On Perturbation Theory, Dyson Series, and Feynman Diagrams

The Interaction Picture of QM and the Dyson Series

Many quantum systems have Hamiltonians of the form $\hat{H} = \hat{H}_0 + \hat{V}$ where $\hat{H}_0$ is a *free Hamiltonian* with known spectrum — which is used to classify the states of the full theory — while $\hat{V}$ is a *perturbation* which causes transitions between the eigenstates of the $\hat{H}_0$. For example, in scattering theory

$$\hat{H}_0 = \frac{\hat{\mathbf{P}}^2_{\text{red}}}{2M_{\text{red}}}, \qquad \hat{V} = \text{potential } V(\hat{\mathbf{x}}_{\text{rel}}). \tag{1}$$

Similarly, for quantum scalar field $\hat{\Phi}$ with a $(\lambda/24)\hat{\Phi}^4$ self-interaction we have

$$\begin{aligned} \hat{H}_0 &= \int d^3\mathbf{x} \left(\tfrac{1}{2}\hat{\Pi}^2(\mathbf{x}) + \tfrac{1}{2}\left(\nabla\hat{\Phi}(\mathbf{x})\right)^2 + \frac{m^2}{2}\,\hat{\Phi}^2\right) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\,E_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} + \text{const} \\ \text{while } \hat{V} &= \int d^3\mathbf{x}\,\frac{\lambda}{24}\,\hat{\Phi}^4(\mathbf{x}). \end{aligned} \tag{2}$$

To study the transitions (scattering, making new particles, decays, *etc.*) caused by $\hat{V}$ we want to use a fixed basis of $\hat{H}_0$ eigenstates, but we want to keep the transitions separate from wave-packet spreading and other effects due to Schrödinger phases e^{-iEt} of the $\hat{H}_0$ itself. The picture of QM which separates these effects is the *interaction picture.*

In the Schrödinger picture, the operators are time-independent while the quantum states evolve with time as $|\psi\rangle_S(t) = e^{-i\hat{H}t}\,|\psi\rangle(0)$. In the Heisenberg picture it's the other way around — the quantum states are time independent while the operators evolve with time — and the two pictures are related by a time-dependent unitary operator $e^{i\hat{H}t}$,

$$|\Psi\rangle_H(t) = e^{+i\hat{H}t}\,|\Psi\rangle_S(t) \equiv |\Psi\rangle_S(0)\ \forall t, \qquad \hat{A}_H(t) = e^{+i\hat{H}t}\hat{A}_S e^{-i\hat{H}t}. \tag{3}$$

The interaction picture has a similar relation to the Schrödinger's, but using the $\exp(i\hat{H}_0t)$ instead of the $\exp(i\hat{H}t)$,

$$\begin{aligned} \hat{A}_I(t) &= e^{+i\hat{H}_0t}\hat{A}_S e^{-\hat{H}_0t}, \\ |\psi\rangle_I(t) &= e^{+i\hat{H}_0t}\,|\psi\rangle_S(t) = e^{+i\hat{H}_0t}e^{-i\hat{H}t}\,|\psi\rangle_H \neq \text{const}. \end{aligned} \tag{4}$$

In the interaction picture, quantum fields depend on time as if they were free fields, for example

$$\hat{\Phi}_I(\mathbf{x},t) \;=\; \int \frac{d^3\mathbf{p}}{16\pi^3 E_\mathbf{p}} \left(e^{-ipx}\hat{a}_p \,+\, e^{+ipx}\hat{a}_p^\dagger \right)^{p^0=+E_\mathbf{p}}, \tag{5}$$

regardless of the interactions. This is different form the Heisenberg picture where non-free fields depend on time in a much more complicated way.

The time-dependence of quantum states in the interaction picture is governed by the perturbation $\hat{V}$ according to Schrödinger-like equation

$$i\frac{d}{dt}\,|\psi\rangle_I(t) \;=\; \hat{V}_I(t)\,|\Psi\rangle_I(t). \tag{6}$$

The problem with this equation is that the $\hat{V}_I$ operator here is itself in the interaction picture, so it depends on time as $\hat{V}_I(t) = e^{+i\hat{H}_0 t}\hat{V}_S e^{-\hat{H}_0 t}$. Consequently, the *evolution operator* for the interaction picture

$$\hat{U}_I(t,t_0): \quad |\psi\rangle_I(t) \;=\; \hat{U}_I(t,t_0)\,|\psi\rangle_I(t_0) \tag{7}$$

is much more complicated than simply $e^{-i\hat{V}(t-t_0)}$. Specifically, $\hat{U}_I(t,t_0)$ satisfies

$$i\frac{\partial}{\partial t}\hat{U}_I(t,t_0) \;=\; \hat{V}_I(t)\,\hat{U}_I(t,t_0), \qquad \hat{U}_I(t=t_0) \;=\; 1, \tag{8}$$

and the formal solution to these equations is the **Dyson series**

$$\begin{aligned}\hat{U}_I(t,t_0) \;&=\; 1 \;-\; i\int_{t_0}^{t} dt_1\,\hat{V}_I(t_1) \;-\; \int_{t_0}^{t} dt_2\,\hat{V}_I(t_2)\int_{t_0}^{t_2} dt_1\,\hat{V}_I(t_1)\\ &\qquad +\; i\int_{t_0}^{t} dt_3\,\hat{V}_I(t_3)\int_{t_0}^{t_3} dt_2\,\hat{V}_I(t_2)\int_{t_0}^{t_2} dt_1\,\hat{V}_I(t_1) \;+\; \cdots\\ &=\; 1 \;+\; \sum_{n=1}^{\infty}(-i)^n \underset{t_0<t_1<\cdots<t_n<t}{\int\cdots\int} dt_n\cdots dt_1\,\hat{V}_I(t_n)\cdots\hat{V}_I(t_1).\end{aligned} \tag{9}$$

Note the time ordering of operators $\hat{V}_I(t_n)\cdots\hat{V}_I(t_1)$ in each term.

The Dyson series obviously obeys the initial condition $\hat{U}_I(t=t_0)=1$. To see that is also satisfies the Schrödinger-like differential equation (8), note that in each term of the series, the only thing which depends on t is the upper limit of the leftmost dt_n integral. Thus, taking $\partial/\partial t$ of the term amounts to skipping that integral and letting $t_n=t$,

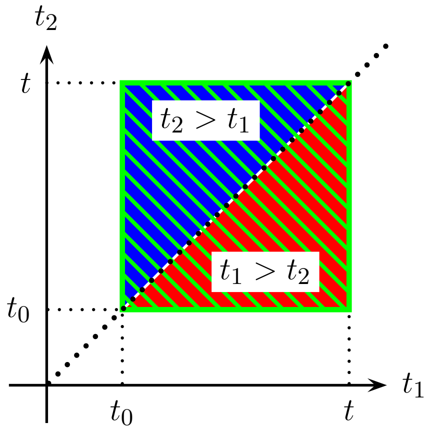
$$
\begin{aligned}
i\frac{\partial}{\partial t}\left((-i)^n\int\limits_{t_0}^{t}dt_n\,\hat{V}_I(t_n)\int\limits_{t_0}^{t_n}dt_{n-1}\,\hat{V}_I(t_{n-1})\;\cdots\int\limits_{t_0}^{t_2}dt_1\,\hat{V}_I(t_1)\right) &= \\
= \hat{V}_I(t_n=t)\times\left((-i)^{n-1}\int\limits_{t_0}^{t_n=t}dt_{n-1}\,\hat{V}_I(t_{n-1})\;\cdots\int\limits_{t_0}^{t_2}dt_1\,\hat{V}_I(t_1)\right)&.
\end{aligned}
\tag{10}
$$

In other words, $i\partial/\partial t$ of the n^{th} term is $\hat{V}_I(t)\times$ the $(n-1)^{\text{st}}$ term. Consequently, the whole series satisfies eq. (8).

Thanks to the time ordering of the $\hat{V}_I(t)$ in each term of the Dyson series — the earliest operator being rightmost so it acts first, the second earliest being second from the right, *etc.*, *etc.*, until the latest operator stands to the left of everything so it acts last — we may rewrite the integrals in a more compact form using the time-orderer $\mathbf{T}$. Earlier in class, I have defined $\mathbf{T}$ of an operator product, but now I would like to extend this by linearity to any sum of operators products. Similarly, we may time-order integrals of operator products and hence products of integrals such as

$$
\begin{aligned}
\mathbf{T}\left(\int\limits_{t_0}^{t}dt'\,\hat{V}_I(t')\right)^2 \;&\stackrel{\text{def}}{=}\; \mathbf{T}\iint\limits_{t_0}^{t}dt_1\,dt_2\,\hat{V}_I(t_1)\hat{V}_I(t_2) \;\stackrel{\text{def}}{=}\; \iint\limits_{t_0}^{t}dt_1\,dt_2\,\mathbf{T}\hat{V}_I(t_1)\hat{V}_I(t_2) \\
&= \iint\limits_{t_0<t_1<t_2<t}dt_1\,dt_2\,\hat{V}_I(t_2)\hat{V}_I(t_1) \;+\; \iint\limits_{t_0<t_2<t_1<t}dt_1\,dt_2\,\hat{V}_I(t_1)\hat{V}_I(t_2)
\end{aligned}
\tag{11}
$$

where the domain of each $dt_1\,dt_2$ integral is color-coded on the diagram below:



(12)

Integral over the blue triangle $t_0 < t_1 < t_2 < t$ is what appears in the Dyson series. But there is $t_1 \leftrightarrow t_2$ symmetry between the blue and red triangles, and the corresponding integrals on the bottom line of eq. (11) are equal to each other. Hence the triangular integral in the Dyson series may be written in a more compact form as

$$\iint\limits_{t_0<t_1<t_2<t} dt_1\, dt_2\, \hat{V}_I(t_2)\hat{V}_I(t_1) \;=\; \frac{1}{2}\,\mathbf{T}\left(\int\limits_{t_0}^{t} dt'\, \hat{V}_1(t')\right)^2 . \tag{13}$$

Similar procedure applies to the higher-order terms in the Dyson series. The n^{th} order term is an integral over a simplex $t_0 < t_1 < t_2 < \cdots < t_n < t$ in the $(t_1, \ldots, t_n)$ space. A hypercube $t_0 < t_1, \ldots, t_n < t$ contains $n!$ such simplexes, and after time-ordering the $\hat{V}$ operators, integrals over all simplexes become equal by permutation symmetry. Thus,

$$\begin{aligned}\int\!\cdots\!\int\limits_{t_0<t_1<\cdots<t_n<t} dt_n \cdots dt_1\, \hat{V}_I(t_n)\cdots\hat{V}_I(t_1) \;&=\; \frac{1}{n!}\int\limits_{t_0}^{t}\!\cdots\!\int dt_n \cdots dt_1\, \mathbf{T}\hat{V}_I(t_n)\cdots\hat{V}_I(t_1) \\ &=\; \frac{1}{n!}\,\mathbf{T}\left(\int\limits_{t_0}^{t} dt'\, \hat{V}_1(t')\right)^n .\end{aligned} \tag{14}$$

Altogether, the Dyson series becomes a *time-ordered exponential*

$$\hat{U}_I(t,t_0) \;=\; 1 \;+\sum_{n=1}^{\infty}\frac{(-i)^n}{n!}\mathbf{T}\left(\int\limits_{t_0}^{t} dt'\, \hat{V}_I(t')\right)^n \;\equiv\; \mathbf{T}\text{-exp}\left(-i\int\limits_{t_0}^{t} dt'\, \hat{V}_I(t')\right). \tag{15}$$

Of particular interest is the evolution operator from the distant past to the distant future,

$$\hat{S} \stackrel{\text{def}}{=} \hat{U}_I(+\infty,-\infty) = \mathbf{T}\text{-exp}\left(-i\int_{-\infty}^{+\infty} dt'\,\hat{V}_I(t')\right). \tag{16}$$

This operator is properly called 'the scattering operator' or 'the S–operator', but everybody calls it 'the S–matrix'. In the scalar field theory where

$$\hat{V}_I(t) = \frac{\lambda}{24}\int d^3\mathbf{x}\,\hat{\Phi}_I^4(\mathbf{x},t), \tag{17}$$

the S–matrix has a Lorentz-invariant form

$$\hat{S} = \mathbf{T}\text{-exp}\left(\frac{-i\lambda}{24}\int\limits_{\substack{\text{whole}\\ \text{spacetime}}} d^4x\,\hat{\Phi}_I^4(x)\right). \tag{18}$$

Note that $\hat{\Phi}_I(x)$ here is the *free* scalar field as in eq. (5). Similar Lorentz-invariant expressions exist for other quantum field theories. For example, in QED

$$\hat{S} = \mathbf{T}\text{-exp}\left(+ie\int\limits_{\substack{\text{whole}\\ \text{spacetime}}} d^4x\,\hat{A}_I^\mu(x)\hat{\overline{\Psi}}_I(x)\gamma_\mu\hat{\Psi}_I(x)\right) \tag{19}$$

where both the EM field $\hat{A}_I^\mu(x)$ and the electron field $\Psi_I(x)$ are in the interaction picture so they evolve with time as free fields.

S–Matrix Elements

The S-matrix *elements* $\langle\text{out}|\,\hat{S}\,|\text{in}\rangle$ should be evaluated between *physical* incoming and outgoing 2-particle (or n-particle) states. In the potential scattering theory, the potential $\hat{V}$ becomes irrelevant when all particles are far away from each other, so the asymptotic states $|\text{in}\rangle$ and $\langle\text{out}|$ are eigenstates of the free Hamiltonian $\hat{H}_0$.⋆ But in the quantum field theory,

⋆ Strictly speaking, the asymptotic states are wave packets moving in space due to $\hat{H}_0$. But once we take the limit of infinitely large distance between wave packets for different particles in the asymptotic past and the asymptotic future, we may then make the wave packets themselves very thick in space but having very narrow ranges of momenta and energies. In this limit, the asymptotic states become eigenvalues of the free Hamiltonian $\hat{H}_0$.

the perturbation $\hat{V}$ is always present, and the asymptotic n–particle states $|k_1,\ldots,k_n\rangle$ of the interacting field theory are quite different from the free theory's n–particle states $\hat{a}^\dagger_{\mathbf{k}_n}\cdots\hat{a}^\dagger_{\mathbf{k}_1}|0\rangle$. Even the physical vacuum state $|\Omega\rangle$ is different from the free theory's vacuum $|0\rangle$.

In the spring semester, we shall learn how to obtain the physical S–matrix elements $\langle\mathrm{out}|\,\hat{S}\,|\mathrm{in}\rangle$ from the correlation functions of fully-interacting quantum fields,

$$\mathcal{F}_n(x_1,\ldots,x_n) \;=\; \langle\Omega|\,\mathbf{T}\hat{\Phi}_H(x_1)\cdots\hat{\Phi}_H(x_n)\,|\Omega\rangle \tag{20}$$

— and also how to calculate those correlation functions in perturbation theory. But this semester, I am taking a short-cut: I will explain the perturbation theory for the naive S-matrix elements

$$\Big\langle\mathrm{free}:k'_1,k'_2\ldots\Big|\,\hat{S}\,\Big|\,\mathrm{free}:k_1,k_2,\ldots\Big\rangle \;=\; \langle 0|\cdots\hat{a}_{k'_2}\hat{a}_{k'_1}\,\hat{S}\,\hat{a}^\dagger_{k_1}\hat{a}^\dagger_{k_2}\cdots|0\rangle \tag{21}$$

between the 2-particle (or n-particle) states of the free Hamiltonian $\hat{H}_0$, and then I will tell you — without proof — how to modify the perturbative expansion to obtain the S-matrix elements between the physical asymptotic states. In terms of the Feynman diagrams (which are explained below), the modification amounts to simply skipping some diagrams that contribute to the naive S-matrix elements (21) but cancel out from the physical matrix elements

$$\Big\langle\mathrm{out}:k'_1,k'_2\ldots\Big|\,\hat{S}\,\Big|\,\mathrm{in}:k_1,k_2,\ldots\Big\rangle \tag{22}$$

Vacuum Sandwiches and Feynman Diagrams

The perturbation theory starts by expanding the $\hat{S}$ operator (18) — and hence its matrix elements (21) — into a power series in the coupling constant λ:

$$\begin{aligned}
\hat{S} \;&=\; \sum_{n=0}^{\infty}\frac{(-i\lambda)^n}{(4!)^n\,n!}\int\! d^4z_1\,\cdots\int\! d^4z_n\,\mathbf{T}\,\hat{\Phi}^4_I(z_1)\cdots\hat{\Phi}^4_I(z_n)\\
&\Downarrow\\
\Big\langle\mathrm{free}:k'_1,k'_2\ldots\Big|\,\hat{S}\,|\mathrm{free}:k_1,k_2,\ldots\big\rangle \;&=\\
=\;\sum_{n=0}^{\infty}\frac{(-i\lambda)^n}{(4!)^n\,n!}\int\! d^4z_1\,\cdots\int\! d^4z_n\;&\Big\langle\mathrm{free}:k'_1,k'_2\ldots\Big|\,\mathbf{T}\hat{\Phi}^4_I(z_1)\cdots\hat{\Phi}^4_I(z_n)\,\Big|\,\mathrm{free}:k_1,k_2,\ldots\Big\rangle\,.
\end{aligned}\tag{23}$$

Our first task is to learn how to calculate the matrix elements inside the integrals on the RHS, or more generally, how to calculate the matrix elements of time-ordered products of free fields $\hat{\Phi}_I(x)$ between eigenstates of the free Hamiltonian. As a warm-up exercise, let's start with the 'vacuum sandwiches'

$$\langle 0|\,\mathbf{T}\,\hat{\Phi}_I(x_1)\ldots\hat{\Phi}_I(x_n)\,|0\rangle$$

of $n = 2, 4, 6, \ldots$ free fields.

For $n = 2$, the field product $\hat{\Phi}_I(x)\hat{\Phi}_I(y)$ contains products of two creation or annihilation operators. As discussed in class, the only products that contribute to the vacuum sandwich $\langle 0|\,\hat{\Phi}_I(x)\hat{\Phi}_I(y)\,|0\rangle$ are $\hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}}$ for the same $\mathbf{p}$, hence $\langle 0|\,\hat{\Phi}_I(x)\hat{\Phi}_I(y)\,|0\rangle = D(x-y)$. After time-ordering of the two fields, this gives us the Feynman propagator

$$\langle 0|\,\mathbf{T}\hat{\Phi}_I(x)\hat{\Phi}_I(y)\,|0\rangle \;=\; G_F(x-y). \tag{24}$$

For $n = 4$, the field product $\hat{\Phi}_I(x)\hat{\Phi}_I(y)\hat{\Phi}_I(z)\hat{\Phi}_I(w)$ contains products of four creation or annihilation operators. There are $2^4 = 16$ types of such products, but only 3 products of two $\hat{a}$'s and two $\hat{a}^\dagger$'s with matching momenta and in the right order contribute to the vacuum sandwich $\langle 0|\,\hat{\Phi}_I(x)\hat{\Phi}_I(y)\hat{\Phi}_I(z)\hat{\Phi}_I(w)\,|0\rangle$. Specifically,

$$\begin{aligned}
&\langle 0|\,\big(\hat{a}_{\mathbf{p}} \in \hat{\Phi}_I(x)\big)\big(\hat{a}^\dagger_{\mathbf{p}} \in \hat{\Phi}_I(y)\big)\big(\hat{a}_{\mathbf{q}} \in \hat{\Phi}_I(z)\big)\big(\hat{a}^\dagger_{\mathbf{q}} \in \hat{\Phi}_I(w)\big)\,|0\rangle \;\to\; D(x-y)\times D(z-w),\\
&\langle 0|\,\big(\hat{a}_{\mathbf{p}} \in \hat{\Phi}_I(x)\big)\big(\hat{a}_{\mathbf{q}} \in \hat{\Phi}_I(y)\big)\big(\hat{a}^\dagger_{\mathbf{q}} \in \hat{\Phi}_I(z)\big)\big(\hat{a}^\dagger_{\mathbf{p}} \in \hat{\Phi}_I(w)\big)\,|0\rangle \;\to\; D(x-w)\times D(y-z),\\
&\langle 0|\,\big(\hat{a}_{\mathbf{p}} \in \hat{\Phi}_I(x)\big)\big(\hat{a}_{\mathbf{q}} \in \hat{\Phi}_I(y)\big)\big(\hat{a}^\dagger_{\mathbf{p}} \in \hat{\Phi}_I(z)\big)\big(\hat{a}^\dagger_{\mathbf{q}} \in \hat{\Phi}_I(w)\big)\,|0\rangle \;\to\; D(x-z)\times D(y-w),
\end{aligned} \tag{25}$$

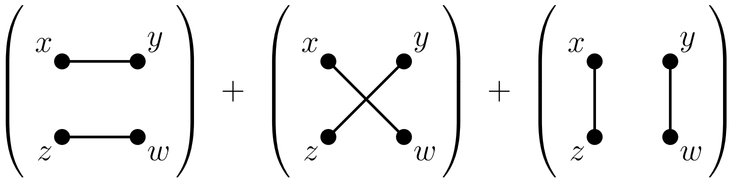
hence

$$\begin{aligned}
&\langle 0|\,\hat{\Phi}_I(x)\hat{\Phi}_I(y)\hat{\Phi}_I(z)\hat{\Phi}_I(w)\,|0\rangle \;= \\
&\qquad = \; D(x-y)\times D(z-w) \;+\; D(x-w)\times D(y-z) \;+\; D(x-z)\times D(y-w).
\end{aligned} \tag{26}$$

After time ordering of the 4 fields, the D-functions become Feynman propagators, thus

$$\begin{aligned}
&\langle 0|\,\mathbf{T}\hat{\Phi}_I(x)\hat{\Phi}_I(y)\hat{\Phi}_I(z)\hat{\Phi}_I(w)\,|0\rangle \;= \\
&\quad = \; G_F(x-y)\times G_F(z-w) \;+\; G_F(x-w)\times G_F(y-z) \;+\; G_F(x-z)\times G_F(y-w).
\end{aligned} \tag{27}$$

Diagrammatically, we may summarize this expression as



(28)

where each line stands for the Feynman propagator $G_F(\text{one\_end} - \text{other\_end})$ and each diagram in () is the product of 2 propagators for its 2 lines.

In the same spirit, for $n = 6$ fields we would have diagrams with 3 lines connecting 6 points, for $n = 8$ the diagrams would have 4 lines connecting 8 points, *etc.*, *etc.* Each diagram corresponds to a different arrangement of n points into $n/2$ pairs, thus

$$\langle 0|\, \mathbf{T}\hat{\Phi}_I(x_1)\cdots\hat{\Phi}_I(x_n)\,|0\rangle \;=\; \sum_{\text{pairings}} \left(\prod_{\text{pairs}} G_F\big(x^{\text{first}}_{\text{in pair}} \,-\, x^{\text{second}}_{\text{in pair}}\big)\right). \tag{29}$$

Note that the number of pairings one has to sum over — *i.e.*, the number of diagrams — grows rather rapidly with the number n of fields in the vacuum sandwich:

$$\#\text{pairings} \;=\; \frac{n!}{2^{n/2}\,(n/2)!} \;=\; \begin{cases} 1 \text{ for } n=2, \\ 3 \text{ for } n=4, \\ 15 \text{ for } n=6, \\ 105 \text{ for } n=8, \\ 945 \text{ for } n=10, \\ \dots\dots\dots\dots\dots \end{cases} \tag{30}$$

For the non-vacuum matrix elements

$$\begin{aligned} &\Big\langle \text{free} : k'_1, k'_2 \ldots \Big|\, \mathbf{T}\hat{\Phi}_I(z_1)\cdots\hat{\Phi}_I(z_n)\,\Big|\, \text{free} : k_1, k_2, \ldots\Big\rangle \;= \\ &\qquad\qquad = \;\langle 0|\cdots \hat{a}_{\mathbf{k}'_2}\hat{a}_{\mathbf{k}'_1}\, \mathbf{T}\hat{\Phi}_I(z_1)\cdots\hat{\Phi}_I(z_n)\,\hat{a}^\dagger_{\mathbf{k}_1}\hat{a}^\dagger_{\mathbf{k}_2}\cdots|0\rangle \end{aligned} \tag{31}$$

we have even more pairings. Besides pairing creation and annihilation operators contained in the n free fields, we may pair a creation operator $\hat{a}^\dagger_{\mathbf{k}}$ for one of the incoming particles with a matching $\hat{a}_{\mathbf{k}}$ in one of the fields, or an annihilation operator $\hat{a}_{\mathbf{k}'}$ for one of the outgoing particles with a matching $\hat{a}^\dagger_{\mathbf{k}'}$ in one of the fields, or even a creation operator for an incoming particle with an annihilation operator for an outgoing particles (if they happen to have equal momenta). Diagrammatically, such pairs are represented by different types of lines:

$$
\begin{array}{rccl}
\text{internal lines} & \overset{x}{\bullet}\!\!\frac{}{}\!\!\overset{y}{\bullet} & \text{for } \hat{\Phi}_I(x)\hat{\Phi}_I(y) \text{ pairs} & \to\ G_F(x-y), \\
\text{incoming lines} & \overset{k}{}\!\!\frac{}{}\!\!\overset{x}{\bullet} & \text{for } \hat{\Phi}_I(x)\,\hat{a}^\dagger_{\mathbf{k}} \text{ pairs} & \to\ e^{-ikx}, \\
\text{outgoing lines} & \overset{x}{\bullet}\!\!\frac{}{}\!\!\overset{k'}{} & \text{for } \hat{a}_{\mathbf{k}'}\,\hat{\Phi}_I(x) \text{ pairs} & \to\ e^{+ik'x}, \\
\text{non-stop lines} & \overset{k}{}\!\!\frac{}{}\!\!\overset{k'}{} & \text{for } \hat{a}_{\mathbf{k}'}\hat{a}^\dagger_{\mathbf{k}} \text{ pairs} & \to\ 2E_{\mathbf{k}}(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}'),
\end{array}
\tag{32}
$$

where the exponential factors e^{-ikx} and $e^{+ik'x}$ for the incoming and outgoing lines follow from the expansion of a free field into creation and annihilation operators,

$$
\hat{\Phi}_I(x) \;=\; \int \frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\left(e^{-ipx}\hat{a}_{\mathbf{p}} \,+\, e^{+ipx}\hat{a}^\dagger_{\mathbf{p}}\right). \tag{33}
$$

Thus, to calculate a matrix element like (31), we draw all possible diagrams where n vertices at $x_1,\ldots,x_n$ are connected to each other and to the incoming and outgoing particles by various lines, multiply the line factors from the right column of the table (32), and then total up such products for all the diagrams,

$$
\langle 0|\cdots\hat{a}_{\mathbf{k}'_2}\hat{a}_{\mathbf{k}'_1}\,\mathbf{T}\hat{\Phi}_I(z_1)\cdots\hat{\Phi}_I(z_n)\,\hat{a}^\dagger_{\mathbf{k}_1}\hat{a}^\dagger_{\mathbf{k}_2}\cdots|0\rangle \;=\; \sum_{\text{diagrams}}\,\prod_{\text{lines}}\text{line factors from table (32)}. \tag{34}
$$

Thus far, I have put each free field $\hat{\Phi}_I(z_i)$ at its own location z_i. But in the perturbative expansion (23), there are 4 fields at each z_i. When all 4 of them are paired up with other fields or incoming / outgoing particles, we end up with 4 lines connected to the same vertex at z_i. In other words, the vertices in the diagrams for the perturbative expansion have valence $=4$.

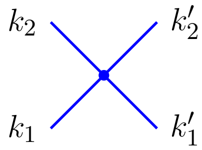
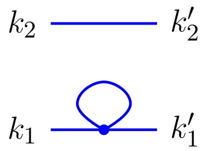
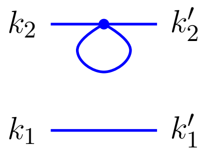
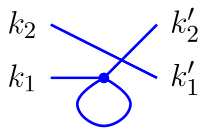
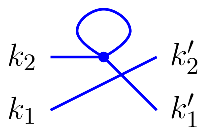
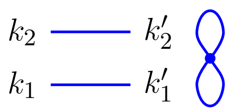
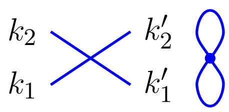
Moreover, for evaluating a diagram it does not matter which of the four $\hat{\Phi}_I(z_i)$ is paired up with which other field or incoming / outgoing particle, all that matter is the pattern of connections. Consequently, each diagram describes many nominally distinct but physically equivalent pairings, so the number of in-equivalent diagrams contributing to a matrix element

$$\langle 0| \cdots \hat{a}_{\mathbf{k}_2'} \hat{a}_{\mathbf{k}_1'} \, \mathbf{T} \hat{\Phi}_I^4(z_1) \cdots \hat{\Phi}_I^4(z_n) \, \hat{a}^\dagger_{\mathbf{k}_1} \hat{a}^\dagger_{\mathbf{k}_2} \cdots |0\rangle \tag{35}$$

is much smaller than the total number of pairings (30) of $N = 4n + \#\text{incoming} + \#\text{outgoing}$ operators. For example, for the matrix element

$$\langle 0| \, \hat{a}_{\mathbf{k}_2'} \hat{a}_{\mathbf{k}_1'} \, \hat{\Phi}_I^4(z) \, \hat{a}^\dagger_{\mathbf{k}_1} \hat{a}^\dagger_{\mathbf{k}_2} \, |0\rangle \tag{36}$$

there are 8 operators $\hat{a}\hat{a}\hat{\Phi}\hat{\Phi}\hat{\Phi}\hat{\Phi}\hat{a}^\dagger\hat{a}^\dagger$ and hence 105 pairings, although without the un-physical $\hat{a}^\dagger\hat{a}^\dagger$ or $\hat{a}\hat{a}$ pairs this number is reduced to 78. But all these 78 pairings make for only 7 different diagrams shown the following table

	24 pairings	1 symmetries
	12 pairings	2 symmetries
	12 pairings	2 symmetries
	12 pairings	2 symmetries
	12 pairings	2 symmetries
	3 pairings	8 symmetries
	3 pairings	8 symmetries

(37)

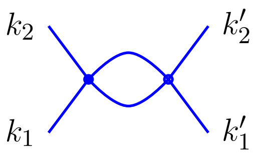
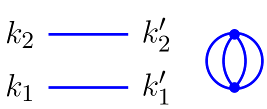
For each diagram, the table lists the number of pairings — which multiplies the overall contribution of the diagram to the matrix element — and the number of graphical symmetries due to permutations of similar lines (connected to same vertices) or of two ends of the same line that happen to be connected to the same vertex. Note that for each diagram

$$\#\text{pairings} \times \#\text{symmetries} \;=\; 24 \;=\; 4!. \tag{38}$$

At the next order of perturbation theory, the matrix element

$$\langle 0|\, \hat{a}_{\mathbf{k}'_2} \hat{a}_{\mathbf{k}'_1}\, \mathbf{T}\hat{\Phi}^4_I(z_1)\hat{\Phi}^4_I(z_2)\, \hat{a}^\dagger_{\mathbf{k}_1} \hat{a}^\dagger_{\mathbf{k}_2}\, |0\rangle \tag{39}$$

involves 8610 pairings but only 33 distinct diagrams. I am not going to list all 33 diagrams here, but the following table shows two examples:

	576 pairings	2 symmetries
	24 pairings	48 symmetries

(40)

This time, the number of pairings and the number of graphic symmetries for each diagram — including the 31 diagrams I do not list — are related as

$$\#\text{pairings} \times \#\text{symmetries} \;=\; 1152 \;=\; 2 \times (4!)^2. \tag{41}$$

Note: the symmetries include permutations of similar vertices since at the end of the day I am going to integrate both d^4z_1 and d^4z_2 over the whole spacetime. However, the external lines (incoming or outgoing) should not be permuted since the incoming or outgoing particles may be distinguished by their momenta.

At higher orders n of perturbation theory, we generally have

$$\forall \text{diagram}, \quad \#\text{pairings} \;=\; \frac{n! \times (4!)^n}{\#\text{symmetries}}\,. \tag{42}$$

Combining this combinatorial factor for a diagram with the overall coefficient

$$\frac{(-i\lambda)^n}{n!\,(4!)^n} \tag{43}$$

for the n^{th} order term in the perturbative expansion (23), we find the product of lines for each diagram should be multiplied by

$$\frac{(-i\lambda)^n}{\#\text{symmetries}}\,. \tag{44}$$

Attributing the numerator here to the n vertices of the diagram, we arrive at the following **Feynman rules**: To calculate the matrix element

$$\left\langle \text{free} : k_1', k_2' \ldots \middle| \,\hat{S}\, \middle| \text{free} : k_1, k_2, \ldots \right\rangle \tag{45}$$

in the $\lambda\Phi^4$ theory to order $N_{\max}$ of perturbation theory,

1. Draw all diagrams with $n = 1, 2, \ldots, N_{\max}$ vertices labeled z_1 through z_n and any pattern of lines obeying the following requirements: (1) an incoming (or non-stop) line for each incoming particle, (2) an outgoing (or non-stop) line for each outgoing particle, (3) each vertex is connected to 4 lines (internal, incoming, or outgoing).
   Make sure to draw *all* diagrams obeying these conditions.

2. For each diagram, multiply together the following factors:

   - $(-i\lambda)$ for each vertex $z_1, \ldots, z_n$;
   - ⋆ $G_F(z_i - z_j)$ for an internal line connecting vertices z_i and z_j.
   - ⋆ $\exp(-ikz)$ for an incoming line connecting an incoming particle with momentum k^μ to the vertex z;
   - ⋆ $\exp(+ik'z)$ for an outgoing line connecting vertex z to an outgoing particle with momentum k'^μ;

⋆ $\langle k'|k\rangle = 2E_k(2\pi)^3\delta^{(3)}(\mathbf{k'}-\mathbf{k})$ for a non-stop line;

⋄ the symmetry factor 1/#symmetries of the diagram.

3. For each diagram, $\int d^4z_1 \cdots \int d^4z_n$ over the locations of all the vertices.

4. Total up the diagrams.

Vacuum Bubbles

The above Feynman rules give us the S-matrix elements between 2-particle or m-particle eigenstates of the free Hamiltonian $\hat{H}_0$. The S-matrix elements between the physical asymptotic states are related to these as

$$\begin{aligned}\Big\langle \text{out}: k_1', k_2' \ldots \Big| \hat{S} \Big| \text{in}: k_1, k_2, \ldots \Big\rangle \;&=\; \Big\langle \text{free}: k_1', k_2' \ldots \Big| \hat{S} \Big| \text{free}: k_1, k_2, \ldots \Big\rangle \times \\ &\times C_{\text{vac}} \times \prod_{\substack{\text{incoming}\\ \text{particles}}} F(k_i) \times \prod_{\substack{\text{outgoing}\\ \text{particles}}} F(k_i')\end{aligned} \tag{46}$$

where C_{vac} is a common overall factor for all matrix elements while the $F(k)$ factors depends on the momenta of incoming and outgoing particles, but each factor depends on only one particle's momentum. All factors on the right hand side of eq. (46) are badly divergent and need to be regularized, so eq. (46) is just a short-hand for a more accurate formula

$$\begin{aligned}\Big\langle \text{out}: k_1', k_2' \ldots \Big| \hat{S} \Big| \text{in}: k_1, k_2, \ldots \Big\rangle \;&=\; \lim_{\text{reg}\to\text{away}} \Big\langle \text{free}: k_1', k_2' \ldots \Big| \hat{S} \Big| \text{free}: k_1, k_2, \ldots \Big\rangle_{\text{reg}} \times \\ &\times C_{\text{vac}}(\text{reg}) \times \prod_{\substack{\text{incoming or}\\ \text{outgoing}\\ \text{particles}}} F(k_i \text{ or } k_i', \text{reg}).\end{aligned} \tag{47}$$

In the Spring semester, I shall explain where this formula is coming from and how to regulate various divergences. For the moment, let's ignore the divergences and focus on formal sums of Feynman diagrams.

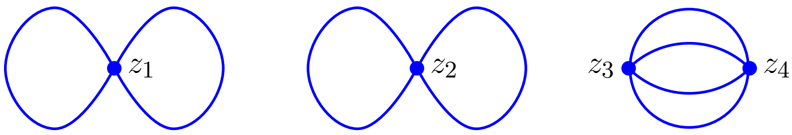
The C_{vac} factor in eq. (46) appears in all S-matrix elements, even in the vacuum-to-vacuum element

$$\langle\Omega|\, \hat{S}\, |\Omega\rangle \;=\; \langle 0|\, \hat{S}\, |0\rangle \;\times C_{\text{vac}}\,. \tag{48}$$

Since in any theory with stable vacuum state $\langle\Omega|\,\hat{S}\,|\Omega\rangle = 1$, it follows that

$$\frac{1}{C_{\text{vac}}} \;=\; \langle 0|\,\hat{S}\,|0\rangle \;=\; \sum \text{vacuum diagrams} \tag{49}$$

— *i.e.*, Feynman diagrams without any incoming, outgoing, or non-stop lines. Some vacuum diagrams are connected (any two vertices are connected to each other through some sequence of lines) while others comprise two or more disconnected bubbles (each bubble is internally connected but there are no connections between the bubbles), for example



(50)

When we evaluate a disconnected diagram, all factors that depend on z_i locations from one bubble do not depend on the z_j from any other bubble. Consequently, the $\int d^4z_i$ integrals over vertices of different bubbles are completely independent form each other, so the whole calculation factorizes to the product of bubbles. That is, we may evaluate each bubble as a standalone diagram, and then simply multiply the bubbles together,

$$\text{for } \mathcal{D} \;=\; \mathcal{B}_1 \oplus \mathcal{B}_2 \oplus \cdots \oplus \mathcal{B}_m\,, \quad \text{eval}(\mathcal{D}) \;=\; \text{eval}(\mathcal{B}_1) \times \text{eval}(\mathcal{B}_2) \times \cdots \times \text{eval}(\mathcal{B}_m). \tag{51}$$

However, one has to be careful of the symmetry factors. Evaluating each bubble as a standalone diagram accounts for symmetries involving permutations of similar vertices, lines, or line ends *within the same bubble only*, but when a disconnected diagram like (50) has two or more similar bubbles, we get additional symmetries of permuting the whole bubbles. Accounting for those extra symmetries gives us

$$\text{eval}\bigl(\mathcal{D} \;=\; \mathcal{B}_1 \oplus \mathcal{B}_2 \oplus \cdots \oplus \mathcal{B}_m\bigr) \;=\; \frac{\text{eval}(\mathcal{B}_1) \times \text{eval}(\mathcal{B}_2) \times \cdots \times \text{eval}(\mathcal{B}_m)}{\#\text{permutations of similar bubbles}}\,. \tag{52}$$

Using this formula, we may re-organize the formal sum over vacuum diagrams in eq. (49) into a power series in connected vacuum bubbles $\mathcal{B}$:

$$\frac{1}{C_{\text{vac}}} \;=\; 1 \;+\; \sum_{\mathcal{B}}^{\text{bubbles}} \text{eval}(\mathcal{B}) \;+\; \sum_{m=2}^{\infty} \sum_{\mathcal{B}_1,\ldots\mathcal{B}_m}^{\text{bubbles}} \frac{\text{eval}(\mathcal{B}_1) \times \cdots \times \text{eval}(\mathcal{B}_m)}{\#\text{bubble permutations}}\,. \tag{53}$$

By the way, it's often convenient to re-express this series as exponential of the sum over connected bubbles only. Indeed, let $\mathcal{B}_\alpha$ run over different kinds of connected vacuum bubbles, then a generic N-bubble vacuum diagram has form

$$\mathcal{D} \;=\; \bigoplus_{\alpha} n_\alpha \times \mathcal{B}_\alpha \tag{54}$$

for some non-negative integers n_α with $\sum_\alpha n_\alpha = N$. Consequently,

$$\operatorname{eval}(\mathcal{D}) \;=\; \prod_\alpha \frac{\big(\operatorname{eval}(\mathcal{B}_\alpha)\big)^{n_\alpha}}{n_\alpha!} \tag{55}$$

and therefore

$$\begin{aligned}
\frac{1}{C_{\text{vac}}} \;&=\; \sum_{N=0}^{\infty} \sum_{\mathcal{D}}^{\substack{N\,\text{bubble}\\ \text{diagrams}}} \operatorname{eval}(\mathcal{D}) \;=\; \sum_{\substack{\text{sets of}\\ \{n_\alpha\}}} \left(\prod_\alpha \frac{\big(\operatorname{eval}(\mathcal{B}_\alpha)\big)^{n_\alpha}}{n_\alpha!} \right) \\
&=\; \prod_\alpha \left(\sum_{n_\alpha=0}^{\infty} \frac{\big(\operatorname{eval}(\mathcal{B}_\alpha)\big)^{n_\alpha}}{n_\alpha!} \;=\; \exp\big(\operatorname{eval}(\mathcal{B}_\alpha)\big) \right) \\
&=\; \exp\left(\sum_\alpha \operatorname{eval}(\mathcal{B}_\alpha) \right).
\end{aligned} \tag{56}$$

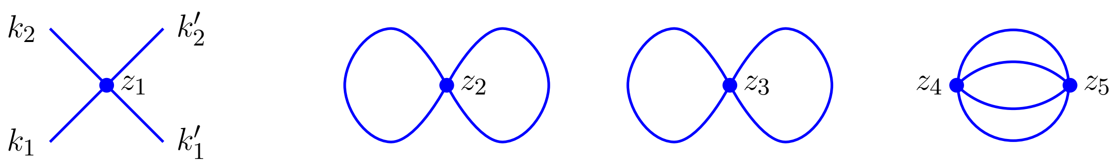
Now consider the perturbative expansion of a non-vacuum S-matrix element such as

$$\big\langle \text{free} : k_1', k_2' \big| \, \hat{S} \, \big| \, \text{free} : k_1, k_2 \big\rangle \;=\; \sum_{\mathcal{D}}^{\substack{\text{diagrams with 4}\\ \text{external lines}}} \operatorname{eval}(\mathcal{D}). \tag{57}$$

The sum here is over all diagrams with 4 external lines (two incoming and 2 outgoing), including both connected and disconnected diagrams. In a disconnected diagram, each connected component must have an even number of external lines, so ether one component has all 4 external lines and the rest are vacuum bubbles, or two components have two external lines apiece, and the rest are vacuum bubbles,

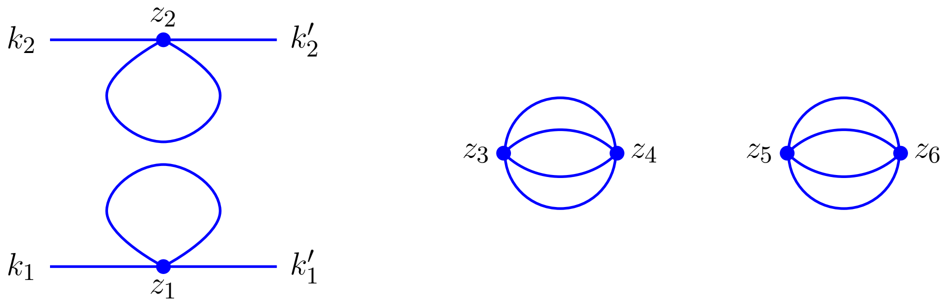
$$\mathcal{D} \;=\; \mathcal{C} \oplus \mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_m \quad \text{or} \quad \mathcal{D} \;=\; \mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_m\,, \tag{58}$$

for example



(59)

or



(60)

Again, evaluating a disconnected diagram like that amounts to evaluating each connected part as a standalone diagram, multiplying them together, and then dividing by an extra symmetry factor due to permutations of similar parts. Since the external lines are not permuted, the extra symmetries comes from permuting similar vacuum bubbles only, thus

$$\begin{aligned}
\mathrm{eval}(\mathcal{C}\oplus\mathcal{B}_1\oplus\cdots\oplus\mathcal{B}_m) &= \mathrm{eval}(\mathcal{C})\times\frac{\mathrm{eval}(\mathcal{B}_1)\times\cdots\times\mathrm{eval}(\mathcal{B}_m)}{\#\text{permutations of similar bubbles}}\,,\\
\mathrm{eval}(\mathcal{C}_1\times\mathcal{C}_2\oplus\mathcal{B}_1\oplus\cdots\oplus\mathcal{B}_m) &= \mathrm{eval}(\mathcal{C}_1)\times\mathrm{eval}(\mathcal{C}_2)\times\frac{\mathrm{eval}(\mathcal{B}_1)\times\cdots\times\mathrm{eval}(\mathcal{B}_m)}{\#\text{permutations of similar bubbles}}\,.
\end{aligned}\tag{61}$$

Consequently, when in eq. (57) we sum over all diagrams — connected or disconnected — with 4 external lines, the sum factorizes into a sum of bubble-less diagrams (connected or two-component) and a sum over vacuum bubbles:

$$\begin{aligned}
\sum_{\mathcal{D}}^{\substack{\text{all diagrams with}\\ \text{4 external lines}}}\mathrm{eval}(\mathcal{D}) &= \sum_{\mathcal{C}\,\mathrm{or}\,\mathcal{C}_1\oplus\mathcal{C}_2}^{\substack{\text{4 external lines,}\\ \text{no vacuum bubbles}}}\mathrm{eval}(\mathcal{C}\,\mathrm{or}\,\mathcal{C}_1\oplus\mathcal{C}_2)\\
&\quad+\sum_{\mathcal{C}\,\mathrm{or}\,\mathcal{C}_1\oplus\mathcal{C}_2}^{\substack{\text{4 external lines,}\\ \text{no vacuum bubbles}}}\ \sum_{m=1}^{\infty}\ \sum_{\mathcal{B}_1,\ldots,\mathcal{B}_m}^{\text{bubbles}}\mathrm{eval}(\mathcal{C}\,\mathrm{or}\,\mathcal{C}_1\oplus\mathcal{C}_2)\times\frac{\mathrm{eval}(\mathcal{B}_1)\times\cdots\times\mathrm{eval}(\mathcal{B}_m)}{\#\text{bubble permutations}}\\
&= \sum_{\mathcal{C}\,\mathrm{or}\,\mathcal{C}_1\oplus\mathcal{C}_2}^{\substack{\text{4 external lines,}\\ \text{no vacuum bubbles}}}\mathrm{eval}(\mathcal{C}\,\mathrm{or}\,\mathcal{C}_1\oplus\mathcal{C}_2)\times\\
&\qquad\times\left(1+\sum_{m=1}^{\infty}\ \sum_{\mathcal{B}_1,\ldots,\mathcal{B}_m}^{\text{bubbles}}\frac{\mathrm{eval}(\mathcal{B}_1)\times\cdots\times\mathrm{eval}(\mathcal{B}_m)}{\#\text{bubble permutations}}\right).
\end{aligned}\tag{62}$$

Moreover, the second factor here is precisely the sum (53) for the $1/C_{\text{vac}}$ factor, hence

$$C_{\text{vac}} \times \left\langle \text{free}: k_1', k_2' \middle| \hat{S} \middle| \text{free}: k_1, k_2 \right\rangle \;=\; \sum_{\mathcal{C}\,\text{or}\,\mathcal{C}_1\oplus\mathcal{C}_2}^{\substack{\text{4 external lines,}\\ \text{no vacuum bubbles}}} \text{eval}(\mathcal{C}\,\text{or}\,\mathcal{C}_1 \oplus \mathcal{C}_2) \tag{63}$$

Likewise, for any number of the incoming or outgoing particles, we may account for the overall C_{vac} factor in eq. (46) for the physical S-matrix element by simply skipping the Feynman diagrams with any vacuum bubbles,

$$C_{\text{vac}} \times \left\langle \text{free}: k_1', k_2', \ldots \middle| \hat{S} \middle| \text{free}: k_1, k_2, \ldots \right\rangle \;=\; \sum \begin{bmatrix} \text{Feynman diagrams with appropriate exter-} \\ \text{nal lines and without any vacuum bubbles.} \end{bmatrix} \tag{64}$$

Besides the vacuum bubbles, there are other types of 'bad diagrams' which do not contribute to physical S-matrix elements. But the reason those diagrams are 'bad' depends on momentum conservation, so let me first re-formulate the Feynman rules in momentum space.

Momentum Space Feynman Rules

Consider an internal line connecting vertices x and y of some diagram. The Feynman propagator corresponding to this line is

$$G_F(x-y) \;=\; \int\!\frac{d^4q}{(2\pi)^4}\,\frac{ie^{-iq(x-y)}}{q^2-m^2+i\epsilon}\,. \tag{65}$$

In coordinate-space Feynman rules we saw earlier in these notes, we should evaluate all such propagators before integrating over the vertex locations $\int d^4x \int d^4y \cdots$. But let's change the order of integration: Once we spell all the propagators as momentum integrals (65), let's integrate over the vertex locations before we integrate over the momenta in propagators.

For a fixed momentum q^μ, the integrand of the Feynman propagator (65) depends on the vertex locations x and y as $e^{-iqx} \times e^{+iqy}$ — for each vertex, the location-dependent factor is simply e^{-iqx} or e^{+iqy}. Similar exponential factors e^{-ikz} or $e^{+ik'z}$ accompany the external (incoming or outgoing) lines connected to a vertex z. Thus, each line connected to a vertex depends on its location as $e^{\pm ipz}$ where $p = q, k, k'$, depending on the type of the line.

Combining 4 lines (internal or internal) connected to the same vertex, we have

$$e^{\pm ip_1 z} \times e^{\pm ip_2 z} \times e^{\pm ip_3 z} \times e^{\pm ip_4 z}, \tag{66}$$

and nothing else in the diagram depends on that vertex's location z. Consequently, integrating over z produces a delta-function in momenta,

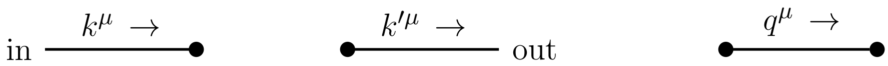
$$\int d^4z\, e^{\pm ip_1 z}\, e^{\pm ip_2 z}\, e^{\pm ip_3 z}\, e^{\pm ip_4 z} \;=\; (2\pi)^4 \delta^{(4)}(\pm p_1 \pm p_2 \pm p_3 \pm p_4). \tag{67}$$

These delta-functions lead to the *Kirchhoff Current Law* for momenta in a Feynman diagram. Think of a Feynman diagram as an electric circuit: the vertices are circuit junctions, the lines are conducting elements, and the momenta q^μ, k^μ, or k'^μ are currents flowing through those elements. The net currents flowing in and out of any junction must balance each other, and that's precisely what eqs. (67) say about the momenta: the net momenta flowing in or out of any vertex must balance each other.

So here are the momentum-space Feynman rules for evaluating a diagram:

1. First of all, assign momenta to all lines of the diagram: *fixed* momenta k^μ and k'^μ for the incoming and outgoing lines, and *variable* momenta q^μ for the internal lines.

   For each internal line, choose a direction in which its momentum q^μ flows from one vertex into another; use arrows to indicate the directions of momentum flow. For the external lines, the directions are fixed: inflow for incoming particles' lines, and outflow for the outgoing particles' lines.



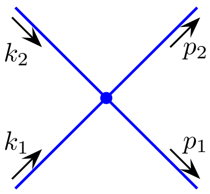
2. Multiply the following factors:
   - $\dfrac{i}{q^2 - m^2 + i\epsilon}$ for each *internal* line, but not for the external lines.
   - $(-i\lambda) \times (2\pi)^4 \delta^{(4)}(\pm p_1 \pm p_2 \pm p_3 \pm p_4)$ for each vertex. Here $p_1^\mu, \ldots, p_4^\mu$ stand for the momenta of 4 lines connected to the vertex; depending on the type of each line, $p^\mu = k^\mu, k'^\mu, q^\mu$. The sign of each p^μ is $+$ if the momentum flows into the vertex and $-$ if it flows out.

- Combinatorial factor for the whole diagram, 1/#symmetries.

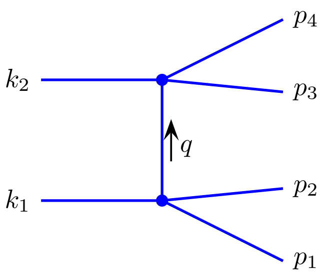
3. Integrate $\displaystyle\int\frac{d^4q}{(2\pi)^4}$ over all internal lines' momenta.

⋆ Most of the momentum integrals for the internal lines are 'eaten-up' by the δ-functions in the vertices — that's how we get the Kirchhoff Current Law for the momenta. Ultimately, there is one non-trivial $\displaystyle\int\frac{d^4q}{(2\pi)^4}$ integral for each *closed loop* in the diagram, and one un-integrated $(2\pi)^4\delta^{(4)}(\sum \text{momenta})$ for each *connected component* of the diagram.

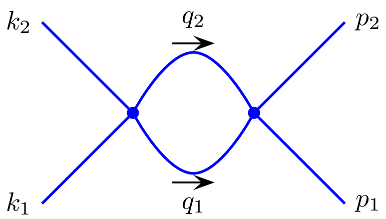
Here are some examples:



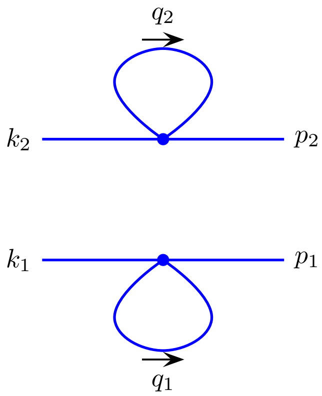
$$= \; (-i\lambda)\times(2\pi)^4\delta^{(4)}(k_1+k_2-p_1-p_2) \tag{68}$$



$$\begin{aligned} &= \int\frac{d^4q}{(2\pi)^4}\,\frac{i}{q^2-m^2+i\epsilon}\times(-i\lambda)(2\pi)^4\delta^{(4)}(k_1-q-p_1-p_2) \\ &\qquad\qquad\qquad\qquad\times(-i\lambda)(2\pi)^4\delta^{(4)}(k_2+q-p_3-p_4) \\ &= (2\pi)^4\delta^{(4)}(k_1+k_2-p_1-p_2-p_3-p_4)\times\left.\frac{-i\lambda^2}{q^2-m^2}\right|_{q=k_1-p_1-p_2=p_3+p_4-k_2} \end{aligned} \tag{69}$$



$$\begin{aligned} &= \int\frac{d^4q_1}{(2\pi)^4}\,\frac{i}{q_1^2-m^2+i\epsilon}\times\int\frac{d^4q_2}{(2\pi)^4}\,\frac{i}{q_2^2-m^2+i\epsilon}\times\frac{1}{2} \\ &\qquad\times(-i\lambda)\,(2\pi)^4\delta^{(4)}(k_1+k_2-q_1-q_2) \\ &\qquad\times(-i\lambda)\,(2\pi)^4\delta^{(4)}(q_1+q_2-p_1-p_2) \\ &= (2\pi)^4\delta^{(4)}(k_1+k_2-p_1-p_2)\times\frac{\lambda^2}{2}\times\int\frac{d^4q_1}{(2\pi)^4}\,\frac{1}{q_1^2-m^2+i\epsilon}\,\frac{1}{(k_1+k_2-q_1)^2-m^2+i\epsilon} \end{aligned} \tag{70}$$



$$
\begin{aligned}
&= \int \frac{d^4 q_1}{(2\pi)^4} \frac{i}{q_1^2 - m^2 + i\epsilon} \times \int \frac{d^4 q_2}{(2\pi)^4} \frac{i}{q_2^2 - m^2 + i\epsilon} \times \frac{1}{4} \\
&\qquad \times (-i\lambda)\,(2\pi)^4 \delta^{(4)}(k_1 + q_1 - q_1 - p_1) \\
&\qquad \times (-i\lambda)\,(2\pi)^4 \delta^{(4)}(k_2 + q_2 - q_2 - p_2) \\
&= (2\pi)^4 \delta^{(4)}(k_1 - p_1) \times (2\pi)^4 \delta^{(4)}(k_2 - p_2) \times \frac{\lambda^2}{4} \times \left[\int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2 - m^2 + i\epsilon} \right]^2
\end{aligned}
\tag{71}
$$

Connected Diagrams

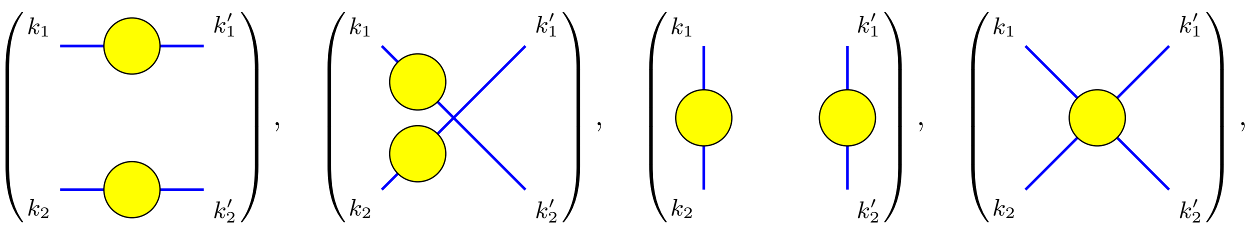
In a scattering experiment, most of the times the particles miss each other and do not scatter at all. In the formal scattering theory, this corresponds to decomposition

$$
\hat{S} = 1 + i\hat{T} \tag{72}
$$

where the $\hat{T}$ operator takes care of the scattering events that do happen every now and then. For a 2 body $\to$ 2 body process involving identical bosons, this means

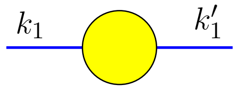
$$
\begin{aligned}
\langle \mathrm{in} : k_1', k_2' | \, \hat{S} \, | \mathrm{out} : k_1, k_2 \rangle &= \langle k_1' | k_1 \rangle \times \langle k_2' | k_2 \rangle + \langle k_1' | k_2 \rangle \times \langle k_2' | k_1 \rangle \\
&\quad + i \langle \mathrm{in} : k_1', k_2' | \, \hat{T} \, | \mathrm{out} : k_1, k_2 \rangle .
\end{aligned}
\tag{73}
$$

Diagrammatically, this formula corresponds to distinguishing between connected and disconnected diagrams. Note that while we have ruled out the vacuum bubbles, we still allow disconnected diagrams where each connected piece has some external lines. For two incoming and two outgoing particles, this allows for 4 topologically different classes of diagrams:



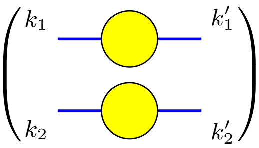
(74)

where each yellow circle stands for a connected diagram or sub-diagram. For the first type of diagrams, each connected sub-diagram imposes its own energy-momentum conservation — $k'_1 = k_1$, $k'_2 = k_2$ or else the diagram $= 0$. In the S-matrix decomposition (73), such diagrams corresponds to the first non-scattering term $\langle k'_1|k_1\rangle \times \langle k'_2|k_2\rangle$. Or rather, the sum of such diagrams — times the external-line factors $F(k)$ and $F(k')$ from eq. (46) for the physical S-matrix elements — produces that non-scattering term,

$$F(k_1) \times F(k'_1) \times \sum$$ 

$$= \big\langle \mathrm{out} : k'_1 \big| \hat{S} \big| \mathrm{in} : k_2 \big\rangle = \big\langle k'_1 | k_1 \big\rangle , \qquad (75)$$

likewise for the $k_2 \to k'_2$ process, and therefore

$$F(k_1)F(k'_1)F(k_2)F(k'_2) \times \sum \left(\right.$$ 

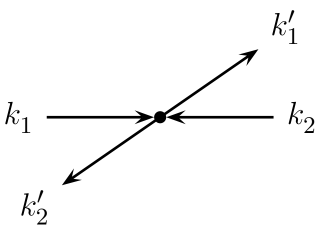
$$\left. \right) = \big\langle k'_1 | k_1 \big\rangle \times \big\langle k'_2 | k_2 \big\rangle . \qquad (76)$$

Similarly, the diagrams of the second disconnected type on figure (74) require $k'_2 = k_1$, $k'_1 = k_2$, so they contribute to the second non-scattering term $\langle k'_2|k_1\rangle \times \langle k'_1|k_2\rangle$ in eq. (73). As to the third disconnected type of diagrams from figure (74), they require $k_1 + k_2 = k'_1 + k'_2 = 0$, which is quite impossible on-shell. Indeed, the net energy of physical particles is always positive and cannot vanish. Consequently, the diagrams of the third disconnected type do not contribute to the S-matrix element at all.

On the other hand, the connected diagrams — the fourth type on figure (74) — impose the overall energy–momentum conservation

$$k'^{\mu}_1 + k'^{\mu}_2 = k^{\mu}_1 + k^{\mu}_2 \qquad (77)$$

but allow for scattering in which this net energy-momentum is re-distributed between the two particles. For example, in the center-of-mass frame we may have



$$k^{\mu}_{1,2} = (E, \pm\mathbf{k}), \quad k'^{\mu}_{1,2} = (E, \pm\mathbf{k}'). \qquad (78)$$

Thus, it's the connected Feynman diagrams which are solely responsible for the scattering!

Formally,

$$i\left\langle \mathrm{in}: k_1', k_2' \right| \hat{T} \left| \mathrm{out}: k_1, k_2 \right\rangle \;=\; F(k_1)F(k_1')F(k_2)F(k_2') \times \sum \begin{bmatrix}\text{Connected diagrams}\\ \text{with 4 external lines}\end{bmatrix}, \tag{79}$$

where the $F(k)$ factors for the external lines may be extracted from eq. (75):

$$\frac{\langle k'|k\rangle}{F^2(k)} \;=\; \sum \begin{bmatrix}\text{Connected diagrams}\\ \text{with 2 external lines}\end{bmatrix}. \tag{80}$$

We shall return to the $F(k)$ factors in the Spring semester. For the moment, let me simply say that to the lowest order of perturbation theory $F(k) = 1$.

Scattering Amplitudes

According to momentum-space Feynman rules, a *connected* diagram has precisely one uncanceled delta function of momenta enforcing the overall energy-momentum conservation (77), but all the other factors are analytic functions of the incoming and outgoing momenta. Hence, the sum of connected diagrams in eq. (79) for the T-matrix element produces

$$i\left\langle \mathrm{in}: k_1', k_2' \right| \hat{T} \left| \mathrm{out}: k_1, k_2 \right\rangle \;=\; (2\pi)^4\delta^{(4)}(k_1' + k_2' - k_1 - k_2) \times i\mathcal{M}(k_1, k_2 \to k_1', k_2') \tag{81}$$

for some analytic function of momenta $\mathcal{M}$ called the *scattering amplitude,* or to be precise, the *relativistically normalized scattering amplitude.* It differs from the non-relativistic amplitude f you have learned about in an undergraduate QM class by an overall factor

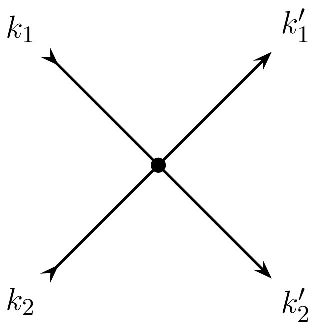
$$f \;=\; \frac{\mathcal{M}}{8\pi E_{\mathrm{cm}}} \tag{82}$$

where E_{cm} is the net energy of both incoming particles (or both outgoing particles) in the center of mass frame. As explained in painful detail in §4.5 of the *Peskin and Schroeder* textbook, the partial cross-section for an elastic 2 body $\to$ 2 body scattering is

$$\frac{d\sigma}{d\Omega_{\mathrm{cm}}} \;=\; \frac{|\mathcal{M}|^2}{64\pi^2\, E_{\mathrm{cm}}^2} \tag{83}$$

where $d\Omega_{\mathrm{cm}}$ is the element of body angle for the direction of one of the final particles in the CM frame (the other final particle flies in the opposite direction).

At the lowest order of the perturbation theory, the scattering amplitude $\mathcal{M}$ is particularly simple. Indeed, for $n = 1$ there is only one connected diagram with 4 external lines, namely



$$= \; -i\lambda \times (2\pi)^4 \delta^{(4)}(k_1' + k_2' - k_1 - k_2). \tag{84}$$

Dropping the net energy-momentum conservation factor, we obtain the scattering amplitude as simply

$$i\mathcal{M} \;=\; -i\lambda \quad\Longrightarrow\quad \mathcal{M} \;=\; -\lambda \quad \forall\,\text{momenta}. \tag{85}$$

Consequently, to the lowest order of perturbation theory, the partial scattering cross-section

$$\frac{d\sigma}{d\Omega_{\text{cm}}} \;=\; \frac{\lambda^2}{64\pi^2\, E_{\text{cm}}^2} \tag{86}$$

is isotropic, and the total scattering cross-section is

$$\sigma_{\text{tot}} \;=\; \frac{4\pi}{2} \times \frac{\lambda^2}{64\pi^2\, E_{\text{cm}}^2} \;=\; \frac{\lambda^2}{32\pi\, E_{\text{cm}}^2}\,. \tag{87}$$

Note: since the two outgoing particles are identical bosons and we cannot tell which particle goes up and which goes down, the net solid angle for one of the two opposite directions is $4\pi/2$ rather than the whole 4π.

Alas, at the higher orders of perturbation theory, the $O(\lambda^2)$, $O(\lambda^3)$, *etc.*, terms in the scattering amplitude are much more complicated than the leading term $-\lambda$. And in theories other than $\lambda\Phi^4$, even the lowest-order scattering amplitudes are fairly complicated functions of the particles' energies and directions.

Summary

I conclude these notes with the summary of Feynman rules for the $\lambda\Phi^4$ theory. To obtain the scattering amplitude for a 2 particle $\to$ 2 particle or 2 particle $\to m$ particle process to order N in perturbation theory:

1. Draw all *connected* Feynman diagrams with 2 incoming and m outgoing lines, and $n \leq N$ vertices. Each vertex must be connected to precisely 4 lines, external or internal. Make sure you draw all the diagrams obeying these conditions.

2. Evaluate the diagrams. For each diagram:

   (a) Assign momenta to all the lines: fixed momenta for the incoming and outgoing lines, and variable momenta for the internal lines. For the internal lines, specify the direction of momentum flow (from which vertex to which vertex); the choice of those directions is arbitrary, but you must stick to the same choice while you evaluate the diagram.

   (b) Multiply the following factors:

   - $\displaystyle\int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 - m^2 + i\epsilon}$ for each *internal* line, but not for the external lines.
   - $(-i\lambda) \times (2\pi)^4 \delta^{(4)}(\pm q_1 \pm q_2 \pm q_3 \pm q_4)$ for each vertex. The sign of each q^μ is $+$ if the momentum flows into the vertex and $-$ if it flows out.
   - Combinatorial factor for the whole diagram, 1/#symmetries.

   (c) Do the integrals over the momenta q^μ of internal lines.
   Many of these integrals will be 'eaten up' by the δ-functions at the vertices, but for each closed loop in the graph there is one un-eaten $\int d^4q$ integral that needs to be calculated the hard way.

   (d) Drop the overall energy-momentum conservation factor $(2\pi)^4 \delta^{(4)}\big(k_1 + k_2 - \sum k'\big)$.

3. Total up the diagrams.

4. If you are working beyond the lowest non-vanishing order of perturbation theory, multiply the whole amplitude by the $F(k)$ factors for each of the incoming or outgoing particles.

These rules should give you the relativistically normalized scattering amplitude, or rather $i\times\mathcal{M}(k_1, k_2 \to k'_1, \ldots, k'_m)$. The partial and total scattering cross sections follow from this amplitude as

$$d\sigma = |\mathcal{M}|^2 \times d\mathcal{P} \tag{88}$$

where $d\mathcal{P}$ is the phase-space factor. Most generally,

$$d\mathcal{P} = \frac{1}{4\,|E_1\mathbf{k}_2 - E_2\mathbf{k}_1|} \times \prod_{i=1}^{m} \frac{d^3\mathbf{k}'_i}{(2\pi)^3\, 2E'_i} \times (2\pi)^4 \delta^{(4)}\Big(k_1^\mu + k_2^\mu - \sum k'^\mu\Big), \tag{89}$$

which should be integrated over enough final-state parameters to eat up the δ-function. To find where this phase-space formula comes from, please read §4.5 of the *Peskin and Schroeder* textbook. Or for a quick 4-page summary, read my notes on phase space.