

1. First, an exercise in Dirac matrices γ^μ . In this problem, you should not assume any explicit matrices for the γ^μ but simply use the anticommutation relations

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}. \quad (1)$$

When necessary, you may also assume that the Dirac matrices are 4×4 , and the γ^0 matrix is hermitian while the $\gamma^1, \gamma^2, \gamma^3$ matrices are antihermitian, $(\gamma^0)^\dagger = +\gamma^0$ while $(\gamma^i)^\dagger = -\gamma^i$ for $i = 1, 2, 3$.

- (a) Show that $\gamma^\alpha \gamma_\alpha = 4$, $\gamma^\alpha \gamma^\nu \gamma_\alpha = -2\gamma^\nu$, $\gamma^\alpha \gamma^\mu \gamma^\nu \gamma_\alpha = 4g^{\mu\nu}$, and $\gamma^\alpha \gamma^\lambda \gamma^\mu \gamma^\nu \gamma_\alpha = -2\gamma^\nu \gamma^\mu \gamma^\lambda$.
Hint: use $\gamma^\alpha \gamma^\nu = 2g^{\nu\alpha} - \gamma^\nu \gamma^\alpha$ repeatedly.
- (b) The electron field in the EM background obeys the *covariant Dirac equation*
 $(i\gamma^\mu D_\mu - m)\Psi(x) = 0$ where $D_\mu \Psi = \partial_\mu \Psi - ieA_\mu \Psi$. Show that this equation implies

$$(D_\mu D^\mu + m^2 + qF_{\mu\nu} S^{\mu\nu}) \Psi(x) = 0. \quad (2)$$

Besides the 4 Dirac matrices γ^μ , there is another useful matrix $\gamma^5 \stackrel{\text{def}}{=} i\gamma^0 \gamma^1 \gamma^2 \gamma^3$.

- (c) Show that the γ^5 anticommutes with each of the γ^μ matrices — $\gamma^5 \gamma^\mu = -\gamma^\mu \gamma^5$ — and commutes with all the spin matrices, $\gamma^5 S^{\mu\nu} = +S^{\mu\nu} \gamma^5$.
- (d) Show that the γ^5 is hermitian and that $(\gamma^5)^2 = 1$.
- (e) Show that $\gamma^5 = (i/24)\epsilon_{\kappa\lambda\mu\nu} \gamma^\kappa \gamma^\lambda \gamma^\mu \gamma^\nu$ and that $\gamma^{[\kappa} \gamma^\lambda \gamma^\mu \gamma^{\nu]}$ $= +24i\epsilon^{\kappa\lambda\mu\nu} \gamma^5$.
- (f) Show that $\gamma^{[\lambda} \gamma^\mu \gamma^{\nu]}$ $= +6i\epsilon^{\kappa\lambda\mu\nu} \gamma_\kappa \gamma^5$.
- (g) Show that any 4×4 matrix Γ is a unique linear combination of the following 16 matrices:
 $1, \gamma^\mu, \frac{1}{2}\gamma^{[\mu} \gamma^{\nu]}$ $= -2iS^{\mu\nu}, \gamma^5 \gamma^\mu$, and γ^5 .
- * My conventions here are: $\epsilon^{0123} = -1, \epsilon_{0123} = +1, \gamma^{[\mu} \gamma^{\nu]}$ $= \gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu$,
 $\gamma^{[\lambda} \gamma^\mu \gamma^{\nu]}$ $= \gamma^\lambda \gamma^\mu \gamma^\nu - \gamma^\lambda \gamma^\nu \gamma^\mu + \gamma^\mu \gamma^\nu \gamma^\lambda - \gamma^\mu \gamma^\lambda \gamma^\nu + \gamma^\nu \gamma^\lambda \gamma^\mu - \gamma^\nu \gamma^\mu \gamma^\lambda$, etc.

1*. For extra challenge, let's generalize the Dirac matrices to spacetime dimensions $d \neq 4$. Such matrices always satisfy the Clifford algebra (1), but their sizes depend on d .

Let $\Gamma = i^n \gamma^0 \gamma^1 \dots \gamma^{d-1}$ be the generalization of the γ^5 to d dimensions; the pre-factor $i^n = \pm i$ or ± 1 is chosen such that $\Gamma = \Gamma^\dagger$ and $\Gamma^2 = +1$.

(a) For even d , Γ anticommutes with all the γ^μ . Prove this, then use this fact to show that there are 2^d independent products of the γ^μ matrices, and consequently the matrices should be $2^{d/2} \times 2^{d/2}$.

(b) For odd d , Γ commutes with all the γ^μ — prove this. Consequently, one can set $\Gamma = +1$ or $\Gamma = -1$; the two choices lead to in-equivalent sets of the γ^μ .

Classify the independent products of the γ^μ for odd d and show that their net number is 2^{d-1} ; consequently, the matrices should be $2^{(d-1)/2} \times 2^{(d-1)/2}$.

2. Now let's go back to $d = 3 + 1$ and learn about the Weyl spinors and Weyl spinor fields. Since all the spin matrices $S^{\mu\nu}$ commute with the γ^5 , for all *continuous* Lorentz symmetries L^μ_ν their Dirac-spinor representations $M_D(L) = \exp(-\frac{i}{2} \Theta_{\alpha\beta} S^{\alpha\beta})$ are block-diagonal in the eigenbasis of the γ^5 . This makes the Dirac spinor Ψ a *reducible* multiplet of the continuous Lorentz group $SO^+(3,1)$ — it comprises two different irreducible 2-component spinor multiplets called the left-handed Weyl spinor ψ_L and the right-handed Weyl spinor ψ_R .

This decomposition becomes clear in the Weyl convention for the Dirac matrices where

$$\gamma^\mu = \begin{pmatrix} 0 & \bar{\sigma}^\mu \\ \sigma^\mu & 0 \end{pmatrix} \quad \text{where} \quad \begin{aligned} \sigma^\mu &\stackrel{\text{def}}{=} (\mathbf{1}_{2 \times 2}, -\boldsymbol{\sigma}), \\ \bar{\sigma}^\mu &\stackrel{\text{def}}{=} (\mathbf{1}_{2 \times 2}, +\boldsymbol{\sigma}), \end{aligned} \quad (3)$$

and consequently

$$\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix} \implies M_D(L) = \begin{pmatrix} M_L(L) & 0 \\ 0 & M_R(L) \end{pmatrix}. \quad (4)$$

(a) Check that the γ^5 matrix indeed has this form and write down explicit matrices for the $S^{\mu\nu}$ in the Weyl convention.

- (b) Show that the 2×2 matrices $M_L(L)$ and $M_R(L)$ in eq. (4) are precisely the $M(L)$ and the $\bar{M}(L)$ matrices from eqs. (11–12) of the [previous homework](#). Specifically, for a pure rotation by angle θ around axis \mathbf{n} ,

$$M_L = M_R = \exp\left(-\frac{i}{2} \theta \mathbf{n} \cdot \boldsymbol{\sigma}\right), \quad (5)$$

while for a pure Lorentz boost of rapidity r in the direction \mathbf{n} ,

$$M_L(B) = \exp\left(-\frac{1}{2} r \mathbf{n} \cdot \boldsymbol{\sigma}\right) \quad \text{while} \quad M_R(B) = \exp\left(+\frac{1}{2} r \mathbf{n} \cdot \boldsymbol{\sigma}\right). \quad (6)$$

In terms of the β and γ parameters of the boost,

$$M_L = \sqrt{\gamma} \times \sqrt{1 - \beta \mathbf{n} \cdot \boldsymbol{\sigma}}, \quad M_R = \sqrt{\gamma} \times \sqrt{1 + \beta \mathbf{n} \cdot \boldsymbol{\sigma}}. \quad (7)$$

In the Weyl convention for the Dirac matrices,

$$\Psi_{\text{Dirac}}(x) = \begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix} \quad \text{where} \quad \begin{aligned} \psi'_L(x') &= M_L(L) \psi_L(x), \\ \psi'_R(x') &= M_R(L) \psi_R(x). \end{aligned} \quad (8)$$

In other words, the left-handed Weyl spinor field $\psi_L(x)$ is in the $\mathbf{2}$ multiplet of the $\text{Spin}(3, 1) = \text{SL}(2, \mathbf{C})$ symmetry while the right-handed Weyl spinor field $\psi_R(x)$ is in the conjugate $\bar{\mathbf{2}}$ multiplet.

- (c) In the [previous homework](#) (eq. (13)) we saw that $M_R = \sigma_2 \times M_L^* \times \sigma_2$. Use this fact to show that $\sigma_2 \times \psi_L^*(x)$ transforms under continuous Lorentz symmetries like the $\psi_R(x)$, while the $\sigma_2 \times \psi_R^*(x)$ transforms like the $\psi_L(x)$.

Finally, consider the Dirac Lagrangian $\bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi$.

- (d) Express this Lagrangian in terms of the Weyl spinor fields $\psi_L(x)$ and $\psi_R(x)$ (and their conjugates $\psi_L^\dagger(x)$ and $\psi_R^\dagger(x)$).
- (e) Show that for $m = 0$ — and only for $m = 0$ — the two Weyl spinor fields become independent from each other.

3. The third problem is about the plane-wave solutions $e^{-ipx}u(p, s)$ and $e^{+ipx}v(p, s)$ of the Dirac equation. In all these waves $p^0 = +E_{\mathbf{p}} = +\sqrt{\mathbf{p}^2 + m^2}$ while the 4-component spinors $u(p, s)$ and $v(p, s)$ satisfy

$$(\not{p} - m)u(p, s) = 0, \quad (\not{p} + m)v(p, s) = 0 \quad (9)$$

and are normalized to

$$u^\dagger(p, s)u(p, s') = v^\dagger(p, s)v(p, s') = 2E\delta_{s, s'}. \quad (10)$$

Let's write down explicit formulae for these spinors in the Weyl basis for the γ^μ matrices.

- (a) Show that for $\mathbf{p} = 0$,

$$u(\mathbf{p} = \mathbf{0}, s) = \begin{pmatrix} \sqrt{m}\xi_s \\ \sqrt{m}\xi_s \end{pmatrix} \quad (11)$$

where ξ_s is a two-component $SO(3)$ spinor encoding the electron's spin state. The ξ_s are normalized to $\xi_s^\dagger \xi_{s'} = \delta_{s, s'}$.

- (b) For other momenta, $u(p, s) = M_D(\text{boost}) \times u(\mathbf{p} = 0, s)$ for the boost that turns $(m, \vec{0})$ into p^μ . Use eqs. (7) to show that

$$u(p, s) = \begin{pmatrix} \sqrt{E - \mathbf{p} \cdot \boldsymbol{\sigma}} \xi_s \\ \sqrt{E + \mathbf{p} \cdot \boldsymbol{\sigma}} \xi_s \end{pmatrix} = \begin{pmatrix} \sqrt{p_\mu \bar{\sigma}^\mu} \xi_s \\ \sqrt{p_\mu \sigma^\mu} \xi_s \end{pmatrix}. \quad (12)$$

- (c) Use similar arguments to show that

$$v(p, s) = \begin{pmatrix} +\sqrt{E - \mathbf{p} \cdot \boldsymbol{\sigma}} \eta_s \\ -\sqrt{E + \mathbf{p} \cdot \boldsymbol{\sigma}} \eta_s \end{pmatrix} = \begin{pmatrix} +\sqrt{p_\mu \bar{\sigma}^\mu} \eta_s \\ -\sqrt{p_\mu \sigma^\mu} \eta_s \end{pmatrix} \quad (13)$$

where η_s are two-component $SO(3)$ spinors normalized to $\eta_s^\dagger \eta_{s'} = \delta_{s, s'}$.

Physically, the η_s should have opposite spins from ξ_s — the holes in the Dirac sea have opposite spins (as well as p^μ) from the missing negative-energy particles. Mathematically, this requires $\eta_s^\dagger \mathbf{S} \eta_s = -\xi_s^\dagger \mathbf{S} \xi_s$; we may solve this condition by letting $\eta_s = \sigma_2 \xi_s^* = \pm i \xi_{-s}$.

- (d) Check that this is a solution, then show that it leads to $v(p, s) = \gamma^2 u^*(p, s)$.
- (e) Show that for ultra-relativistic electrons or positrons of definite helicity $\lambda = \pm\frac{1}{2}$, the Dirac plane waves become *chiral* — *i.e.*, dominated by one of the two irreducible Weyl spinor components $\psi_L(x)$ or $\psi_R(x)$ of the Dirac spinor $\Psi(x)$, while the other component becomes negligible. Specifically,

$$\begin{aligned} u(p, -\tfrac{1}{2}) &\approx \sqrt{2E} \begin{pmatrix} \xi_L \\ 0 \end{pmatrix}, & u(p, +\tfrac{1}{2}) &\approx \sqrt{2E} \begin{pmatrix} 0 \\ \xi_R \end{pmatrix}, \\ v(p, -\tfrac{1}{2}) &\approx -\sqrt{2E} \begin{pmatrix} 0 \\ \eta_L \end{pmatrix}, & v(p, +\tfrac{1}{2}) &\approx \sqrt{2E} \begin{pmatrix} \eta_R \\ 0 \end{pmatrix}. \end{aligned} \quad (14)$$

Note that for the electrons the helicity and the chirality are both left or both right, but for the positrons the chirality is opposite from the helicity.

Back in problem 2(b) we saw that for $m = 0$ the LH and the RH Weyl spinor fields decouple from each other. Now this exercise show us which particle modes comprise each Weyl spinor: *The $\psi_L(x)$ and its hermitian conjugate $\psi_L^\dagger(x)$ contain the left-handed fermions and the right-handed antifermions, while the $\psi_R(x)$ and the $\psi_R^\dagger(x)$ contain the right-handed fermions and the left-handed antifermions.*

4. Finally, let's establish some basis-independent properties of the Dirac spinors $u(p, s)$ and $v(p, s)$ — although you may use the Weyl basis to verify them.

- (a) Show that

$$\bar{u}(p, s)u(p, s') = +2m\delta_{s,s'}, \quad \bar{v}(p, s)v(p, s') = -2m\delta_{s,s'}; \quad (15)$$

note that the normalization here is different from eq. (10) for the $u^\dagger u$ and $v^\dagger v$.

- (b) There are only two independent $SO(3)$ spinors, hence $\sum_s \xi_s \xi_s^\dagger = \sum_s \eta_s^\dagger \eta_s = \mathbf{1}_{2 \times 2}$. Use this fact to show that

$$\sum_{s=1,2} u_\alpha(p, s) \bar{u}_\beta(p, s) = (\not{p} + m)_{\alpha\beta} \quad \text{and} \quad \sum_{s=1,2} v_\alpha(p, s) \bar{v}_\beta(p, s) = (\not{p} - m)_{\alpha\beta}. \quad (16)$$