Poisson Brackets and Commutator Brackets

Both classical mechanics and quantum mechanics use bi-linear brackets of variables with similar algebraic properties. In classical mechanics the variables are functions of the canonical coordinates and momenta, and the Poisson bracket of two such variables $A(q,p)$ and $B(q,p)$ are defined as

$$[A,B]_P \stackrel{\text{def}}{=} \sum_i \left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}\right). \tag{1}$$

In quantum mechanics the variables are linear operators in some Hilbert space, and the commutator bracket of two operators is

$$[A,B]_C \stackrel{\text{def}}{=} AB - BA. \tag{2}$$

Both types of brackets have similar algebraic properties:

1. Linearity: $[\alpha_1 A_1 + \alpha_2 A_2, B] = \alpha_1[A_1,B] + \alpha_2[A_2,B]$ and $[A, \beta_1 B_1 + \beta_2 B_2] = \beta_1[A,B_1] + \beta_2[A,B_2]$.
2. Antisymmetry: $[A,B] = -[B,A]$.
3. Leibniz rules: $[AB,C] = A[B,C] + [A.C]B$ and $[A,BC] = B[A,C] + [A,B]C$.
4. Jacobi Identity: $[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$.

Theorem: For non-commuting variables, any bracket $[A,B]$ with the above algebraic properties 1 through 4 is proportional to the commutator bracket:

$$[A,B] = c(AB - BA) \tag{3}$$

for a universal constant c (same c for all variables). In particular, generalization of classical Poisson brackets to quantum mechanics leads to

$$[\hat{A},\hat{B}]_P = \frac{\hat{A}\hat{B} - \hat{B}\hat{A}}{i\hbar}. \tag{4}$$

Proof: Take any 4 variables A, B, U, V and calculate $[AU, BV]$ using the Leibniz rules, first for the AU and then for the BV:

$$\begin{aligned} [AU, BV] &= A[U, BV] + [A, BV]U \\ &= AB[U, V] + A[U, B]V + B[A, V]U + [A, B]VU. \end{aligned} \tag{5}$$

OOH, if we use the two Leibniz rules in the opposite order we get a different expression

$$\begin{aligned} [AU, BV] &= B[AU, V] + [AU, B]V \\ &= BA[U, V] + B[A, V]U + A[U, B]V + [A, B]UV. \end{aligned} \tag{6}$$

To make sure the two expressions are equal to each other we need

$$\begin{aligned} AB[U, V] + [A, B]VU &= BA[U, V] + [A, B]UV \\ &\Downarrow \\ (AB - BA)[U, V] &= [A, B](UV - VU) \\ &\Downarrow \\ [U, V](UV - VU)^{-1} &= (AB - BA)^{-1}[A, B] \end{aligned} \tag{7}$$

On the last line here, the LHS depends only on the U and V while the RHS depends only on the A and B, and the only way a relation like that can work for any *unrelated* variables is if the ratios on both sides of equations are equal to the same universal constant c, thus

$$[A, B] = c(AB - BA) \quad \text{and} \quad [U, V] = c(UV - VU). \tag{8}$$

Quod erat demonstrandum.