1. First, an exercise in bosonic commutation relations

$$[\hat{a}_\alpha, \hat{a}_\beta] \;=\; 0, \qquad [\hat{a}^\dagger_\alpha, \hat{a}^\dagger_\beta] \;=\; 0, \qquad [\hat{a}_\alpha, \hat{a}^\dagger_\beta] \;=\; \delta_{\alpha\beta}\,. \tag{1}$$

   (a) Calculate the commutators $[\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}^\dagger_\gamma]$, $[\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}_\delta]$, $[\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}^\dagger_\gamma \hat{a}_\delta]$, and $[\hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\gamma \hat{a}_\delta, \hat{a}^\dagger_\mu \hat{a}_\nu]$.

   (b) For a single pair of $\hat{a}$ and $\hat{a}^\dagger$ operators, show that for any analytic function $f(x) = f_0 + f_1 x + f_2 x^2 + \cdots$,

$$[\hat{a}, f(\hat{a}^\dagger)] \;=\; +f'(\hat{a}^\dagger) \quad \text{and} \quad [\hat{a}^\dagger, f(\hat{a})] \;=\; -f'(\hat{a}) \tag{2}$$

   where $f(\hat{a}) \stackrel{\text{def}}{=} f_0 + f_1\hat{a} + f_2(\hat{a})^2 + \cdots$ and likewise $f(\hat{a}^\dagger) \stackrel{\text{def}}{=} f_0 + f_1\hat{a}^\dagger + f_2(\hat{a}^\dagger)^2 + \cdots$.

   (c) Show that $e^{c\hat{a}}\hat{a}^\dagger e^{-c\hat{a}} = \hat{a}^\dagger + c$, $e^{c\hat{a}^\dagger}\hat{a} e^{-c\hat{a}^\dagger} = \hat{a} - c$, hence for any analytic function f,

$$e^{c\hat{a}} f(\hat{a}^\dagger) e^{-c\hat{a}} \;=\; f(\hat{a}^\dagger + c) \quad \text{and} \quad e^{c\hat{a}^\dagger} f(\hat{a}) e^{-c\hat{a}^\dagger} \;=\; f(\hat{a} - c). \tag{3}$$

   (d) Now generalize (b) and (c) to any set of creation and annihilation operators $\hat{a}^\dagger_\alpha$ and $\hat{a}_\alpha$. Show that for any analytic function $f(\text{multiple } \hat{a}^\dagger_\alpha)$ of creation operators but not of the annihilation operators or a function $f(\text{multiple } \hat{a}_\alpha)$ of the annihilation operators but not of the creation operators,

$$\begin{aligned}
[\hat{a}_\alpha, f(\hat{a}^\dagger)] \;&=\; +\frac{\partial f(\hat{a}^\dagger)}{\partial \hat{a}^\dagger_\alpha}\,, \qquad [\hat{a}^\dagger_\alpha, f(\hat{a})] \;=\; -\frac{\partial f(\hat{a})}{\partial \hat{a}_\alpha}\,, \\
\exp\Bigl(\sum\nolimits_\alpha c_\alpha \hat{a}_\alpha\Bigr) f(\hat{a}^\dagger) \exp\Bigl(-\sum\nolimits_\alpha c_\alpha \hat{a}_\alpha\Bigr) \;&=\; f(\text{each } \hat{a}^\dagger_\alpha \to \hat{a}^\dagger_\alpha + c_\alpha), \\
\exp\Bigl(\sum\nolimits_\alpha c_\alpha \hat{a}^\dagger_\alpha\Bigr) f(\hat{a}) \exp\Bigl(-\sum\nolimits_\alpha c_\alpha \hat{a}^\dagger_\alpha\Bigr) \;&=\; f(\text{each } \hat{a}_\alpha \to \hat{a}_\alpha - c_\alpha).
\end{aligned} \tag{4}$$

2. Now consider an $O(N)$ symmetric Lagrangian for N interacting real scalar fields,

$$\mathcal{L} = \frac{1}{2}\sum_{a=1}^{N}\bigl(\partial_\mu \Phi_a\bigr)^2 - \frac{m^2}{2}\sum_{a=1}^{N}\Phi_a^2 - \frac{\lambda}{24}\left(\sum_{a=1}^{N}\Phi_a^2\right)^2. \tag{5}$$

By the Noether theorem, the continuous $SO(N)$ subgroup of the $O(N)$ symmetry gives rise to $\frac{1}{2}N(N-1)$ conserved currents

$$J^\mu_{ab}(x) = -J^\mu_{ba}(x) = \Phi_a(x)\,\partial^\mu\Phi_b(x) - \Phi_b(x)\,\partial^\mu\Phi_a(x). \tag{6}$$

In the quantum field theory, these currents become operators

$$\begin{aligned} \hat{\mathbf{J}}_{ab}(\mathbf{x},t) &= -\hat{\mathbf{J}}_{ba}(\mathbf{x},t) = -\hat{\Phi}_a(\mathbf{x},t)\nabla\hat{\Phi}_b(\mathbf{x},t) + \hat{\Phi}_b(\mathbf{x},t)\nabla\hat{\Phi}_a(\mathbf{x},t), \\ \hat{J}^0_{ab}(\mathbf{x},t) &= -\hat{J}^0_{ba}(\mathbf{x},t) = \hat{\Phi}_a(\mathbf{x},t)\hat{\Pi}_b(\mathbf{x},t) - \hat{\Phi}_b(\mathbf{x},t)\hat{\Pi}_a(\mathbf{x},t). \end{aligned} \tag{7}$$

This problem is about the net charge operators

$$\hat{Q}_{ab}(t) = -\hat{Q}_{ba}(t) = \int d^3\mathbf{x}\,\hat{J}^0_{ab}(\mathbf{x}) = \int d^3\mathbf{x}\left(\hat{\Phi}_a(\mathbf{x},t)\hat{\Pi}_b(\mathbf{x},t) - \hat{\Phi}_b(\mathbf{x},t)\hat{\Pi}_a(\mathbf{x},t)\right). \tag{8}$$

(a) Write down the equal-time commutation relations for the quantum $\hat{\Phi}_a$ and $\hat{\Pi}_a$ fields. Also, write down the Hamiltonian operator for the interacting fields.

(b) Show that

$$\begin{aligned} \left[\hat{Q}_{ab}(t), \hat{\Phi}_c(\mathbf{x}, \text{same } t)\right] &= i\delta_{bc}\hat{\Phi}_a(\mathbf{x},t) - i\delta_{ac}\hat{\Phi}_b(\mathbf{x},t), \\ \left[\hat{Q}_{ab}(t), \hat{\Pi}_c(\mathbf{x}, \text{same } t)\right] &= i\delta_{bc}\hat{\Pi}_a(\mathbf{x},t) - i\delta_{ac}\hat{\Pi}_b(\mathbf{x},t), \end{aligned} \tag{9}$$

(c) Show that the all the $\hat{Q}_{ab}$ commute with the Hamiltonian operator $\hat{H}$. In the Heisenberg picture, this makes all the charge operators $\hat{Q}_{ab}$ time independent.

(d) Verify that the $\hat{Q}_{ab}$ obey commutation relations of the $SO(N)$ generators,

$$\left[\hat{Q}_{ab}, \hat{Q}_{cd}\right] = -i\delta_{[c[b}\hat{Q}_{a]d]} \equiv -i\delta_{bc}\hat{Q}_{ad} + i\delta_{ac}\hat{Q}_{bd} + i\delta_{bd}\hat{Q}_{ac} - i\delta_{ad}\hat{Q}_{bc}\,. \tag{10}$$

Now let's take $\lambda \to 0$ and focus on the free fields. Let's work in the Schrödinger picture and expand all the fields into creation and annihilation operators $\hat{a}^\dagger_{\mathbf{p},a}$ and $\hat{a}_{\mathbf{p},a}$ $(a = 1, \ldots, N)$.

(e) Show that in terms of creation and annihilation operators, the charges (8) become

$$\hat{Q}_{ab} = \sum_{\mathbf{p}} \left(-i\hat{a}^\dagger_{\mathbf{p},a}\hat{a}_{\mathbf{p},b} + i\hat{a}^\dagger_{\mathbf{p},b}\hat{a}_{\mathbf{p},b} \right). \tag{11}$$

(f) Use the commutation relations (1) for the creation and annihilation operators (and the results of problem 1.a) to verify that the operators (11) obey the commutation relations (10).

Finally, for $N = 2$ the $SO(2)$ symmetry is the phase symmetry of one complex field $\Phi = (\Phi_1 + i\Phi_2)/\sqrt{2}$ and its conjugate $\Phi^* = (\Phi_1 - i\Phi_2)/\sqrt{2}$. In the Fock space, they give rise to particles and anti-particles of opposite charges.

(g) Expand the fields $\Phi(\mathbf{x})$ and $\Phi^\dagger(\mathbf{x})$ into the creation and annihilation operators for the particles and antiparticles,

$$\begin{aligned}
\hat{a}_{\mathbf{p}} &= \frac{\hat{a}_{\mathbf{p},1} + i\hat{a}_{\mathbf{p},2}}{\sqrt{2}} && \text{are particle annihilation operators,}\\
\hat{b}_{\mathbf{p}} &= \frac{\hat{a}_{\mathbf{p},1} - i\hat{a}_{\mathbf{p},2}}{\sqrt{2}} && \text{are antiparticle annihilation operators,}\\
\hat{a}^\dagger_{\mathbf{p}} &= \frac{\hat{a}^\dagger_{\mathbf{p},1} - i\hat{a}^\dagger_{\mathbf{p},2}}{\sqrt{2}} && \text{are particle creation operators,}\\
\hat{b}^\dagger_{\mathbf{p}} &= \frac{\hat{a}^\dagger_{\mathbf{p},1} + i\hat{a}^\dagger_{\mathbf{p},2}}{\sqrt{2}} && \text{are antiparticle creation operators.}
\end{aligned} \tag{12}$$

(h) Show that in terms of the operators (12),

$$\hat{Q}_{21} = -\hat{Q}_{12} = \hat{N}_{\text{particles}} - \hat{N}_{\text{antiparticles}} = \sum_{\mathbf{p}} \left(\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} - \hat{b}^\dagger_{\mathbf{p}}\hat{b}_{\mathbf{p}} \right). \tag{13}$$

3. An operator acting on identical bosons can be described in terms of N–particle wave functions (the *first-quantized* formalism) or in terms of creation and annihilation operators in the Fock space (the *second-quantized* formalism). This problem is about converting the operators from one formalism to another.

   Let's start with a discrete orthonormal basis $\{|\alpha\rangle\}_\alpha$ of single-particle wave states with wave-functions $\phi_\alpha(\mathbf{x})$. (By abuse of notations, $\mathbf{x} = (x, y, z, \text{spin}, \textit{etc.})$. The corresponding basis of the N-boson Hilbet space comprises the states

$$|\alpha, \beta, \cdots, \omega\rangle \;=\; \frac{1}{\sqrt{T}}\, \hat{a}^\dagger_\omega \cdots \hat{a}^\dagger_\beta \hat{a}^\dagger_\alpha\, |0\rangle \tag{14}$$

   with totally-symmetrized wave functions

$$\begin{aligned}
\phi_{\alpha\beta\cdots\omega}(\mathbf{x}_1, \mathbf{x}_2\, \ldots, \mathbf{x}_N) \;&=\; \frac{1}{\sqrt{D}} \sum_{(\tilde{\alpha},\tilde{\beta},\ldots,\tilde{\omega})}^{\substack{\text{distinct permutations}\\ \text{of } (\alpha,\beta,\ldots,\omega)}} \phi_{\tilde{\alpha}}(\mathbf{x}_1) \times \phi_{\tilde{\beta}}(\mathbf{x}_2) \times \cdots \times \phi_{\tilde{\omega}}(\mathbf{x}_N) \\
&=\; \frac{1}{\sqrt{T \times N!}} \sum_{(\tilde{\alpha},\tilde{\beta},\ldots,\tilde{\omega})}^{\substack{\text{all } N! \text{ permutations}\\ \text{of } (\alpha,\beta,\ldots,\omega)}} \phi_{\tilde{\alpha}}(\mathbf{x}_1) \times \phi_{\tilde{\beta}}(\mathbf{x}_2) \times \cdots \times \phi_{\tilde{\omega}}(\mathbf{x}_N),
\end{aligned} \tag{15}$$

   where $T = \prod_\gamma n_\gamma!$ is the number of trivial permutations between *coincident* entries of the list $(\alpha, \beta, \ldots, \omega)$ (for example, $\alpha \leftrightarrow \beta$ when α and β happen to be equal), and $D = N!/T$ is the number of *distinct permutations.*

   To make sure that the states (14) have the wavefunctions (15), the wave-function picture of the creation and the annihilation operators should be as follows: Given an N-boson state $|N, \psi\rangle$ with a totally-symmetric wavefunction $\psi(\mathbf{x}_1, \ldots, \mathbf{x}_N)$,
   the state $|N+1, \psi'\rangle = \hat{a}^\dagger_\alpha\, |N, \psi\rangle$ has a totally-symmetric $(N+1)$-particle wave function

$$\psi'(\mathbf{x}_1, \ldots, \mathbf{x}_{N+1}) \;=\; \frac{1}{\sqrt{N+1}} \sum_{i=1}^{N+1} \phi_\alpha(\mathbf{x}_i) \times \psi(\mathbf{x}_1, \ldots, \cancel{\mathbf{x}_i}, \ldots, \mathbf{x}_{N+1}). \tag{16}$$

   In particular, for $N = 0$, $\psi'(x_1) = \phi_\alpha(x_1)$. Also, the state $|N-1,\psi''\rangle = \hat{a}_\alpha |N,\psi\rangle$ has a totally-symmetric $(N-1)$-particle wave function

$$\psi''(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1}) \;=\; \sqrt{N}\int d^3\mathbf{x}_N\,\phi^*_\alpha(\mathbf{x}_N)\times\psi(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1},\mathbf{x}_N). \tag{17}$$

In particular, for $N=1$, $\psi''(\text{no arguments}) = \langle\phi_\alpha|\psi\rangle$. Also, for $N=0$ we simply define $\hat{a}_\alpha|0\rangle \stackrel{\text{def}}{=} 0$.

(a) Verify the commutation relations (1) for these operators.

(b) Verify that the $\hat{a}_\alpha$ and the $\hat{a}^\dagger_\alpha$ are hermitian conjugates of each other by checking that

$$\langle N-1,\widetilde{\psi}|\,\hat{a}_\alpha\,|N,\psi\rangle \;=\; \langle N,\psi|\,\hat{a}^\dagger_\alpha\,|N-1,\widetilde{\psi}\rangle^* \tag{18}$$

for any $N\geq 1$ and any totally-symmetric wave functions $\psi(\mathbf{x}_1,\ldots,\mathbf{x}_N)$ and $\widetilde{\psi}(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1})$.

(c) Verify that the states (14) indeed have the wavefunctions (15).

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Now let's move on to the the next subject, namely the one-body operators — the additive operators acting on one particle at a time. In the first-quantized formalism they act on N–particle states according to

$$\hat{A}^{(1)}_{\text{net}} \;=\; \sum_{i=1}^{N}\hat{A}_1(i^{\text{th}}\ \text{particle}) \tag{19}$$

where $\hat{A}_1$ is some kind of a one-particle operator (such as momentum $\hat{\mathbf{p}}$, or kinetic energy $\frac{1}{2m}\hat{\mathbf{p}}^2$, or potential $V(\hat{\mathbf{x}})$, *etc.*, *etc.*). In the second-quantized formalism such operators become

$$\hat{A}^{(2)}_{\text{net}} \;=\; \sum_{\alpha,\beta}\langle\alpha|\,\hat{A}_1\,|\beta\rangle\;\hat{a}^\dagger_\alpha\hat{a}_\beta\,. \tag{20}$$

(d) Verify that the two operators have the same matrix elements between any two N-boson states $|N,\psi\rangle$ and $|N,\widetilde{\psi}\rangle$, $\langle N,\widetilde{\psi}|\,\hat{A}^{(1)}_{\text{net}}\,|N,\psi\rangle = \langle N,\widetilde{\psi}|\,\hat{A}^{(2)}_{\text{net}}\,|N,\psi\rangle$.
Hint: use $\hat{A}_1 = \sum_{\alpha,\beta}|\alpha\rangle\langle\alpha|\,\hat{A}_1\,|\beta\rangle\langle\beta|$.

(e) Now let $\hat{A}^{(2)}_{\rm net}$, $\hat{B}^{(2)}_{\rm net}$, and $\hat{C}^{(2)}_{\rm net}$ be three second-quantized net one-body operators corresponding to the single-particle operators $\hat{A}_1$, $\hat{B}_1$, and $\hat{C}_1$.
Show that if $\hat{C}_1 = [\hat{A}_1, \hat{B}_1]$ then $\hat{C}^{(2)}_{\rm net} = \left[\hat{A}^{(2)}_{\rm net}, \hat{B}^{(2)}_{\rm net}\right]$.

Finally, consider the two-body operators, *i.e.* additive operators acting on two particles at a time. Given a two-particle operator $\hat{B}_2$ — such as $V(\hat{\mathbf{x}}_1 - \hat{\mathbf{x}}_2)$ — the *net* B operator acts in the first-quantized formalism according to

$$\hat{B}^{(1)}_{\rm net} \;=\; \tfrac{1}{2}\sum_{i\neq j} \hat{B}_2(i^{\rm th} \text{ and } j^{\rm th} \text{ particles}), \tag{21}$$

and in the second-quantized formalism according to

$$\hat{B}^{(2)}_{\rm net} \;=\; \tfrac{1}{2}\sum_{\alpha,\beta,\gamma,\delta} (\langle\alpha| \otimes \langle\beta|)\hat{B}_2(|\gamma\rangle \otimes |\delta\rangle)\, \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\delta \hat{a}_\gamma\,. \tag{22}$$

Note: in this formula, it is OK to use the un-symmetrized 2-particle states $\langle\alpha| \otimes \langle\beta|$ and $|\gamma\rangle \otimes |\delta\rangle$, and hence the un-symmetrized matrix elements of the $\hat{B}_2$. At the level of the second-quantized operator $\hat{B}^{(2)}_{\rm net}$, the Bose symmetry is automatically provided by $\hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta = \hat{a}^\dagger_\beta \hat{a}^\dagger_\gamma$ and $\hat{a}_\delta \hat{a}_\gamma = \hat{a}_\gamma \hat{a}_\delta$, even for un-symmetrized matrix elements of the $\hat{B}_2$.

(f) Similar to part (d), show the operators (21) and (22) have the same matrix elements between any two N-boson states, $\langle N, \widetilde{\psi}|\, \hat{A}^{(1)}_{\rm net}\, |N, \psi\rangle = \langle N, \widetilde{\psi}|\, \hat{A}^{(2)}_{\rm net}\, |N, \psi\rangle$ for any $\langle N, \widetilde{\psi}|$ and $|N, \psi\rangle$.

(g) Now let $\hat{A}_1$ be a one-particle operator, let $\hat{B}_2$ and $\hat{C}_2$ be two-body operators, and let $\hat{A}^{(2)}_{\rm net}$, $\hat{B}^{(2)}_{\rm net}$, and $\hat{C}^{(2)}_{\rm net}$ be the corresponding second-quantized operators according to eqs. (20) and (22).
Show that if $\hat{C}_2 = \left[\left(\hat{A}_1(1^{\rm st}) + \hat{A}_1(2^{\rm nd})\right), \hat{B}_2\right]$ then $\hat{C}^{(2)}_{\rm net} = \left[\hat{A}^{(2)}_{\rm net}, \hat{B}^{(2)}_{\rm net}\right]$.