

# Relativistic Causality for Fermions

In a quantum field theory, relativistic causality requires that **any two measurable local operators**  $\hat{O}_1(x)$  and  $\hat{O}_2(y)$  acting at points separated by a spacelike interval  $x - y$  must commute with each other,  $\hat{O}_1(x)\hat{O}_2(y) = +\hat{O}_2(y)\hat{O}_1(x)$ . In general, a local operator is a product of quantum field and/or their derivatives, so if all fields at points separated by spacelike intervals commute with each other,  $\hat{\phi}_A(x)\hat{\phi}_B(y) = +\hat{\phi}_B(y)\hat{\phi}_A(x)$  when  $(x-y)^2 < 0$ , then so do all local operators at  $x$  and  $y$ , and the relativistic causality is upheld.

However, not all local operators are necessarily measurable, and this gives us more options for the relativistic causality. In particular, in fermionic theories, the fermionic quantum fields  $\hat{\Psi}_\alpha(x)$  and  $\hat{\Psi}_\alpha^\dagger(x)$  themselves are not measurable. Only the bilinears such as the current  $\hat{J}^\mu(x) = \hat{\Psi}(x)\gamma^\mu\hat{\Psi}(x)$  are measurable. In general, **local measurable operators are products of even numbers of fermionic fields and their derivatives** — as well as any number of bosonic fields and their derivatives,

$$\begin{aligned} \text{measurable } \hat{O}(x) = & \prod_{i=1}^{\text{even } N} (F_i(x) \text{ or } \partial F_i(x) \text{ or } \partial\partial F_i(x) \text{ or } \dots) \times \\ & \times \prod_{i=1}^{\text{any } M} (B_i(x) \text{ or } \partial B_i(x) \text{ or } \partial\partial B_i(x) \text{ or } \dots). \end{aligned} \tag{1}$$

Consequently, to assure that all such *measurable* operators commute at spacelike-separated points, the fermionic fields should either commute or anticommute with each other. Altogether, we need

For any bosonic fields  $\hat{B}_1$  and  $\hat{B}_2$ , and any fermionic fields  $\hat{F}_1$  and  $\hat{F}_2$ ,

$$\text{when } (x - y)^2 < 0,$$

$$\hat{B}_1(x) \times \hat{B}_2(y) = +\hat{B}_2(y) \times \hat{B}_1(x), \tag{2}$$

$$\hat{F}_i(x) \times \hat{B}_j(y) = +\hat{B}_j(y) \times \hat{F}_i(x), \tag{3}$$

$$\hat{F}_1(x) \times \hat{F}_2(y) = -\hat{F}_2(y) \times \hat{F}_1(x). \tag{4}$$

By itself, the relativistic causality is consistent with either ‘+’ or ‘−’ sign on the last line (4), but other considerations fix that sign to be negative. Thus, at spacelike separations, the fermionic fields anticommute rather than commute with each other.

The simplest reason why the antifermionic fields should anticommute is the fermionic Fock space. To realize the Pauli principle, we need anticommuting creation and annihilation operators. The quantum fields are linear combinations of such operators, so at equal times but  $\mathbf{x} \neq \mathbf{y}$ , the fields should anticommute with each other rather than commute, and by relativistic causality, the same sign should extend to any spacelike  $x - y$ .

Another reason for the anticommutation relation is the [spin-statistics theorem](#) — the spinor fields should anticommute with each other rather than commute. In this writeup I shall explain how this works for the free Dirac fields; the general case is explained in the [separate notes](#) (which I had used for the supplementary lecture on 10/16).

Before we do anything else, let's by assume the free Dirac spinor fields anticommute at equal times, and show that they also anticommute at all spacelike separations. Expanding the spinor fields into creation and annihilation operators, we get

$$\begin{aligned}\widehat{\Psi}_\alpha(x) &= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2E_p} \sum_s \left( e^{-ipx} u_\alpha(\mathbf{p}, s) \hat{a}_{\mathbf{p},s} + e^{+ipx} v_\alpha(\mathbf{p}, s) \hat{b}_{\mathbf{p},s}^\dagger \right)^{p_0=+E_p}, \\ \widehat{\Psi}_\beta(y) &= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2E_p} \sum_s \left( e^{-ipy} \bar{v}_\beta(\mathbf{p}, s) \hat{b}_{\mathbf{p},s} + e^{+ipy} \bar{u}_\beta(\mathbf{p}, s) \hat{a}_{\mathbf{p},s}^\dagger \right)^{p_0=+E_p},\end{aligned}\tag{5}$$

where the creation/annihilation operators obey the fermionic anticommutation relations,

$$\{\hat{a}_{\mathbf{p},s}, \hat{a}_{\mathbf{p}',s'}^\dagger\} = \{\hat{b}_{\mathbf{p},s}, \hat{b}_{\mathbf{p}',s'}^\dagger\} = +2E_p(2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}') \times \delta_{ss'},\tag{6}$$

everybody else anticommutes.

Consequently,

$$\{\widehat{\Psi}_\alpha(x), \widehat{\Psi}_\beta(y)\} \equiv 0 \equiv \{\widehat{\Psi}_\alpha(x), \widehat{\Psi}_\beta(y)\} \quad \text{for any } x - y,\tag{7}$$

but the anticommutator of a  $\widehat{\Psi}$  with a  $\widehat{\Psi}$  is not so trivial. Indeed, such anticommutator gets non-trivial terms from  $\hat{a}_{\mathbf{p},s} \in \widehat{\Psi}(x)$  and matching  $\hat{a}_{\mathbf{p},s}^\dagger \in \widehat{\Psi}(y)$ , and also from  $\hat{b}_{\mathbf{p},s}^\dagger \in \widehat{\Psi}(x)$

and matching  $\hat{b}_{\mathbf{p},s} \in \widehat{\Psi}(y)$ . Specifically,

$$\begin{aligned} \{\widehat{\Psi}_\alpha(x), \widehat{\Psi}_\beta(y)\} &= \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_p} \sum_s \int \frac{d^3\mathbf{p}'}{(2\pi)^3 2E_{p'}} \sum_{s'} \\ &\quad \left( \begin{aligned} &e^{-ipx+ip'y} u_\alpha(\mathbf{p}, s) \bar{u}_\beta(\mathbf{p}', s') \times \{\hat{a}_{\mathbf{p},s}, \hat{a}_{\mathbf{p}',s'}^\dagger\} \\ &+ e^{+ipx-ip'y} v_\alpha(\mathbf{p}, s) \bar{v}_\beta(\mathbf{p}', s') \times \{\hat{b}_{\mathbf{p},s}^\dagger, \hat{b}_{\mathbf{p}',s'}\} \end{aligned} \right) \quad (8) \\ &= \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_p} \sum_s \left( \begin{aligned} &e^{-ip(x-y)} \times u_\alpha(\mathbf{p}, s) \bar{u}_\beta(\mathbf{p}, s) \\ &+ e^{+ip(x-y)} \times v_\alpha(\mathbf{p}, s) \bar{v}_\beta(\mathbf{p}, s) \end{aligned} \right). \end{aligned}$$

At this point, let's make use of the spin sums worked out in [homework set#7](#) (problem 4):

$$\sum_s u_\alpha(\mathbf{p}, s) \bar{u}_\beta(\mathbf{p}, s) = (\not{p} + m)_{\alpha\beta}, \quad \sum_s v_\alpha(\mathbf{p}, s) \bar{v}_\beta(\mathbf{p}, s) = (\not{p} - m)_{\alpha\beta}, \quad (9)$$

where  $\not{p} = \gamma^\mu p_\mu$  and  $p_0 = +E_p$ . Plugging these spin sums into eq. (8), we obtain

$$\begin{aligned} \{\widehat{\Psi}_\alpha(x), \widehat{\Psi}_\beta(y)\} &= \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_p} \left( e^{-ip(x-y)} \times (\not{p} + m)_{\alpha\beta} + e^{+ip(x-y)} \times (\not{p} - m)_{\alpha\beta} \right)^{p_0=+E_p} \\ &= \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_p} \left( (+i \not{\partial}_x + m)_{\alpha\beta} e^{-ip(x-y)} + (-i \not{\partial}_x - m)_{\alpha\beta} e^{+ip(x-y)} \right)^{p_0=+E_p} \\ &= (i \not{\partial}_x + m)_{\alpha\beta} \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_p} \left( e^{-ip(x-y)} - e^{+ip(x-y)} \right)^{p_0=+E_p} \\ &= (i \not{\partial}_x + m)_{\alpha\beta} \left( D(x-y) - D(y-x) \right), \end{aligned} \quad (10)$$

where  $D(x-y)$  is the good old

$$D(x-y) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_p} e^{-ip(x-y)} \Big|^{p_0=+E_p}. \quad (11)$$

As we have learned a while ago,  $D(x-y)$  is invariant under orthochronous Lorentz transforms; in particular, for spacelike  $x-y$ ,  $D(x-y) = D(y-x)$ , Consequently, thanks to the ‘-’ sign on the bottom line of eq. (10),

$$\{\widehat{\Psi}_\alpha(x), \widehat{\Psi}_\beta(y)\} = (i \not{\partial}_x + m)_{\alpha\beta} \left( D(x-y) - D(y-x) \right) = 0 \quad \text{for spacelike } x-y, \quad (12)$$

which upholds the relativistic causality for the fermionic fields.

Now suppose for a moment that the spinor fields were commuting (instead of anti-commuting) at equal times, so in the expansion (5), the creation/annihilation operators  $\hat{a}_{\mathbf{p},s}^\dagger, \hat{b}_{\mathbf{p},s}^\dagger, \hat{a}_{\mathbf{p},s}, \hat{b}_{\mathbf{p},s}$  were obeying the bosonic commutation relations (instead of the fermionic anticommutation relations (6)). Of course, such expansion would have screwed up the Dirac Hamiltonian

$$\hat{H} = \int d^3\mathbf{x} \Psi^\dagger (-i\gamma^0 \vec{\gamma} \cdot \nabla + \gamma^0 m) \Psi \neq \int \frac{d^3\mathbf{p}}{(2\pi)^3} \sum_s \frac{1}{2E_p} \left( E_p \times \hat{a}_{\mathbf{p},s}^\dagger \hat{a}_{\mathbf{p},s} + E_p \times \hat{b}_{\mathbf{p},s}^\dagger \hat{b}_{\mathbf{p},s} \right) \quad (13)$$

since the particle-hole formalism does not work for the bosons. But even apart from that issue, we would get problems with relativistic causality. Indeed, we would get

$$\begin{aligned} [\widehat{\Psi}_\alpha(x), \widehat{\Psi}_\beta(y)] &= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \sum_s \frac{1}{2E_p} \int \frac{d^3\mathbf{p}'}{(2\pi)^3} \sum_{s'} \frac{1}{2E_{p'}} \\ &\quad \left( \begin{aligned} &e^{-ipx+ip'y} u_\alpha(\mathbf{p}, s) \bar{u}_\beta(\mathbf{p}', s') \times [\hat{a}_{\mathbf{p},s}, \hat{a}_{\mathbf{p}',s'}^\dagger] \\ &+ e^{+ipx-ip'y} v_\alpha(\mathbf{p}, s) \bar{v}_\beta(\mathbf{p}', s') \times [\hat{b}_{\mathbf{p},s}^\dagger, \hat{b}_{\mathbf{p}',s'}] \end{aligned} \right) \quad (14) \\ &= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \sum_s \frac{1}{2E_p} \left( \begin{aligned} &e^{-ip(x-y)} \times u_\alpha(\mathbf{p}, s) \bar{u}_\beta(\mathbf{p}, s) \\ &- e^{+ip(x-y)} \times v_\alpha(\mathbf{p}, s) \bar{v}_\beta(\mathbf{p}, s) \end{aligned} \right) \end{aligned}$$

with a wrong sign on the bottom line compared to eq. (8), and consequently

$$[\widehat{\Psi}_\alpha(x), \widehat{\Psi}_\beta(y)] = (i\partial_x + m)_{\alpha\beta} \left( D(x-y) + D(y-x) \right) \neq 0 \quad \text{for spacelike } x-y. \quad (15)$$

**Bottom line:** Relativistic causality works for Dirac spinor fields quantized as fermions, and does not work when the same fields are quantized as bosons. This is a special case of the [spin-statistics theorem](#).

## Feynman Propagator for the Dirac Spinor Field

For any kind of a free field, the Feynman propagator obtains as the ‘vacuum sandwich’ of the time-ordered product of two fields, one at  $x$  and one at  $y$ . For the charged fields, the two fields should have opposite charges, for example, for a charged scalar field

$$G_F(x - y) = \langle 0 | \mathbf{T} \hat{\Phi}^\dagger(x) \hat{\Phi}(y) | 0 \rangle. \quad (16)$$

For the fermionic fields — such as Dirac spinor fields — the time-orderer  $\mathbf{T}$  flips sign when it exchanges two fields,

$$\mathbf{T} \hat{\Psi}_\alpha(x) \hat{\Psi}_\beta^\dagger(y) = \begin{cases} +\hat{\Psi}_\alpha(x) \hat{\Psi}_\beta^\dagger(y) & \text{when } x^0 > y^0, \\ -\hat{\Psi}_\beta^\dagger(y) \hat{\Psi}_\alpha(x) & \text{when } y^0 > x^0. \end{cases} \quad (17)$$

By relativistic causality, the fermionic fields anticommute when separated by spacelike intervals, so the time ordered product — with the signs as in eq. (17) — does not have a discontinuity when  $x^0 = y^0$  (except maybe at  $\mathbf{x} = \mathbf{y}$ ).

So, keeping the signs in eq. (17) in mind, we define the Feynman propagator for the free Dirac field as

$$S_{\alpha\beta}^F(x - y) \stackrel{\text{def}}{=} \langle 0 | \mathbf{T} \hat{\Psi}_\alpha(x) \times \hat{\Psi}_\beta(y) | 0 \rangle. \quad (18)$$

Note Dirac indices  $\alpha, \beta$  of the spinor fields, so the propagator is a  $4 \times 4$  matrix.

To work out what this propagator looks like, let’s consider separate cases of  $x^0 > y^0$  and  $y^0 > x^0$ .

$$\text{For } x^0 > y^0, \quad S_{\alpha\beta}^F = + \langle 0 | \hat{\Psi}_\alpha(x) \times \hat{\Psi}_\beta(y) | 0 \rangle, \quad (19)$$

where the vacuum sandwich obtains from  $\hat{a}_{\mathbf{p},s}^\dagger \in \hat{\Psi}_\beta(y)$  and matching  $\hat{a}_{\mathbf{p},s} \in \hat{\Psi}_\alpha(x)$ ; all other combinations of creation and annihilation operators make for  $\langle 0 | (\text{op})(\text{op}) | 0 \rangle = 0$ .

Consequently,

$$\begin{aligned}
S_{\alpha\beta}^F(x-y) &= +\int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2E_p} \sum_s e^{-ip(x-y)} \times u_\alpha(\mathbf{p}, s) \bar{u}_\beta(\mathbf{p}, s) \\
&= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2E_p} e^{-ip(x-y)} \times (\not{p} + m)_{\alpha\beta} \\
&= (i \not{\partial} + m)_{\alpha\beta} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2E_p} e^{-ip(x-y)} \\
&= (i \not{\partial} + m)_{\alpha\beta} D(x-y).
\end{aligned} \tag{20}$$

On the other hand,

$$\text{For } x^0 < y^0, \quad S_{\alpha\beta}^F = -\langle 0 | \widehat{\Psi}_\beta(y) \times \widehat{\Psi}_\alpha(x) | 0 \rangle, \tag{21}$$

and this vacuum sandwich obtains from  $\hat{b}_{\mathbf{p},s}^\dagger \in \widehat{\Psi}_\alpha(x)$  and matching  $\hat{b}_{\mathbf{p},s} \in \widehat{\Psi}_\beta(y)$ . Consequently,

$$\begin{aligned}
S_{\alpha\beta}^F(x-y) &= -\int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2E_p} \sum_s e^{+ip(x-y)} \times v_\alpha(\mathbf{p}, s) \bar{v}_\beta(\mathbf{p}, s) \\
&= -\int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2E_p} e^{+ip(x-y)} \times (\not{p} - m)_{\alpha\beta} \\
&= +(i \not{\partial} + m)_{\alpha\beta} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2E_p} e^{+ip(x-y)} \\
&= +(i \not{\partial} + m)_{\alpha\beta} D(y-x).
\end{aligned} \tag{22}$$

Comparing formulae (20) and (22) to the scalar propagator

$$G^F(x-y) = \begin{cases} D(x-y) & \text{when } x^0 > y^0, \\ D(y-x) & \text{when } x^0 < y^0, \end{cases} \tag{23}$$

we immediately see that for both  $x^0 > y^0$  and  $x^0 < y^0$  we have

$$S_{\alpha\beta}^F(x-y) = +(i \not{\partial} + m)_{\alpha\beta} G^F(x-y). \tag{24}$$

The only subtlety here concerns equal times  $x^0 = y^0$ , but thanks to relativistic causality, the scalar propagator is not just continuous but analytic at equal times, and even at

$(x^0, \mathbf{x}) = (y^0, \mathbf{x})$  the first time derivative is continuous. Since the derivative operator in eq. (24) is first-order, this allows us to extend the formula (24) to all  $x - y$  without exceptions.

Finally, let's go to the momentum space. For the scalar propagator, we have

$$G^F(x - y) = \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i0} \times e^{-ip(x-y)}, \quad (25)$$

where the  $i0$  in the denominator means the limit of  $i\epsilon$  for  $\epsilon \rightarrow +0$ , and its purpose is to regularizes the integration over the poles along the mass shells  $p^0 = \pm\sqrt{\mathbf{p}^2 + m^2}$ . There are different ways to regularize the poles, but this particular regulator corresponds to the Feynman propagator.

In light of eqs. (24) and (25), the Dirac propagator becomes

$$\begin{aligned} S_{\alpha\beta}^F(x - y) &= +(i\not{\partial} + m)_{\alpha\beta} \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} \times e^{-ip(x-y)} \\ &= \int \frac{d^4 p}{(2\pi)^4} \frac{i(\not{p} + m)_{\alpha\beta}}{p^2 - m^2 + i0} \times e^{-ip(x-y)}. \end{aligned} \quad (26)$$

Furthermore, using

$$(\not{p} + m) \times (\not{p} - m) = \not{p}\not{p} - m^2 = (p^2 - m^2) \times \mathbf{1}_{4 \times 4}, \quad (27)$$

we may rewrite

$$\frac{(\not{p} + m)_{\alpha\beta}}{p^2 - m^2 + i0} = \frac{(\not{p} + m - i0)_{\alpha\beta}}{p^2 - (m - i0)^2} = \left( \frac{1}{\not{p} - m + i0} \right)_{\alpha\beta}, \quad (28)$$

so the Feynman propagator for the Dirac field becomes

$$S_{\alpha\beta}^F(x - y) = \int \frac{d^4 p}{(2\pi)^4} \left( \frac{i}{\not{p} - m + i0} \right)_{\alpha\beta} \times e^{-ip(x-y)}. \quad (29)$$

Naturally, this propagator is a Green's function of the Dirac equation,

$$(i\not{\partial} - m)_{\alpha\beta} S_{\beta\gamma}^F(x - y) = i\delta^{(4)}(x - y) \times \delta_{\alpha\gamma}. \quad (30)$$

Indeed,

$$\begin{aligned}(i \not{\partial} - m)_{\alpha\beta} S_{\beta\gamma}^F(x - y) &= (i \not{\partial} - m)_{\alpha\beta} \times (i \not{\partial} + m)_{\beta\gamma} G^F(x - y) \\ &= ((\partial^2 + m^2) \times \mathbf{1})_{\alpha\gamma} G^F(x - y) \\ &= \delta_{\alpha\gamma} \times (\partial^2 + m^2) G^F(x - y) \\ &= \delta_{\alpha\gamma} \times i\delta^{(4)}(x - y).\end{aligned}\tag{31}$$