0. First of all, refresh your memory of special relativity. Make sure you understand index summation conventions in Minkowski or Euclidean spaces. If you don't understand (or have hard time deciphering) expressions such as $B_i = \epsilon_{ijk}\partial_j A_k$ (in 3 space dimensions) or $\partial_\mu F^{\mu\nu} = J^\nu$ (in the Minkowski spacetime), *get up to speed ASAP* or you would not be able to follow the class.

1. Consider a *massive* relativistic vector field $A^\mu(x)$ with the Lagrangian density

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}m^2 A_\mu A^\mu - A^\mu J_\mu \tag{1}$$

where $c = \hbar = 1$, $F_{\mu\nu} \stackrel{\text{def}}{=} \partial_\mu A_\nu - \partial_\nu A_\mu$, and the current $J^\mu(x)$ is a fixed source for the $A^\mu(x)$ field. Note that because of the mass term, the Lagrangian (1) is *not* gauge invariant.

(a) Derive the Euler–Lagrange field equations for the massive vector field $A^\mu(x)$.

(b) Show that this field equation *does not require* current conservation; however, if the current happens to satisfy $\partial_\mu J^\mu = 0$, then the field $A^\mu(x)$ satisfies

$$\partial_\mu A^\mu = 0 \qquad \text{and} \qquad (\partial^2 + m^2)A^\mu = J^\mu. \tag{2}$$

2. In spacetimes of higher dimensions $D > 4$ there are antisymmetric-tensor fields analogous to the EM-like vector fields; such tensor fields play important roles in supergravity and string theory.

For example, consider a free 2-index antisymmetric tensor field $B_{\mu\nu}(x) \equiv -B_{\nu\mu}(x)$, where μ and ν are D-dimensional Lorentz indices running from 0 to $D-1$. To be precise, $B_{\mu\nu}(x)$ is the *tensor potential*, analogous to the electromagnetic vector potential $A_\mu(x)$. The analog of the EM tension fields $F_{\mu\nu}(x)$ is is the 3–index tension tensor

$$H_{\lambda\mu\nu}(x) = \partial_\lambda B_{\mu\nu} + \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu}\,. \tag{3}$$

(a) Show that this tensor is totally antisymmetric in all 3 indices.

(b) Show that regardless of the Lagrangian, the H fields satisfy Jacobi identities

$$\tfrac{1}{6}\partial_{[\kappa}H_{\lambda\mu\nu]} \;\equiv\; \partial_\kappa H_{\lambda\mu\nu} \;-\; \partial_\lambda H_{\mu\nu\kappa} \;+\; \partial_\mu H_{\nu\kappa\lambda} \;-\; \partial_\nu H_{\kappa\lambda\mu} \;=\; 0. \tag{4}$$

(c) The Lagrangian for the free $B_{\mu\nu}(x)$ fields is simply

$$\mathcal{L}(B,\partial B) \;=\; \tfrac{1}{12}H_{\lambda\mu\nu}H^{\lambda\mu\nu} \tag{5}$$

where $H_{\lambda\mu\nu}$ are as in eq. (3). Treating the $B_{\mu\nu}(x)$ as $\frac{1}{2}D(D-1)$ independent fields, derive their equations of motion.

Similar to the EM fields, the $B_{\mu\nu}$ fields are subject to *gauge transforms*

$$B'_{\mu\nu}(x) \;=\; B_{\mu\nu}(x) \;+\; \partial_\mu\Lambda_\nu(x) \;-\; \partial_\nu\Lambda_\mu(x) \tag{6}$$

where $\Lambda_\mu(x)$ is an arbitrary vector field.

(d) Show that the tension fields $H_{\lambda\mu\nu}(x)$ — and hence the Lagrangian (5) — are invariant under such gauge transforms.

In spacetimes of sufficiently high dimensions D, one may have similar tensor fields with more indices. Generally, the potentials form a p-index totally antisymmetric tensor $C_{\mu_1\mu_2\cdots\mu_p}(x)$, the tensions form a $p+1$ index tensor

$$\begin{aligned} G_{\mu_1\mu_2\cdots\mu_{p+1}} \;&=\; \frac{1}{p!}\,\partial_{[\mu_1}C_{\mu_2\cdots\mu_p\mu_{p+1}]} \\ &\equiv\; \partial_{\mu_1}C_{\mu_2\cdots\mu_{p+1}} \;-\; \partial_{\mu_2}C_{\mu_1\mu_3\cdots\mu_{p+1}} \;+\; \cdots \;+\; (-1)^p\partial_{\mu_{p+1}}C_{\mu_1\cdots\mu_p}\,, \end{aligned} \tag{7}$$

also totally antisymmetric in all its indices, and the Lagrangian is

$$\mathcal{L}(C,\partial C) \;=\; \frac{(-1)^p}{2(p+1)!}\,G_{\mu_1\mu_2\cdots\mu_{p+1}}G^{\mu_1\mu_2\cdots\mu_{p+1}}. \tag{8}$$

(e) Derive the Jacobi identities and the equations of motion for the G fields.

(f) Show that the tension fields $G_{\mu_1\mu_2\cdots\mu_{p+1}}(x)$ — and hence the Lagrangian (8) — are invariant under gauge transforms of the potentials $C_{\mu_1\mu_2\cdots\mu_p}(x)$ which act as

$$C'_{\mu_1\mu_2\cdots\mu_p}(x) \;=\; C_{\mu_1\mu_2\cdots\mu_p}(x) \;+\; \frac{1}{(p-1)!}\,\partial_{[\mu_1}\Lambda_{\mu_2\cdots\mu_p]}(x) \tag{9}$$

where $\Lambda_{\mu_2\cdots\mu_p}(x)$ is an arbitrary $(p-1)$-index tensor field (totally antisymmetric).