1. The first problem is about the angular momentum in presence of a magnetic monopole. For simplicity, we assume a single monopole of magnetic charge M which does not move but remains static at $\mathbf{x} = \mathbf{0}$.

   Let's put a static electric charge Q at some point $\mathbf{x} = \mathbf{a}$ near the monopole. Together, the electric field of the charge and the magnetic field of the monopole have non-zero Poynting vector $\mathbf{S} = \frac{1}{4\pi c}\,\mathbf{E} \times \mathbf{B}$. When integrated over the space, this Poynting vector has zero net momentum but non-zero angular momentum

   $$\mathbf{L} \;=\; \int d^3\mathbf{x}\,\mathbf{x} \times \mathbf{S} \;=\; -\frac{QM}{c}\,\mathbf{n_a} \tag{1}$$

   where $\mathbf{n_a}$ is the unit vector in the direction of the electric charge (from the monopole's position at the origin). Note that this integral is independent from the distance a between the monopole and the charge — they could be light-years apart, and still produce the same angular momentum. Also note that when the magnetic charge M and the electric charge Q obey Dirac's quantization condition

   $$Q \times M \;=\; \frac{\hbar c}{2} \times \text{an integer}, \tag{2}$$

   the angular momentum (1) has integral or half-integral magnitude in units of $\hbar$.

   (a) In lieu of working out the integral (1) by yourself, read the solution in these notes by Prof. James Wheeler at the University of Utah.

   Now consider a quantum particle of electric charge q moving in the monopole's magnetic field superimposed on some spherically symmetric electric potential $V(r)$. For simplicity, take the particle to be spinless and non-relativistic, so its Hamiltonian has form

   $$\hat{H} \;=\; \frac{1}{2m}\,\hat{\vec{\pi}}^2 \;+\; qV(\hat{r}) \tag{3}$$

   where $\hat{\vec{\pi}} = \hat{\mathbf{p}} - \frac{q}{c}\mathbf{A}(\mathbf{x})$ is the kinematic momentum of the particle rather that its canonical momentum $\mathbf{p}$. In the coordinate basis, $\vec{\hat{\pi}} = -i\hbar\vec{\mathcal{D}}$, the covariant gradient. And in any basis,

the kinematic momenta and the coordinate operators obey the following commutation relations:

$$\begin{aligned}
[\hat{x}_i, \hat{x}_j] &= 0, \\
[\hat{x}_i, \hat{\pi}_j] &= i\hbar\delta_{ij}, \\
[\hat{\pi}_i, \hat{\pi}_j] &= \frac{i\hbar q}{c}\epsilon_{ijk}\hat{B}_k = \frac{i\hbar qM}{c} \cdot \frac{\epsilon_{ijk}\hat{x}^k}{\hat{r}^3}.
\end{aligned} \tag{4}$$

Because of the angular momentum (1), the net *orbital* angular momentum operator for the moving particle has form

$$\hat{\mathbf{L}} = \hat{\mathbf{x}} \times \hat{\vec{\pi}} - \frac{qM}{c}\frac{\hat{\mathbf{x}}}{\hat{r}}. \tag{5}$$

(b) Verify that this angular momentum operator obeys the usual commutation relations:

$$\begin{aligned}
[\hat{L}_i, \hat{x}_j] &= i\hbar\epsilon_{ijk}\,\hat{x}_k, \\
[\hat{L}_i, \hat{\pi}_j] &= i\hbar\epsilon_{ijk}\,\hat{\pi}_k, \\
[\hat{L}_i, \hat{L}_j] &= i\hbar\epsilon_{ijk}\,\hat{L}_k.
\end{aligned} \tag{6}$$

(c) Verify that the angular momentum (5) is conserved, that is, it commutes with the Hamiltonian (3).

(d) In spherical coordinates (r, θ, ϕ), the vector potential of the magnetic monopole has form

$$\vec{A}(r, \theta, \phi) = m \cdot \frac{\pm 1 - \cos\theta}{r\sin\theta} \cdot \vec{e}_\phi, \tag{7}$$

where the two signs correspond to two possible gauge choices as to whether the Dirac string should hang below the dyon $(\theta = \pi)$ or above it $(\theta = 0)$.

Show that with these gauge choices the operator $\hat{L}_z$ becomes

$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi} \mp \frac{qM}{c} \tag{8}$$

(in the spherical coordinate representation).

(e) Use eq. (8) — plus the usual quantization rules for the eigenvalues $\hbar^2 \ell(\ell + 1)$ of $\hat{\mathbf{L}}^2$ and $\hbar m$ of $\hat{L}_z$ — to show that the allowed values of ℓ for the particle moving around the monopole are

$$\ell = \frac{|qM|}{\hbar c}, \ \frac{|qM|}{\hbar c} + 1, \ \frac{|qM|}{\hbar c} + 2, \ \ldots. \tag{9}$$

Note: for a half-integral $QM/\hbar c$, the allowed values of the *orbital* angular momentum are half-integral rather than integral!

2. Next, consider a *massive* relativistic vector field $A^\mu(x)$ with the Lagrangian density

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \tfrac{1}{2} m^2 A_\mu A^\mu - A^\mu J_\mu \tag{10}$$

(in $\hbar = c = 1$ units) where the current $J^\mu(x)$ is a fixed source for the $A^\mu(x)$ field. Because of the mass term, the Lagrangian (10) is *not* gauge invariant. However, we *assume* that the current $J^\mu(x)$ is conserved, $\partial_\mu J^\mu(x) = 0$.

Back in homework set#1 (problem 1) we have derived the Euler–Lagrange equations for the massive vector field. In this problem, we develop the Hamiltonian formalism for the $A^\mu(x)$. Our first step is to identify the canonically conjugate "momentum" fields.

(a) Show that $\partial \mathcal{L}/\partial \dot{\mathbf{A}} = -\mathbf{E}$ but $\partial \mathcal{L}/\partial \dot{A}_0 \equiv 0$.

In other words, the canonically conjugate field to $\mathbf{A}(\mathbf{x})$ is $-\mathbf{E}(\mathbf{x})$ but the $A_0(\mathbf{x})$ does not have a canonical conjugate! Consequently,

$$H = \int d^3\mathbf{x} \left(-\dot{\mathbf{A}}(\mathbf{x}) \cdot \mathbf{E}(\mathbf{x}) - \mathcal{L} \right). \tag{11}$$

(b) Show that in terms of the $\mathbf{A}$, $\mathbf{E}$, and A_0 fields, and their *space* derivatives,

$$H = \int d^3\mathbf{x} \left\{ \tfrac{1}{2}\mathbf{E}^2 + A_0 \left(J_0 - \nabla \cdot \mathbf{E} \right) - \tfrac{1}{2} m^2 A_0^2 + \tfrac{1}{2} (\nabla \times \mathbf{A})^2 + \tfrac{1}{2} m^2 \mathbf{A}^2 - \mathbf{J} \cdot \mathbf{A} \right\}. \tag{12}$$

Because the A_0 field does not have a canonical conjugate, the Hamiltonian formalism does not produce an equation for the time-dependence of this field. Instead, it gives us a time-independent equation relating the $A_0(\mathbf{x}, t)$ to the values of other fields *at the same time t.*

Specifically, we have

$$\frac{\delta H}{\delta A_0(\mathbf{x})} \equiv \left.\frac{\partial \mathcal{H}}{\partial A_0}\right|_{\mathbf{x}} - \nabla \cdot \left.\frac{\partial \mathcal{H}}{\partial (\nabla A_0)}\right|_{\mathbf{x}} = 0. \tag{13}$$

At the same time, the vector fields $\mathbf{A}$ and $\mathbf{E}$ satisfy the Hamiltonian equations of motion,

$$
\begin{aligned}
\frac{\partial}{\partial t}\mathbf{A}(\mathbf{x},t) &= -\left.\frac{\delta H}{\delta \mathbf{E}(\mathbf{x})}\right|_t \equiv -\left[\frac{\partial \mathcal{H}}{\partial \mathbf{E}} - \nabla_i \frac{\partial \mathcal{H}}{\partial (\nabla_i \mathbf{E})}\right]_{(\mathbf{x},t)}, \\
\frac{\partial}{\partial t}\mathbf{E}(\mathbf{x},t) &= +\left.\frac{\delta H}{\delta \mathbf{A}(\mathbf{x})}\right|_t \equiv +\left[\frac{\partial \mathcal{H}}{\partial \mathbf{A}} - \nabla_i \frac{\partial \mathcal{H}}{\partial (\nabla_i \mathbf{A})}\right]_{(\mathbf{x},t)}.
\end{aligned} \tag{14}
$$

(c) Write down the explicit form of all these equations.

(d) Verify that the equations you have just written down are equivalent to the relativistic Euler–Lagrange equations for the $A^\mu(x)$, namely

$$(\partial^\mu \partial_\mu + m^2)A^\nu = \partial^\nu(\partial_\mu A^\mu) + J^\nu \tag{15}$$

and hence $\partial_\mu A^\mu(x) = 0$ and $(\partial^\nu \partial_\nu + m^2)A^\mu = 0$ when $\partial_\mu J^\mu \equiv 0$, *cf.* homework #1.

3. Finally, let's quantize the massive vector field. Since classically the $-\mathbf{E}(\mathbf{x})$ fields are canonically conjugate momenta to the $\mathbf{A}(\mathbf{x})$ fields, the corresponding quantum fields $\hat{\mathbf{E}}(\mathbf{x})$ and $\hat{\mathbf{A}}(\mathbf{x})$ satisfy the canonical equal-time commutation relations

$$
\begin{aligned}
[\hat{A}_i(\mathbf{x},t), \hat{A}_j(\mathbf{y},t)] &= 0, \\
[\hat{E}_i(\mathbf{x},t), \hat{E}_j(\mathbf{y},t)] &= 0, \\
[\hat{A}_i(\mathbf{x},t), \hat{E}_j(\mathbf{y},t)] &= -i\delta_{ij}\delta^{(3)}(\mathbf{x}-\mathbf{y})
\end{aligned} \tag{16}
$$

(in the $\hbar = c = 1$ units). The currents also become quantum fields $\hat{J}^\mu(\mathbf{x},t)$, but they are composed of some kind of charged degrees of freedom rather than the vector fields in question. Consequently, *at equal times* the currents $\hat{J}^\mu(\mathbf{x},t)$ commute with both the $\hat{\mathbf{E}}(\mathbf{y},t)$ and the $\hat{\mathbf{A}}(\mathbf{y},t)$ fields.

The classical $A^0(\mathbf{x}, t)$ field does not have a canonical conjugate and its equation of motion does not involve time derivatives. In the quantum theory, $\hat{A}^0(\mathbf{x}, t)$ satisfies a similar time-independent constraint

$$m^2 \hat{A}^0(\mathbf{x}, t) = \hat{J}^0(\mathbf{x}, t) - \nabla \cdot \hat{\mathbf{E}}(\mathbf{x}, t), \tag{17}$$

but from the Hilbert space point of view this is an operatorial identity rather than an equation of motion. Consequently, the commutation relations of the scalar potential field follow from eqs. (16); in particular, at equal times the $\hat{A}^0(\mathbf{x}, t)$ commutes with the $\hat{\mathbf{E}}(\mathbf{y}, t)$ but does not commute with the $\hat{\mathbf{A}}(\mathbf{y}, t)$.

Finally, the Hamiltonian operator follows from the classical eq. (12), namely

$$
\begin{aligned}
\hat{H} &= \int d^3\mathbf{x} \left\{ \tfrac{1}{2}\hat{\mathbf{E}}^2 + \hat{A}_0 \left(\hat{J}_0 - \nabla \cdot \hat{\mathbf{E}} \right) - \tfrac{1}{2}m^2 \hat{A}_0^2 + \tfrac{1}{2}\left(\nabla \times \hat{\mathbf{A}} \right)^2 + \tfrac{1}{2}m^2 \hat{\mathbf{A}}^2 - \hat{\mathbf{J}} \cdot \hat{\mathbf{A}} \right\} \\
&= \int d^3\mathbf{x} \left\{ \tfrac{1}{2}\hat{\mathbf{E}}^2 + \frac{1}{2m^2} \left(\hat{J}_0 - \nabla \cdot \hat{\mathbf{E}} \right)^2 + \tfrac{1}{2}\left(\nabla \times \hat{\mathbf{A}} \right)^2 + \tfrac{1}{2}m^2 \hat{\mathbf{A}}^2 - \hat{\mathbf{J}} \cdot \hat{\mathbf{A}} \right\}
\end{aligned}
\tag{18}
$$

where the second line follows from the first and eq. (17).

Your task is to calculate the commutators $[\hat{A}_i(\mathbf{x}, t), \hat{H}]$ and $[\hat{E}_i(\mathbf{x}, t), \hat{H}]$ and write down the Heisenberg equations for the quantum vector fields. Make sure those equations are similar to the Hamilton equations for the classical fields.