\mu H^*_\alpha$ from eq. (1). The scalar potential $V = \frac{\lambda}{2}\left(H^\dagger H \;-\; \frac{v^2}{2}\right)^2$ has a local maximum rather than a minimum at $H = 0$, while its minima form a spherical shell $H^\dagger H = \frac{v^2}{2}$ in the scalar field space $\mathbf{C}^2 = \mathbf{R}^4$. All such minima

are related to each other by the gauge symmetry, so without loss of generality we assume the Higgs fields have Vacuum Expectation Values (VEVs)

$$\langle H \rangle \;=\; \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}. \tag{4}$$

Note that this expectation value breaks 3 out of 4 gauge symmetries of the theory, but one combination of the $U(1)_Y$ and an $U(1)$ subgroup of the $SU(2)_2$ remains unbroken. Indeed, the $U(1)_Y$ symmetry $\exp\bigl(i\Theta(x)\hat{Y}\bigr)$ acts on the Higgs fields as $H(x) \to \exp\bigl(iy\Theta(x)\bigr)H(x) = \exp\bigl(\frac{i}{2}\Theta(x)\bigr)H(x)$ since H has $y = +\frac{1}{2}$, while the $SU(2)$ symmetry $\exp\bigl(i\Theta(x)\hat{T}^3\bigr)$ — for the same $\Theta(x)$ — ants on the $SU(2)$ double H as $H(x) \to \exp\bigl(\frac{i}{2}\Theta(x)\tau^3\bigr)H(x)$. Combining the two symmetries, we have

$$H(x) \;\to\; \exp\bigl(\tfrac{i}{2}\Theta(x)\bigr)\exp\bigl(\tfrac{i}{2}\Theta(x)\tau^3\bigr)H(x) \;= \begin{pmatrix} e^{i\Theta(x)} & 0 \\ 0 & 1 \end{pmatrix} H(x), \tag{5}$$

which indeed leaves the vacuum expectation value (4) invariant. Thus, the $U(1)$ subgroup of the electroweak $SU(2)_W \times U(1)_Y$ generated by the operator

$$\hat{Q} \;=\; \hat{Y} \;+\; \hat{T}^3 \tag{6}$$

remains unbroken. Physically, this subgroup is the $U(1)_Q$ gauge symmetry of the electromagnetism and $\hat{Q}$ is the electric charge operator (or rather electric charge in units of e).

We shall see in a moment that one linear combination of the four $SU(2)_W \times U(1)_Y$ gauge fields corresponding to the $\hat{Q}$ generator remains massless while the other 3 combinations become massive via the Higgs mechanism. The same mechanism also eliminates 3 scalar fields, which becomes the longitudinal components of the 3 massive vector fields. Since the 2 complex Higgs fields are equivalent to 4 real scalars, we end up with $4 - 3 = 1$ physical scalar field $h(x)$; its quanta — called the *physical Higgs particles* — were experimentally discovered at the LHC in 2013.

The simplest way to see how this works is to fix the unitary gauge for the spontaneously broken symmetries. Note that any complex doublet $H(x)$ can be $SU(2)$–rotated to

$$H'(x) \;=\; U(x)H(x) \;=\; \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \tilde{h}(x) \end{pmatrix} \tag{7}$$

for a *real* $\tilde{h}(x) \geq 0$. This gauge transform would be singular for $H(x) \approx 0$ but it is nice and

smooth for $H(x)$ in the vicinity of the vacuum expectation value (4), so we may use it to fix the unitary gauge $H_1(x) \equiv 0$, $\Im(H_2(x)) \equiv 0$. Once we fix this gauge, we are left with a single real scalar field $\tilde{h}(x)$, which we may now shift by its VEV,

$$\tilde{h}(x) \;=\; v \;+\; h(x). \tag{8}$$

In terms of this shifted field,

$$H^\dagger H \;-\; \frac{v^2}{2} \;=\; \frac{(v+h)^2}{2} \;-\; \frac{v^2}{2} \;=\; vh \;+\; \tfrac{1}{2}h^2, \tag{9}$$

so the scalar potential becomes

$$V(h) \;=\; \frac{\lambda}{2}\left(H^\dagger H \;-\; \frac{v^2}{2}\right)^2 \;=\; \frac{\lambda}{2}\left(vh \;+\; \tfrac{1}{2}h^2\right)^2 \;=\; \frac{\lambda v^2}{2}\times h^2 \;+\; \frac{\lambda v}{2}\times h^3 \;+\; \frac{\lambda}{8}\times h^4 \tag{10}$$

with a positive mass$^2 = \lambda v^2 > 0$ for the physical Higgs field. Experimentally, $v = 247$ GeV while the physical Higgs mass is 125 GeV, which means $\lambda \approx 0.26$.

The mass terms for the vector fields emerge from the kinetic term $D_\mu H^\dagger D^\mu H$ for the Higgs doublets. Indeed, in the unitary gauge

$$D_\mu H \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{i}{2}g_2\big(W^1_\mu - iW^2_\mu\big)\tilde{h} \\ \partial_\mu\tilde{h} \;+\; \frac{i}{2}\big(g_1B_\mu - g_2W^3_\mu\big)\tilde{h}\end{pmatrix} \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{i}{2}g_2\big(W^1_\mu - iW^2_\mu\big)(v+h) \\ \partial_\mu h \;+\; \frac{i}{2}\big(g_1B_\mu - g_2W^3_\mu\big)(v+h)\end{pmatrix} \tag{11}$$

and hence

$$\begin{aligned} D_\mu H^\dagger D^\mu H \;&=\; \tfrac{1}{2}\Big|\partial_\mu h \;+\; \tfrac{i}{2}\big(g_1B_\mu - g_2W^3_\mu\big)(v+h)\Big|^2 \;+\; \tfrac{1}{2}\Big|\tfrac{i}{2}g_2\big(W^1_\mu - iW^2_\mu\big)(v+h)\Big|^2 \\ &=\; \tfrac{1}{2}(\partial_\mu h)^2 \;+\; \frac{(v+h)^2}{8}\Big(g_1B_\mu - g_2W^3_\mu\Big)^2 \;+\; \frac{g_2^2(v+h)^2}{8}\Big(\big(W^1_\mu\big)^2 + \big(W^2_\mu\big)^2\Big). \end{aligned} \tag{12}$$

The first term on the last line here is the kinetic term for the physical Higgs field while the rest are the mass terms for the vector fields and also their interactions with the physical Higgs field $h(x)$. In particular, the vector mass terms obtain from truncating the $(v + h(x))^2$ factors

to simply v^2, thus

$$\mathcal{L}^{\text{vector}}_{\text{masses}} \;=\; \frac{g_2^2 v^2}{8} \times \Big(\big(W^1_\mu\big)^2 + \big(W^2_\mu\big)^2 \Big) \;+\; \frac{v^2}{8} \times \Big(g_1 B_\mu - g_2 W^3_\mu \Big)^2 . \tag{13}$$

In particular, the W^1_μ and W^2_μ vector fields have masses

$$M_W^2 \;=\; \frac{g_2^2 v^2}{4} \quad \Longrightarrow \quad M_W \;=\; \frac{g_2 v}{2} \,, \tag{14}$$

while the W^3_μ and B_μ vector fields have a 2×2 mass matrix

$$M^2 \;=\; \frac{v^2}{4} \begin{pmatrix} g_2^2 & -g_2 g_1 \\ -g_2 g_1 & g_1^2 \end{pmatrix} . \tag{15}$$

This matrix has eigenvalues

$$M_Z^2 \;=\; \frac{(g_2^2 + g_1^2) v^2}{4} \quad \text{and} \quad M_A^2 \;=\; 0 \tag{16}$$

— as promised, there is one massless vector field — while the mass eigenstates correspond to the vector fields

$$\begin{aligned} \text{massive } Z_\mu(x) \;&=\; \cos\theta \times W^3_\mu(x) \;-\; \sin\theta \times B_\mu(x), \\ \text{massless } A_\mu(x) \;&=\; \sin\theta \times W^3_\mu(x) \;+\; \cos\theta \times B_\mu(x), \end{aligned} \tag{17}$$

where

$$\theta \;=\; \arctan \frac{g_1}{g_2} \tag{18}$$

is the *weak mixing angle* or the Weinberg angle; experimentally, $\sin^2\theta \approx 0.23$.

Physically, the $A_\mu(x)$ is the EM field whose quanta are massless photons, the $Z_\mu(x)$ is the neutral weak field whose quanta are Z^0 particles of mass $M_Z \approx 91$ GeV, and the $W^{1,2}_\mu(x)$ — or rather their linear combinations

$$W^+_\mu(x) \;=\; \frac{W^1_\mu(x) \,+\, i W^2_\mu(x)}{\sqrt{2}} \quad \text{and} \quad W^-_\mu(x) \;=\; \frac{W^1_\mu(x) \,-\, i W^2_\mu(x)}{\sqrt{2}} \tag{19}$$

— are the charged weak fields (electric charges $q = \pm 1$) whose quanta are the W^+ and W^- particles of mass $M_W \approx 80$ GeV. The experimentally found mass ratio between the $W^\pm$ and

Z^0 particles gives us the value of the weak mixing angle:

$$\frac{M_W^2}{M_Z^2} \;=\; \frac{g_2^2}{g_2^2+g_1^2} \;=\; \frac{1}{1+\tan^2\theta} \;=\; \cos^2\theta \quad\Longrightarrow\quad \cos^2\theta\approx 0.77 \;\Longrightarrow\; \sin^2\theta\approx 0.23. \tag{20}$$

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Now lets find the currents to which the vector fields $W_\mu^\pm$, Z_μ, and A_μ couple and the strengths of those couplings. Of particular importance is the EM coupling strength e since it acts as the unit of the conventionally normalized electric charge, so we would like to relate it to the original $SU(2)_W\times U(1)_Y$ couplings g_2 and g_1. But the weak currents and couplings are also important.

Our starting point is the $SU(2)_W\times U(1)_Y$ symmetry currents J_μ^Y, J_μ^{T1}, J_μ^{T2}, J_μ^{T3} of the fermionic fields. Without going into the details of these currents — they are described in detail in my notes on quarks and leptons — we can say that the original gauge fields $B_\mu(x)$ and $W_\mu^a(x)$ couple to these currents according to

$$\mathcal{L}_{\text{net}} \;\supset\; \mathcal{L}_{\text{current}} \;=\; -g_2W_\mu^1\times J_{T1}^\mu \;-\; g_2W_\mu^2\times J_{T2}^\mu \;-\; g_2W_\mu^3\times J_{T3}^\mu \;-\; g_1B_\mu\times J_Y^\mu\,. \tag{21}$$

Now let's relate the original gauge fields to the vector fields of definite masses and electric charges. Inverting eqs. (19) and (17), we obtain

$$\begin{aligned} W_\mu^1 &= \frac{1}{\sqrt{2}}\times W_\mu^- \;+\; \frac{1}{\sqrt{2}}\times W_\mu^+\,,\\ W_\mu^2 &= \frac{i}{\sqrt{2}}\times W_\mu^- \;-\; \frac{i}{\sqrt{2}}\times W_\mu^+\,,\\ W_\mu^3 &= \cos\theta\times Z_\mu \;+\; \sin\theta\times A_\mu\,,\\ B_\mu &= -\sin\theta\times Z_\mu \;+\; \cos\theta\times A_\mu\,. \end{aligned} \tag{22}$$

Plugging these formulae into eq. (21) and re-arranging the terms, we find

$$\begin{aligned} \mathcal{L}_{\text{current}} \;=\; &-\frac{g_2}{\sqrt{2}}\,W_\mu^-\times\left(J_{T1}^\mu - iJ_{T2}^\mu\right) \;-\; \frac{g_2}{\sqrt{2}}\,W_\mu^+\times\left(J_{T1}^\mu + iJ_{T2}^\mu\right)\\ &-\; Z_\mu\times\left(g_2\cos\theta J_{T3}^\mu \;-\; g_1\sin\theta\,J_Y^\mu\right) \;-\; A_\mu\times\left(g_2\sin\theta J_{T3}^\mu \;+\; g_1\cos\theta\,J_Y^\mu\right), \end{aligned} \tag{23}$$

or in other words

$$\mathcal{L}_{\text{current}} \;=\; -\frac{g_2}{\sqrt{2}}\Big(W^+_\mu \times J^{-\mu} \,+\, W^-_\mu \times J^{+\mu}\Big) \;-\; \tilde{g} Z_\mu \times J^\mu_Z \;-\; eA_\mu \times J^\mu_{\text{EM}} \tag{24}$$

where

$$J^{+\mu} \;=\; J^\mu_{T1} \;-\; iJ^\mu_{T2}\,, \qquad J^{-\mu} \;=\; J^\mu_{T1} \;+\; iJ^\mu_{T2}\,, \tag{25}$$

are the charged weak currents,

$$\tilde{g} \times J^\mu_Z \;=\; g_2 \cos\theta J^\mu_{T3} \;-\; g_1 \sin\theta\, J^\mu_Y \tag{26}$$

is the neutral weak current (times the neutral weak coupling constant), and

$$e \times J^\mu_{\text{EM}} \;=\; g_2 \sin\theta J^\mu_{T3} \;+\; g_1 \cos\theta\, J^\mu_Y \tag{27}$$

is the (conventionally normalized) electric current. Note that on the right hand side of this formula $g_1 \cos\theta = g_2 \sin\theta$ because of the way the weak mixing angle θ is related to the gauge couplings, $\tan\theta = g_1/g_2$, *cf.* eq. (18). Consequently, we may identify

$$e \;=\; g_2 \sin\theta \;=\; g_1 \cos\theta \quad\Longrightarrow\quad \frac{1}{e^2} \;=\; \frac{1}{g_2^2}\left(\frac{1}{\sin^2\theta} \;=\; 1 + \frac{1}{\tan^2\theta}\right) \;=\; \frac{1}{g_2^2} \;+\; \frac{1}{g_1^2} \tag{28}$$

and

$$J^\mu_{\text{EM}} \;=\; J^\mu_{T3} \;+\; J^\mu_Y\,. \tag{29}$$

Note that this current does not depend on the gauge couplings or θ; instead, it's the current of the electric charge operator $\hat{Q} = \hat{T}^3 + \hat{Y}$ which is the generator of the *unbroken* $U(1)_{\text{EM}}$ gauge symmetry. Naturally, the EM field $A_\mu(x)$ — which is the gauge field of that $U(1)_{\text{EM}}$ — should couple to precisely this symmetry current.

On the other hand, the Z_μ is the gauge field of a spontaneously broken symmetry, so the specific combination of the symmetry currents that couples to the Z_μ depends on the weak

mixing angle. Indeed, the coefficients of the two terms on the RHS of eq. (26) are quite different and their ratio depends on g_1/g_2; specifically,

$$\begin{aligned} g_2 \times \cos\theta &= \frac{g_2^2}{\sqrt{g_2^2+g_1^2}} = \sqrt{g_2^2+g_1^2} \times \cos^2\theta, \\ g_1 \times \cos\theta &= \frac{g_1^2}{\sqrt{g_2^2+g_1^2}} = \sqrt{g_2^2+g_1^2} \times \sin^2\theta, \\ \frac{g_1 \times \sin\theta}{g_2 \times \cos\theta} &= \tan^2\theta. \end{aligned} \tag{30}$$

Consequently, we may identify

$$\tilde{g} = \sqrt{g_2^2+g_1^2} = \frac{g_2}{\cos\theta} = \frac{g_1}{\sin\theta} = \frac{e}{\sin\theta\cos\theta} \tag{31}$$

and then the neutral weak current becomes

$$\begin{aligned} J_Z^\mu &= \cos^2\theta \times J_{T3}^\mu - \sin^2\theta \times J_Y^\mu \\ &= J_{T3}^\mu - \sin^2\theta \times \left(J_{T3}^\mu + J_Y^\mu\right) \\ &= J_{T3}^\mu - \sin^2\theta \times J_{EM}^\mu \,. \end{aligned} \tag{32}$$

Note that the weak couplings g_2 and $\tilde{g}$ are larger than the EM coupling e. Consequently, at high energies much larger than the masses of W and Z particles, the weak interactions are not weak at all — they are stronger them the EM interactions. But at low energies, the β-decays and other processes mediated by the *virtual* $W^\pm$ or Z^0 involve the propagators

$$\begin{aligned} W^\pm \text{ propagator} &\sim \frac{1}{q^2-M_W^2} \approx \frac{-1}{M_W^2}\,, \\ Z^0 \text{ propagator} &\sim \frac{1}{q^2-M_Z^2} \approx \frac{-1}{M_Z^2}\,, \end{aligned} \tag{33}$$

so the overall weak amplitudes are

$$\mathcal{M} \sim \frac{g_2^2 E^2}{M_W^2} \quad \text{or} \quad \mathcal{M} \sim \frac{\tilde{g}^2 E^2}{M_Z^2}\,. \tag{34}$$

It's not the couplings, it's the small E^2/M_W^2 or E^2/M_Z^2 factors which make the weak interactions weak at low energies!

Let me briefly outline the Fermi's effective theory of the low-energy weak interactions. The low-energy processes do not produce any real $W^\pm$ or Z^0 particles, but they do involve the weak currents $J^{\pm\mu}$ and J^μ_X, which in turn give rise to the small $W^\pm_\mu(x)$ and $Z_\mu(x)$ fields. To see how this works at the classical level, consider the electroweak Lagrangian

$$\begin{aligned}\mathcal{L} \;=\; & -\frac{1}{2}\,W^-_{\mu\nu}W^{+\mu\nu} \;+\; M_W^2 W^-_\mu W^{+\mu} - \frac{g_2}{\sqrt{2}}\Big(W^+_\mu \times J^{-\mu} + W^-_\mu \times J^{+\mu}\Big) \\ & -\; \frac{1}{4}\,Z_{\mu\nu}Z^{\mu\nu} \;+\; \frac{M_Z^2}{2}\,Z_\mu Z^\mu \;-\; \tilde{g} J^\mu_Z Z_\mu \\ & + \text{ terms involving the EM field } A_\mu \text{ and the Higgs } h \\ & \qquad \text{and their interactions with the } W^\pm_\mu \text{ and the } Z_\mu.\end{aligned} \tag{35}$$

To the leading order of the perturbation theory, we may neglect the non-abelian terms here as well as the interactions of the vector fields with the physical Higgs field h. Moreover, at low momenta $k^\mu \ll M_W, M_Z$, the kinetic terms for the $W^\pm_\mu$ and Z_μ fields are much smaller than the respective mass terms. Consequently, we may approximate

$$\mathcal{L} \;\approx\; M_W^2 W^-_\mu W^{+\mu} \;-\; \frac{g_2}{\sqrt{2}}\Big(W^+_\mu \times J^{-\mu} + W^-_\mu \times J^{+\mu}\Big) + \frac{M_Z^2}{2}\,Z_\mu Z^\mu \;-\; \tilde{g} J^\mu_Z Z_\mu\,. \tag{36}$$

The field equations stemming from this approximate Lagrangian are simply

$$M_W^2 \times W^{\pm\mu} \;\approx\; \frac{g_2}{\sqrt{2}} \times J^{\pm\mu} \quad \text{and} \quad M_Z^2 \times Z^\mu \;\approx\; \tilde{g} \times J^\mu_Z\,. \tag{37}$$

Solving these equations and plugging the solutions back into the Lagrangian (36), we obtain the *effective current-current Lagrangian for the low-energy weak interactions,*

$$\mathcal{L}^{\text{weak}}_{\text{effective}} \;=\; -\frac{g_2^2}{2M_W^2} \times J^-_\mu J^{+\mu} \;-\; \frac{\tilde{g}^2}{2M_Z^2} \times J_{Z\mu} J^\mu_Z\,. \tag{38}$$

This effective Lagrangian is called the *Fermi Lagrangian* — and the corresponding effective theory of weak interactions is called the *Fermi Theory* — since Enrico Fermi wrote it down back in 1933. Or rather, he wrote down

$$\mathcal{L} \;=\; -2\sqrt{2}G \times J^-_\mu J^{+\mu} \tag{39}$$

since only the charged-current weak interactions were known in those days, and the weak coupling G was an input parameter to be determined experimentally. Today G is called the

Fermi constant and we know how to relate it to the vacuum expectation value of the Higgs field:

$$G \;=\; \frac{1}{4\sqrt{2}}\,\frac{g_2^2}{M_W^2} \;=\; \frac{1}{\sqrt{2}}\times\frac{1}{v^2}\,. \tag{40}$$

The neutral-coupling weak interactions have a separate Fermi-like constant, but in the GWS electroweak theory it has exactly the same value as the Fermi constant for the charged-current weak interactions:

$$\mathcal{L}^{\text{weak}}_{\text{effective}} \;=\; -2\sqrt{2}G\left(J^-_\mu\,J^{+\mu} \,+\, \rho\times J_{Z\mu}J^\mu_Z\right) \qquad \text{for } \rho = 1. \tag{41}$$

Indeed,

$$G\times\rho \;=\; \frac{1}{4\sqrt{2}}\,\frac{\tilde{g}^2}{M_Z^2} \quad\Longrightarrow\quad \rho \;=\; \frac{\tilde{g}^2}{M_Z^2}\Big/\,\frac{g_2^2}{M_W^2} \;=\; \frac{\tilde{g}^2}{g_2^2}\times\frac{M_W^2}{M_Z^2} \;=\; \frac{1}{\cos^2\theta}\times\cos^2\theta \;=\; 1. \tag{42}$$

Experimentally, $\rho = 1$ with high precision, and it is a strong evidence for the GWS theory.