QFT Dimensional Analysis

In the $\hbar = c = 1$ units, all quantities are measured in units of energy to some power. For example $\big[m\big] = \big[p^\mu\big] = E^{+1}$ while $\big[x^\mu\big] = E^{-1}$ where $[m]$ stands for the *dimensionality* of the mass rather than the mass itself, and ditto for the $\big[p^\mu\big]$, $\big[x^\mu\big]$, *etc.* The action

$$S = \int d^4x\,\mathcal{L}$$

is dimensionless (in $\hbar \neq 1$ units, $\big[S\big] = \hbar$), so the Lagrangian of a 4D field theory has dimensionality $[\mathcal{L}] = E^{+4}$.

Dimensionalities of the quantum fields are subject to quantum corrections,

$$[\text{field}] = E^{\Delta}, \quad \Delta = \Delta_0 + O(\text{coupling}) + O(\text{coupling}^2) + \cdots, \tag{1}$$

where Δ_0 is called the *canonical dimension* — it's the dimensionality of the field in question in the semiclassical limit. The canonical dimensions of different fields follow from their free Lagrangians. For example, a scalar field $\Phi(x)$ has

$$\mathcal{L}_{\text{free}} = \tfrac{1}{2}\partial_\mu\Phi\,\partial^\mu\Phi - \tfrac{1}{2}m^2\Phi^2, \tag{2}$$

so $\big[\mathcal{L}\big] = E^{+4}$, $\big[m^2\big] = E^{+2}$, and $\big[\partial_\mu\big] = E^{+1}$ imply $\big[\Phi\big] = E^{+1}$. Likewise, the EM field has

$$\mathcal{L}^{\text{EM}}_{\text{free}} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad\Longrightarrow\quad \big[F_{\mu\nu}\big] = E^{+2}, \tag{3}$$

and since $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, the $A_\nu(x)$ field has dimension

$$\big[A_\nu\big] = \big[F_{\mu\nu}\big] \,/\, \big[\partial_\mu\big] = E^{+1}. \tag{4}$$

The massive vector fields also have $\big[A_\nu\big] = E^{+1}$, so that both terms in

$$\mathcal{L}_{\text{free}} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}m^2A_\nu A^\nu \tag{5}$$

have dimensions $\big[F^2\big] = \big[m^2A^2\big] = E^{+4}$.

In fact, all the *bosonic* fields in 4D spacetime have canonical dimensions E^{+1} because their kinetic terms are quadratic in ∂_μ(field). On the other hand, the fermionic fields like the Dirac field $\Psi(x)$ have dimensionality $[\Psi] = E^{+3/2}$. Indeed, the kinetic terms in the free Dirac Lagrangian

$$\mathcal{L}_{\rm free} \;=\; \overline{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi \tag{6}$$

involve two fermionic fields Ψ and $\overline{\Psi}$ but only one derivative ∂_μ. Consequently, $\bigl[\mathcal{L}\bigr] = E^{+4}$ implies $\bigl[\overline{\Psi}\Psi\bigr] = E^{+3}$ and hence $\bigl[\Psi\bigr] = \bigl[\overline{\Psi}\bigr] = E^{+3/2}$. Similarly, all other types of fermionic fields in 4D have canonical dimension $E^{+3/2}$.

In QFTs in other spacetime dimensions $d \neq 4$, similar arguments show that the bosonic fields such as scalars and vectors have canonical dimension

$$\bigl[\Phi\bigr] \;=\; \bigl[A_\nu\bigr] \;=\; E^{+(d-2)/2} \tag{7}$$

while the fermionic fields have canonical dimension

$$\bigl[\Psi\bigr] \;=\; E^{+(d-1)/2}. \tag{8}$$

In perturbation theory, dimensionality of coupling parameters such as λ in $\lambda\Phi^4$ theory or e in QED follows from the field's canonical dimensions. For example, in a 4D scalar theory with Lagrangian

$$\mathcal{L} \;=\; \tfrac12\partial_\mu\Phi\,\partial^\mu\Phi \;-\; \tfrac12 m^2\Phi^2 \;-\; \sum_{n\geq3}\frac{C_n}{n!}\,\Phi^n, \tag{9}$$

the coupling C_n of the Φ^n term has dimensionality

$$\bigl[C_n\bigr] \;=\; \bigl[\mathcal{L}\bigr]\Big/\bigl[\Phi\bigr]^n \;=\; E^{4-n}. \tag{10}$$

In particular, the cubic coupling C_3 has positive energy dimension E^{+1}, the quartic coupling $\lambda = C_4$ is dimensionless, while all the higher-power couplings have negative energy dimensions $E^{\rm negative}$.

Now consider a theory with a single coupling g of dimensionality $\bigl[g\bigr] = E^{\Delta}$. The perturbation theory in g amounts to a power series expansion

$$\mathcal{M}(\text{momenta}, g) \;=\; \sum_N \left(\frac{g}{\mathcal{E}^{\Delta}}\right)^N \times F_N(\text{momenta}) \qquad (11)$$

where $\mathcal{E}$ is the overall energy scale of the process in question and all the F_N functions of momenta have the same dimensionality. The power series (11) is asymptotic rather than convergent, so it makes sense only when the expansion parameter is small,

$$\frac{g}{\mathcal{E}^{\Delta}} \;\ll\; 1. \qquad (12)$$

For a dimensionless coupling g this condition is simply $g \ll 1$, but for $\Delta \neq 0$ the situation is more complicated.

For couplings of positive dimensionality $\Delta > 0$, the expansion parameter (12) is always small for for high-energy processes with $\mathcal{E} \gg g^{1/\Delta}$. But for low energies $\mathcal{E} \lesssim g^{1/\Delta}$, the expansion parameter becomes large and the perturbation theory breaks down. This is a major problem for theories with $\Delta > 0$ couplings of *massless particles*. However, if all the particles participating in a $\Delta > 0$ coupling are massive, then all processes have energies $\mathcal{E} \gtrsim M_{\text{lightest}}$, and this makes couplings with $\Delta > 0$ OK as long as

$$g \;\ll\; M_{\text{lightest}}^{\Delta}\,. \qquad (13)$$

Couplings of negative dimensionality $\Delta < 0$ have the opposite problem: The expansion parameter (12) is small at low energies but becomes large at high energies $\mathcal{E} \gtrsim g^{-1/\Delta}$. Beyond the maximal energy

$$E^{\max} \;\sim\; g^{1/(-\Delta)}, \qquad (14)$$

the perturbation theory breaks down and we may no longer compute the S–matrix elements $\mathcal{M}$ using any finite number of Feynman diagrams.

Worse, in Feynman diagrams with loops one must worry not only about momenta k^{μ} of the incoming and outgoing particles but also about momenta q^{μ} of the internal lines.

Basically, an L–loop diagram contributing to the N^{th} term in the expansion (11) produces something like

$$g^N \times \int d^{4L}q\,\mathcal{F}_N(q,k,m) \qquad \text{where} \quad \left[\mathcal{F}_N\right] \;=\; E^{-N\Delta-4L+\mathrm{C}}, \quad C \;=\; \dim[\mathcal{M}] \;=\; \text{const.} \tag{15}$$

For very large loop momenta $q \gg k, m$, dimensionality implies $\mathcal{F}_N \propto q^{-N\Delta-4L+\mathrm{C}}$, so for $N(-\Delta) + C \geq 0$, the integral (15) diverges as $q \to \infty$. Moreover, the degree of divergence increases with the order N of the perturbation theory, so any scattering amplitude becomes divergent at high orders. Therefore, *field theories with* $\Delta < 0$ *couplings do not work as complete theories.*

However, theories with $\Delta < 0$ may be used as approximate *effective theories* (without the divergent loop graphs) for low-energy processes, $\mathcal{E} \lesssim \Lambda$ for some $\Lambda < g^{-1/\Delta}$. For example, the Fermi theory of weak interactions

$$\mathcal{L}_{\text{int}} \;=\; \frac{G_F}{\sqrt{2}} \sum_{\substack{\text{appropriate}\\ \text{fermions}}} \overline{\Psi}\gamma_\mu(1-\gamma^5)\Psi \times \overline{\Psi}\gamma^\mu(1-\gamma^5)\Psi \tag{16}$$

has coupling G_F of dimension $\left[G_G\right] = E^{-2}$; its value is $G_F \approx 1.17 \cdot 10^{-5}\,\text{GeV}^{-2}$. This is a good effective theory for low-energy weak interactions, but it cannot be used for energies $\mathcal{E} \gtrsim 1/\sqrt{G_F} \sim 300$ GeV, not even theoretically. In real life, the Fermi theory works only for $\mathcal{E} \ll M_W \sim 80$ GeV; at higher energies, one should use the complete $SU(2) \times U(1)$ electroweak theory instead of the Fermi theory.

Similar to the Fermi theory, most effective theories with $\Delta < 0$ couplings are low-energy limits of more complicated theories with extra heavy particles of masses $M \lesssim g^{-1/\Delta}$.

On the other hand, a *UV-complete* quantum field theory which may be extrapolated to arbitrarily high energies cannot have any negative-dimensionality couplings This is a major restriction because in $d = 4$ dimensions there are only a few coupling types with $\Delta \geq 0$. To see that, note that any coupling involves 3 or more fields, but the more fields it involves, the less is its dimensionality. Specifically, a coupling involving b bosonic fields (scalar or vector), f fermionic fields, and δ derivatives ∂_μ has dimensionality

$$\Delta \;=\; 4 \;-\; b \;-\; \tfrac{3}{2}f \;-\; \delta. \tag{17}$$

Consequently, the only one $\Delta > 0$ coupling type — the boson^3 without derivatives, — and a few more with $\Delta = 0$, namely boson^4, boson $\times$ fermion^2, and boson^2 $\times$ ∂boson. All other coupling types have $\Delta < 0$ and are not allowed (except in effective theories).

In terms of more specific field and coupling types, here is the complete list of the allowed couplings in 4D.

1. Scalar couplings

$$-\frac{\mu}{3!}\,\Phi^3 \quad \text{and} \quad -\frac{\lambda}{4!}\,\Phi^4. \tag{18}$$

Note: the higher powers Φ^5, Φ^6, *etc.*, are not allowed because the couplings would have $\Delta < 0$.

2. Gauge couplings of vectors to charged scalars

$$-iqA^\mu \times (\Phi^*\partial_\mu\Phi \,-\, \Phi\partial_\mu\Phi^*) \,+\, q^2 A_\mu A^\mu \times \Phi^*\Phi \;\subset\; D_\mu\Phi^*\, D^\mu\Phi, \tag{19}$$

or for non-abelian gauge symmetries

$$-igA^{a\mu} \times \Big(\Phi^\dagger T^a \partial_\mu\Phi \,-\, \partial_\mu\Phi^\dagger T^a\Phi\Big) \,+\, g^2 A^a_\mu A^{b\mu} \times \Phi^\dagger T^a T^b \Phi \;\subset\; D_\mu\Phi^\dagger\, D^\mu\Phi. \tag{20}$$

3. Non-abelian gauge couplings between the vector fields

$$-gf^{abc}(\partial_\mu A^a_\nu)A^{\mu b}A^{\nu c} \,-\, \frac{g^2}{4}\,f^{abc}f^{ade}A^b_\mu A^c_\nu A^{\mu d}A^{\nu e} \;\subset\; -\frac{1}{4}\,F^a_{\mu\nu}F^{\mu\nu a}. \tag{21}$$

4. Gauge couplings of vectors to charged fermions,

$$-qA^\mu \times \overline{\Psi}\gamma_\mu\Psi \quad \text{or} \quad -\, gA^{a\mu} \times \overline{\Psi}\gamma_\mu T^a\Psi \quad \subset \quad \overline{\Psi}(i\gamma_\mu D^\mu)\Psi. \tag{22}$$

If the fermions are massless and chiral, we may also have

$$-gA^a_\mu \times \overline{\Psi}\gamma^\mu\,\frac{1 \mp \gamma^5}{2}\,T^a\Psi, \tag{23}$$

or in the Weyl fermion language

$$-gA^a_\mu \times \psi^\dagger_L\bar{\sigma}_\mu T^a\psi_L \quad \text{or} \quad -\, gA^a_\mu \times \psi^\dagger_R\sigma_\mu T^a\psi_R\,.$$

5. Yukawa couplings of scalars to fermions,

$$-y\Phi\times\overline{\Psi}\Psi \quad \text{or} \quad -iy\Phi\times\overline{\Psi}\gamma^5\Psi. \tag{24}$$

If parity is conserved, in the first terms Φ should be a true scalar, and in the second term a pseudo-scalar.

— And this is it! No other coupling types are allowed in 4D field theories that remain valid to arbitrarily high energies.

In other spacetime dimensions $d\neq 3+1$, a coupling involving b bosonic fields, f fermionic fields, and δ derivatives has dimensionality

$$\Delta\;=\;d\;-\;b\times\frac{d-2}{2}\;-\;f\times\frac{d-1}{2}\;-\;\delta\;=\;b\;+\;\tfrac{1}{2}f\;-\;\delta\;-\;\frac{b+f-2}{2}\times d. \tag{25}$$

Since all interactions involve three or more fields, thus $b+f\geq 3$, the dimensionality of any particular coupling always decreases with d. Consequently, there are more perturbatively-allowed couplings with $\Delta\geq 0$ in lower dimensions $d=2+1$ or $d=1+1$ but fewer allowed couplings in higher dimensions $d>3+1$. In particular,

- *In $d\geq 6+1$ dimensions all couplings have $\Delta<0$ and there are no UV-complete quantum field theories,* or at least no perturbative UV-complete quantum field theories.
- In $d=5+1$ dimensions there is a unique $\Delta=0$ coupling $(\mu/6)\Phi^3$, while all the other couplings have $\Delta<0$. Consequently, the only perturbative UV-complete theories are scalar theories with cubic potentials,

$$\mathcal{L}\;=\;\sum_a\left(\tfrac{1}{2}(\partial_\mu\Phi_a)^2\;-\;\tfrac{1}{2}m_a^2\Phi_a^2\right)\;-\;\tfrac{1}{6}\sum_{a,b,c}\mu_{abc}\Phi_a\Phi_b\Phi_c\,. \tag{26}$$

However, while such theories are perturbatively OK, they do not have stable vacua. Indeed, a cubic potential is un-bounded from below — it goes to $-\infty$ along half of the directions in the field space — so even if it has a *local* minimum at $\Phi_a=0$, it's not the global minimum. Consequently, in the quantum theory, the naive vacuum with $\langle\Phi_a\rangle=0$ would decay by tunneling to a run-away state with $\langle\Phi_a\rangle\to\pm\infty$.

- In $d = 4+1$ dimensions, the $(\mu/6)\Phi^3$ coupling has positive $\Delta = +\frac{1}{2}$ while all the other couplings have negative energy dimensions. Again, the only perturbative UV-complete theories are scalar theories with cubic potentials, but they do not have stable vacua.

⋆ The bottom line is, *in $d > 3+1$ dimensions, all quantum field theories are effective theories* for low-enough energies. At higher energies, a different kind of theory must take over — perhaps a theory in a discrete space, perhaps a string theory, or maybe something more exotic.

On the other hand, in lower dimensions $d = 2+1$ or $d = 1+1$ there are many more allowed couplings with $\Delta \geq 0$. In particular, in $d = 2+1$ dimensions the allowed couplings include:

∘ Scalar couplings $(C_n/n!)\Phi^n$ up to $n = 6$;

∘ gauge and Yukawa couplings like in 4D;

∘ Yukawa-like couplings $\tilde{y}\Phi^2 \times \overline{\Psi}\Psi$ involving 2 scalars;

∘ gauge-like couplings with g_{gauge} linearly dependent on a neutral scalar field:

$$D_\mu\Psi \;=\; \partial_\mu\Psi \;+\; i(g_0 + c\phi)A_\mu\Psi \quad\Longrightarrow\quad \overline{\Psi}(i\gamma^\mu D_\mu)\Psi \;\supset\; -c\phi A_\mu\,\overline{\Psi}\gamma^\mu\Psi, \tag{27}$$

and likewise

$$D^\mu\Phi^* D_\mu\Phi \;\supset\; -ic\phi A^\mu \times (\Phi^*\partial_\mu\Phi \;-\; \Phi\partial_\mu\Phi^*) \;+\; c^2\phi^2 \times \Phi^*\Phi A_\mu A^\mu, \tag{28}$$

or non-abelian

$$-\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} \;\supset\; -c\phi \times f^{abc}(\partial_\mu A^a_\nu)A^{\mu b}A^{\nu c} \;-\; \frac{c^2}{4}\phi^2 \times f^{abc}f^{ade}A^b_\mu A^c_\nu A^{\mu d}A^{\nu e}. \tag{29}$$

There are also combinations of $g = g_0 + c\phi$ with Chern–Simons couplings (like the 3D photon mass term in the mid-term exam and its non-abelian generalizations), but I don't want to get into their details here.

∗ There might be some other allowed couplings in 3D, but never mind for now.

Finally, *in $d = 1+1$ dimensions there is an infinite number of allowed $\Delta \geq 0$ couplings.* Indeed, for $d = 1+1$ the bosonic fields have energy dimension E^0, so Δ of a coupling does not depend on the number b of bosonic fields it involves but only on the numbers of derivatives and fermionic fields,

$$\Delta \;=\; 2 - \;\delta\; - \;\tfrac{1}{2}f. \tag{30}$$

Consequently, all scalar potentials $V(\Phi)$ — including $C_n\Phi^n$ terms for any n, and even the non-polynomial potentials — have $\Delta = +2$, so any $V(\Phi)$ is allowed in 2D. Likewise, all Yukawa-like couplings $\Phi^n\overline{\Psi}\Psi$ have $\Delta = +1$, so we may have terms like $y_{IJ}(\Phi) \times \overline{\Psi}^I\Psi^J$ for any functions $y_{IJ}(\Phi)$.

At the $\Delta = 0$ level, we are allowed field-dependent kinetic terms

$$\mathcal{L}_{\text{kin}} \;=\; \tfrac{1}{2}g_{ij}(\phi) \times \partial^\mu\phi^i\,\partial_\mu\phi^j \tag{31}$$

with any Riemannian metrics $g_{ij}(\phi)$ for the non-linear scalar field space, as well as a whole bunch of fermionic terms with arbitrary scalar-dependent coefficients,

$$\begin{aligned}\mathcal{L}_\Psi \;\supset\; \tfrac{1}{4}g_{IJ}(\Phi) \times \overline{\Psi}^I\gamma^\mu\left(i\,\overrightarrow{\partial}_\mu - i\,\overleftarrow{\partial}_\mu\right)\Psi^J \;+\; \Gamma_{IJk}(\Phi) \times \partial_\mu\Phi^k \times \overline{\Psi}^I\gamma^\mu\Psi^J \\ +\; \tfrac{1}{2}R_{IJKL}(\Phi) \times \overline{\Psi}^I\gamma^\mu\Psi^J \times \overline{\Psi}^K\gamma_\mu\Psi^L.\end{aligned} \tag{32}$$

In addition, there are gauge couplings with arbitrary scalar dependent $g_{\text{gauge}}(\Phi)$, chiral couplings to Weyl or Majorana-Weyl fermions, *etc.*, *etc.* In the String Theory class, you will encounter many of these couplings in the context of the 2D field theory on the world sheet of the string.