1. First, a reading assignment: chapter 3 of *Modern Quantum Mechanics* by J. J. Sakurai, sections 1, 2, 3, second half of section 5 (representations of the rotation operators), and section 10. The other sections 4, 6, 7, 8, and 9 are not relevant to the present class material. The main focus of this assignment are the relations between the rotations and the angular momenta $\hat{J}^{x,y,z}$.

   PS: If you have read the Sakurai's book before but it has been a while, please read it again.

2. The rest of this homework is about the Lorentz group and its representations.

   The continuous Lorentz group $SO^+(3,1)$ has 6 generators $\hat{J}^{\mu\nu} = -\hat{J}^{\nu\mu}$ satisfying

$$\left[\hat{J}^{\alpha\beta},\hat{J}^{\mu\nu}\right] \;=\; ig^{\beta\mu}\hat{J}^{\alpha\nu} - ig^{\alpha\mu}\hat{J}^{\beta\nu} - ig^{\beta\nu}\hat{J}^{\alpha\mu} + ig^{\alpha\nu}\hat{J}^{\beta\mu}. \tag{1}$$

   In 3D terms, the generators comprise three angular momenta $\hat{J}^i = \frac{1}{2}\epsilon^{ijk}\hat{J}^{jk}$ — which generate the rotation of space — plus 3 generators $\hat{K}^i = \hat{J}^{0i} = -\hat{J}^{i0}$ of the Lorentz boosts.

   (a) Show that in 3D terms, the commutation relations (1) become

$$\left[\hat{J}^i,\hat{J}^j\right] \;=\; i\epsilon^{ijk}\hat{J}^k,\quad \left[\hat{J}^i,\hat{K}^j\right] \;=\; i\epsilon^{ijk}\hat{K}^k,\quad \left[\hat{K}^i,\hat{K}^j\right] \;=\; -i\epsilon^{ijk}\hat{J}^k. \tag{2}$$

   Lorentz symmetry dictates commutation relations of $\hat{J}^{\mu\nu}$ with any operators comprising a Lorentz multiplet. In particular, for any Lorentz vector $\hat{V}^\mu$

$$\left[\hat{V}^\lambda,\hat{J}^{\mu\nu}\right] \;=\; ig^{\lambda\mu}\hat{V}^\nu - ig^{\lambda\nu}\hat{V}^\mu. \tag{3}$$

   (b) Spell out these commutation relations in 3D terms, then use them to show that the Lorentz boost generators $\hat{\mathbf{K}}$ do not commute with the Hamiltonian $\hat{H}$.

   (c) Show that even in the non-relativistic limit, the Galilean boosts $t' = t$, $\mathbf{x}' = \mathbf{x} + \mathbf{v}t$ and their generators $\hat{\mathbf{K}}_G$ do not commute with the Hamiltonian.

In general, the *time-independent* symmetries commute with the Hamiltonian. But when the action of a symmetry is manifestly time dependent — like a Galilean boost $\mathbf{x}' = \mathbf{x} + \mathbf{v}t$ or a Lorentz boost — the symmetry operators do not commute with the time evolution and hence with the Hamiltonian.

Next, consider the little group $G(p)$ of Lorentz transforms preserving the momentum vector p^μ of some massive particle, $p_\mu p^\mu = m^2 > 0$. For simplicity, assume the particle moves in z direction with velocity β, thus $p^\mu = (E, 0, 0, p)$ for $E = \gamma m$ and $p = \beta\gamma m$.

(d) Show that there 3 independent combination of $\hat{J}^i$ and $\hat{K}^i$ preserving this momentum, namely

$$\tilde{J}^1 \;=\; \gamma\hat{J}^1 \,-\, \beta\gamma\hat{K}^2, \quad \tilde{J}^2 \;=\; \gamma\hat{J}^2 \,+\, \beta\gamma\hat{K}^1, \quad \text{and} \quad \hat{J}^3. \tag{4}$$

Also show that these three combinations have angular-momentum-like commutators with each other, $[\tilde{J}^i, \tilde{J}^j] = i\epsilon^{ijk}\tilde{J}^k$.

For a massless particle, $\gamma = \infty$ so eqs. (4) do not apply. Instead, the little group of momentum $p^\mu = (E, 0, 0, E)$ is generated by the

$$\hat{I}^1 \;=\; \hat{J}^1 \,-\, \hat{K}^2, \quad \hat{I}^2 \;=\; \hat{J}^2 \,+\, \hat{K}^1, \quad \text{and} \quad \hat{J}^3, \tag{5}$$

which obey the ISO(2) commutation relations

$$[\hat{J}^3, \hat{I}^1] \;=\; +i\hat{I}^2, \quad [\hat{J}^3, \hat{I}^2] \;=\; +i\hat{I}^1, \quad [\hat{I}^1, \hat{I}^2] \;=\; 0. \tag{6}$$

(e) Prove this.

As discussed in class, the finite unitary multiplets of the ISO(2) group generated by the (5) are singlets $|\lambda\rangle$, which are eigenstates of the helicity operator $\hat{J}^3$ (for the momentum in z direction) and are annihilated by the $\hat{I}^{1,2}$ operators,

$$\hat{J}^3\,|\lambda\rangle \;=\; \lambda\,|\lambda\rangle\,, \quad \hat{I}^1\,|\lambda\rangle \;=\; 0, \quad \hat{I}^2\,|\lambda\rangle \;=\; 0. \tag{7}$$

(f) Show that in 4D terms the state $|p, \lambda\rangle$ of a *massless* particle satisfies

$$\epsilon_{\alpha\beta\gamma\delta}\hat{J}^{\beta\gamma}\hat{P}^\delta\,|p, \lambda\rangle \;=\; -2\lambda\hat{P}_\alpha\,|p, \lambda\rangle\,. \tag{8}$$

(g) Use this formula to show that *continuous* Lorentz transforms do not change helicities of *massless* particles,

$$\forall L \in \mathrm{SO}^{+}(3,1), \quad \widehat{\mathcal{D}}(L)\,|p,\lambda\rangle \;=\; |Lp,\,\mathrm{same}\,\lambda\rangle \times e^{i\,\mathrm{phase}}. \tag{9}$$

3. While particle states belong to infinite but unitary multiplets of the Lorentz group, the quantum fields form finite but non-unitary multiplets. In this problem we shall classify all such multiplets of the $SO^{+}(3,1)$ group, or rather of its double cover $\mathrm{Spin}(3,1) \cong \mathrm{SL}(2,\mathbf{C})$.

   (a) Let's re-organize the $\hat{\mathbf{J}}$ and $\hat{\mathbf{K}}$ generators of the continuous Lorentz group into two non-hermitian 3-vectors

$$\hat{\mathbf{J}}_{+} \;=\; \tfrac{1}{2}\big(\hat{\mathbf{J}} + i\hat{\mathbf{K}}\big) \quad\text{and}\quad \hat{\mathbf{J}}_{-} \;=\; \tfrac{1}{2}\big(\hat{\mathbf{J}} - i\hat{\mathbf{K}}\big) \;=\; \hat{\mathbf{J}}_{+}^{\dagger}. \tag{10}$$

   Show that the two 3-vectors commute with each other, $[\hat{J}_{+}^{k}, \hat{J}_{-}^{\ell}] = 0$, while the components of each 3-vector satisfy angular momentum commutation relations, $[\hat{J}_{+}^{k}, \hat{J}_{+}^{\ell}] = i\epsilon^{k\ell m}\hat{J}_{+}^{m}$ and $[\hat{J}_{-}^{k}, \hat{J}_{-}^{\ell}] = i\epsilon^{k\ell m}\hat{J}_{-}^{m}$.

By themselves, the 3 $\hat{J}_{+}^{k}$ generate a symmetry group similar to rotations of a 3D space, but since the $\hat{J}_{+}^{k}$ are non-hermitian, the finite irreducible multiplets of this symmetry are non-unitary analytic continuations (to complex "angles") of the ordinary angular momentum multiplets (j) of spin $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$. Likewise, the finite irreducible multiplets of the symmetry group generated by the $\hat{J}_{-}^{k}$ are analytic continuations of the spin-j multiplets of angular momentum. Moreover, the two symmetry groups commute with each other, so the finite irreducible multiplets of the net Lorentz symmetry are tensor products $(j_{+}) \otimes (j_{-})$ of the $\hat{\mathbf{J}}_{+}$ and $\hat{\mathbf{J}}_{-}$ multiplets. In other words, distinct finite irreducible multiplets of the Lorentz symmetry may be labeled by *two* integer or half-integer 'spins' j_{+} and j_{-}, while the states within such a multiplet are $|j_{+}, j_{-}, m_{+}, m_{-}\rangle$ for $m_{+} = -j_{+}, \ldots, +j_{+}$ and $m_{-} = -j_{-}, \ldots, +j_{-}$.

The simplest non-trivial Lorentz multiplet $\mathbf{2}$ has $j_{+} = \frac{1}{2}$ while $j_{-} = 0$. In this two-component multiplet $\hat{\mathbf{J}}_{+} = \frac{1}{2}\boldsymbol{\sigma}$ while $\hat{\mathbf{J}}_{-} = 0$, or in terms of $\hat{\mathbf{J}}$ and $\hat{\mathbf{K}}$, $\hat{\mathbf{J}} = \frac{1}{2}\boldsymbol{\sigma}$ while $\hat{\mathbf{K}} = -\frac{i}{2}\boldsymbol{\sigma}$. Consequently, the finite Lorentz transforms in this multiplet are represented by

2×2 matrices of the form

$$M = \exp\left(-i\mathbf{a}\cdot\hat{\mathbf{J}} - i\mathbf{b}\cdot\hat{\mathbf{K}}\right) = \exp\left(\tfrac{1}{2}(-i\mathbf{a}+\mathbf{b})\cdot\boldsymbol{\sigma}\right). \tag{11}$$

for some real 3-vectors $\mathbf{a}$ and $\mathbf{b}$. The $\mathbf{a}$ vector here parametrizes a rotation of 3D space while the $\mathbf{b}$ vector parametrizes a Lorentz boost; a general continuous Lorentz transform involves both. The matrices (11) always have unit determinant, $\det(M) = 1$, but there are no other general restrictions: a generic M is complex, non-unitary, non-hermitian, *etc.*, *etc.* The group of such 2×2 complex matrices is called $SL(2, \mathbf{C})$.

The Lorenz group has another *in-equivalent* two-component multiplet $\bar{\mathbf{2}}$ with $j_+ = 0$ and $j_- = \frac{1}{2}$. In this multiplet $\hat{\mathbf{J}}$ acts as $\frac{1}{2}\boldsymbol{\sigma}$ but $\hat{\mathbf{K}}$ acts as $+\frac{i}{2}\boldsymbol{\sigma}$, hence a finite Lorentz transform with the same parameters $\mathbf{a}$ and $\mathbf{b}$ as in eq. (11) is represented by a different 2×2 matrix

$$\overline{M} = \exp\left(\tfrac{1}{2}(-i\mathbf{a}-\mathbf{b})\cdot\boldsymbol{\sigma}\right). \tag{12}$$

Generally this matrix is in-equivalent to M but rather equivalent to the complex conjugate of M,

$$\overline{M} = \left(M^\dagger\right)^{-1} = \sigma_2 M^* \sigma_2. \tag{13}$$

(b) Prove this relation for any $\mathbf{a}$ and $\mathbf{b}$.
Hint: prove and use $\sigma_2\boldsymbol{\sigma}^*\sigma_2 = -\boldsymbol{\sigma}$.

For pure rotations of 3D space, M is unitary and $\overline{M} = M$. For pure Lorentz boosts, M is hermitian and $\overline{M} = M^{-1}$. We shall prove both statements later in this exercise.

Later in class we shall study in great detail the Dirac spinor fields that form a reducible $\mathbf{2} + \bar{\mathbf{2}}$ multiplet. There are also Weyl spinor fields that form irreducible $\mathbf{2}$ or $\bar{\mathbf{2}}$ multiplets. There will be future homeworks about those spinors, but for now let's consider the other Lorentz multiplets.

In the ordinary Spin(3) $= SU(2)$ group, one can construct a multiplet of any spin j from a symmetric tensor product of $2j$ doublets. This procedure gives us an object $\Phi_{\alpha_1,\ldots,\alpha_{2j}}$

with $2j$ spinor indices $\alpha_1, \ldots, \alpha_{2j} = 1, 2$ that's totally symmetric under permutation of those indices and transforms under an $SU(2)$ symmetry U as

$$\Phi_{\alpha_1,\alpha_2\ldots,\alpha_{2j}} \;\to\; U_{\alpha_1}^{\beta_1} U_{\alpha_2}^{\beta_2} \cdots U_{\alpha_{2j}}^{\beta_{2j}}\, \Phi_{\beta_1,\beta_2\ldots,\beta_{2j}}. \tag{14}$$

For integer j, such objects are equivalent to tensors of the $SO(3)$; for example, for $j = 2$ $\Phi_{\alpha\beta} \equiv \Phi_{\beta\alpha}$ is equivalent to an $SO(3)$ vector $\vec{\Phi}$.

In the Lorentz group Spin$(3, 1)$ we have a similar situation — any multiplet can be constructed by tensoring together a bunch of two-component spinors of the $SL(2, \mathbf{C})$. But unlike the $SU(2)$, the $SL(2, \mathbf{C})$ has two different spinors $\mathbf{2} \not\cong \bar{\mathbf{2}}$ transforming under different rules. Notationally, we shall distinguish them by different index types: the un-dotted Greek indices belong to spinor that transform according to $M \in SL(2, \mathbf{C})$ while the dotted Greek indices belong to spinors that transform according to M^* (which is equivalent to $\overline{M}$),

$$\Phi_\alpha \;\to\; M_\alpha^\beta \Phi_\beta \quad \not\cong \quad \Phi_{\dot\gamma} \;\to\; M_{\dot\gamma}^{*\dot\delta} \Phi_{\dot\delta}. \tag{15}$$

Combining such spinors to make a multiplet with 'spins' j_+ and j_-, we make an object $\Phi_{\alpha_1,\ldots,\alpha_{(2j_+)};\dot\gamma_1,\ldots,\dot\gamma_{(2j_-)}}$ with $2j_+$ un-dotted indices and $2j_-$ dotted indices. $\Phi_{\cdots}$ is totally symmetric under permutations of the un-dotted indices with each other or dotted indices with each other, but there is no symmetry between indices of different types. Under an $SL(2, \mathbf{C})$ symmetry M, the un-dotted indices transform according to M while the dotted indices transform according to the M^*, thus

$$\Phi_{\alpha_1,\ldots,\alpha_{(2j_+)};\dot\gamma_1,\ldots,\dot\gamma_{(2j_-)}} \;\to\; M_{\alpha_1}^{\beta_1} \cdots M_{\alpha_{(2j_+)}}^{\beta_{(2j_+)}} \times M_{\dot\gamma_1}^{*M\dot\delta_1} \cdots M_{\dot\gamma_{(2j_-)}}^{*M\dot\delta_{(2j_-)}} \cdots \times \Phi_{\beta_1,\ldots,\beta_{(2j_+)};\dot\delta_1,\ldots,\dot\delta_{(2j_-)}}. \tag{16}$$

Of particular importance among such multi-spinors is the bi-spinor $V_{\alpha\dot\gamma}$ with $j_+ = j_- = \frac{1}{2}$ — it is equivalent to the Lorentz vector V^μ. The map between bi-spinors and Lorentz vectors involves four hermitian 2×2 matrices σ^μ, where σ^0 is the unit matrix while σ^1, σ^2 and σ^3 are the Pauli matrices. In $SL(2, \mathbf{C})$ terms, each σ^μ matrix has one dotted and one un-dotted index, thus $\sigma^\mu_{\alpha\dot\gamma}$. Using the σ^μ, we may re-cast any Lorentz vector V^μ as a matrix

$$V^\mu \ \to\ V_\mu\sigma^\mu \ =\ V^0\ -\ \mathbf{V}\cdot\boldsymbol{\sigma} \tag{17}$$

an hence as a $\left(\frac{1}{2},\frac{1}{2}\right)$ bi-spinor

$$V_{\alpha\dot\gamma}\ =\ \Big(V_\mu\sigma^\mu\Big)_{\alpha\dot\gamma}\ =\ V^0\delta_{\alpha\dot\gamma}\ -\ \mathbf{V}\cdot\boldsymbol{\sigma}_{\alpha\dot\gamma}\,. \tag{18}$$

Under an $SL(2,\mathbf{C})$ symmetry, the bi-spinor transforms as

$$V_{\alpha\dot\gamma}\ \to\ V'_{\alpha\dot\gamma}\ =\ M_\alpha^{\ \beta}M^{*\dot\delta}_{\dot\gamma}\,V_{\beta\dot\delta}\,, \tag{19}$$

or in matrix form,

$$V_\mu\sigma^\mu\ \to\ V'_\mu\sigma^\mu\ =\ M\left(V_\mu\sigma^\mu\right)M^\dagger. \tag{20}$$

Since the four matrices σ^μ form a complete basis of 2×2 matrices, eq. (20) defines a linear transform $V'_\mu = L_\mu^{\ \nu}V_\nu$.

(c) Prove that for any $SL(2,\mathbf{C})$ matrix M, the transform $L_\mu^{\ \nu}(M)$ defined by eq. (20) is real (real V'_μ for real V_μ), Lorentzian (preserves $V'_\mu V'^\mu = V_\mu V^\mu$) and orthochronous.
Hint: prove and use $\det(V_\mu\sigma^\mu) = V_\mu V^\mu$.

- $\star$ For extra challenge, show that this transform is proper, $det(L) = +1$.

(d) Verify that this $SL(2,\mathbf{C})\to SO^+(3,1)$ map respects the group law, $L(M_2M_1) = L(M_2)L(M_1)$.

(e) Verify explicitly that for a unitary $M = \exp\bigl(-\frac{i}{2}\theta\,\mathbf{n}\cdot\boldsymbol{\sigma}\bigr)$, $L(M)$ is a rotation by angle θ around axis $\mathbf{n}$.

(f) Likewise, verify that for an hermitian $M = \exp\bigl(\frac{1}{2}r\,\mathbf{n}\cdot\boldsymbol{\sigma}\bigr)$, $L(M)$ is a boost of *rapidity* r in the direction $\mathbf{n}$.
FYI, the rapidity is related to the β and γ parameters or a Lorentz boost as $\beta=\tanh(r)$, $\gamma=\cosh(r)$. For two successive boosts in the same direction, their rapidities add up, $r_{\text{net}} = a_1 + r_2$.

In general, any (j_+, j_-) multiplet of the $SL(2, \mathbf{C})$ with integer net spin $j_+ + j_-$ is equivalent to some kind of a Lorentz tensor. (Here, we include the scalar and the vector among the tensors.) For example, the $(1,1)$ multiplet is equivalent to a symmetric, traceless 2–index tensor $T^{\mu\nu} = T^{\nu\mu}$, $T^{\mu}_{\mu} = 0$. For $j_+ \neq j_-$ the representation is complex, but one can make a real tensor by combining two multiplets with opposite j_+ and j_-, for example the $(1,0)$ and the $(0,1)$ multiplets are together equivalent to the antisymmetric 2–index tensor $F^{\mu\nu} = -F^{\nu\mu}$.

(g) Verify the above examples.

Hint: For any kind of angular momentum, $(j = \frac{1}{2}) \otimes (j = \frac{1}{2}) = (j = 1) \oplus (j = 0)$.