PHY–396 K.   Problem set #9.   Due November 10, 2015.

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1. As a warm-up exercise, consider two species of scalar fields, Φ and ϕ, with a cubic coupling to each other,

$$\mathcal{L} \;=\; \frac{1}{2}(\partial_\mu\Phi)^2 \;-\; \frac{M^2}{2}\,\Phi^2 \;+\; \frac{1}{2}(\partial_\mu\phi)^2 \;-\; \frac{m^2}{2}\,\phi^2 \;-\; \frac{\mu}{2}\,\Phi\phi^2. \tag{1}$$

   (a) Write down the vertices and the propagators for the Feynman rules for this theory.

   (b) Suppose $M > 2m$, so a single Φ particle may decay to two ϕ particles. Calculate the rate Γ of this decay (in the rest frame of the original Φ) to lowest order in perturbation theory.

2. Now consider N scalar fields ϕ_i of the same mass m and with $O(N)$ symmetric quartic couplings to each other,

$$\mathcal{L} \;=\; \frac{1}{2}\sum_i(\partial_\mu\phi_i)^2 \;-\; \frac{m^2}{2}\sum_i\phi_i^2 \;-\; \frac{\lambda}{8}\left(\sim_i \phi_i^2\right)^2. \tag{2}$$

   (a) Write down the Feynman propagators and vertices for this theory.

   (b) Calculate the tree-level scattering amplitudes $\mathcal{M}$, the partial cross-sections $d\sigma/d\Omega_{\rm cm}$ (in the center-of-mass frame), and the total cross-sections for the following 3 processes:

      (i) $\phi_1 + \phi_2 \to \phi_1 + \phi_2$.

      (ii) $\phi_1 + \phi_1 \to \phi_2 + \phi_2$.

      (iii) $\phi_1 + \phi_1 \to \phi_1 + \phi_1$.

3. Next, consider the so-called *linear sigma model* comprising N massless scalar or pseudoscalar fields π_i and a massive scalar field σ with both quartic and cubic couplings to the pions, specifically

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}\sum\nolimits_i(\partial_\mu\pi_i)^2 \;+\; \frac{1}{2}(\partial_\mu\sigma)^2 \;-\; \frac{\lambda}{8}\left(\sum\nolimits_i \pi_i^2 \;+\; \sigma^2 \;+\; 2f\sigma\right)^2 \\
&= \frac{1}{2}\sum_i(\partial_\mu\pi_i)^2 \;+\; \frac{1}{2}(\partial_\mu\sigma)^2 \;-\; \frac{\lambda f^2}{2}\times\sigma^2 \\
&\qquad\qquad -\; \frac{\lambda f}{2}\times\left(\sigma^3 \;+\; \sigma\sum\nolimits_i \pi_i^2\right) \;-\; \frac{\lambda}{8}\left(\sum\nolimits_i \pi_i^2 \;+\; \sigma^2\right)^2
\end{aligned} \tag{3}$$

Both the masslessness of the π_i fields and the specific relations between the quartic couplings, the cubic couplings, and the sigma's mass $M_\sigma^2 = \lambda f^2$ in this model stem from the *spontaneous breaking down* of the $O(N+1)$ symmetry, which I shall explain in class later this semester. I shall also explain the relation of this model to the approximate chiral symmetry of QCD and hence to the real-life pi-mesons and their low-energy scattering amplitudes.

But in this homework, you should simply take the Lagrangian (3) as it is, and explore its implications for the scattering of π particles.

(a) Write down all the vertices and all the propagators for the Feynman rules for this theory.

(b) Draw *all* the tree diagrams and calculate the tree-level scattering amplitudes of two pions to two pions, $\mathcal{M}_{\text{tree}}(\pi^j + \pi^k \to \pi^\ell + \pi^m)$.

(c) Show that due to specific relations between the quartic and the cubic couplings in the Lagrangian (3), in the low-energy limit $E_{\text{tot}} \ll M_\sigma$, all the amplitudes $\mathcal{M}_{\text{tree}}(\pi^j + \pi^k \to \pi^\ell + \pi^m)$ become small as $O(E_{\text{tot}}^2/M_\sigma^2)$ or smaller.

Then use Mandelstam's variables s, t, u to show that when any of the incoming or outgoing pions' energy becomes small (while the other pions' energies are $O(M_\sigma)$), the scattering amplitudes become small as $O(E_{\text{small}}/M_\sigma)$ or smaller.

Later in class, we shall learn that this behavior stems from the *Goldstone–Nambu theorem.*

(d) Write down specific tree-level amplitudes, partial cross-sections (in the CM frame), and total cross-sections for the processes

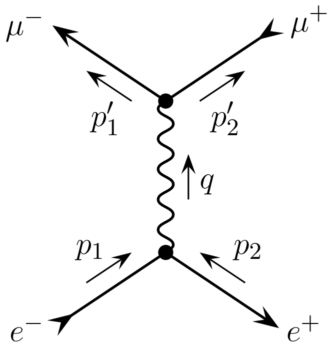
(i) $\pi^1 + \pi^2 \to \pi^1 + \pi^2$

(ii) $\pi^1 + \pi^1 \to \pi^2 + \pi^2$

(iii) $\pi^1 + \pi^1 \to \pi^1 + \pi^1$

in the low-energy limit $E_{\text{cm}} \ll M_\sigma$.

4. For the last problem in this set, consider the muon pair production in QED. At the tree level,



$$\left\langle \mu^-, \mu^+ \right| \mathcal{M} \left| e^-, e^+ \right\rangle \;=\; \frac{e^2}{s} \times \bar{u}(\mu^-)\gamma^\nu v(\mu^+) \times \bar{v}(e^+)\gamma_\nu u(e^-). \tag{4}$$

In 11/5 class we have focused on the un-polarized cross-section for this process; in this exercise we focus on the polarized amplitudes for definite helicities of all 4 particles involved.

For simplicity, let us assume that all the particles are ultra-relativistic so that their Dirac spinors $u(e^-)$, $v(e^+)$, $u(\mu^-)$, $v(\mu^+)$ all have definite chiralities,

$$\begin{aligned} u_L \;&\approx\; \sqrt{2E}\begin{pmatrix} \xi_L \\ 0 \end{pmatrix}, \qquad & u_R \;&\approx\; \sqrt{2E}\begin{pmatrix} 0 \\ \xi_R \end{pmatrix}, \\ v_L \;&\approx\; -\sqrt{2E}\begin{pmatrix} 0 \\ \eta_L \end{pmatrix}, \qquad & v_R \;&\approx\; \sqrt{2E}\begin{pmatrix} \eta_R \\ 0 \end{pmatrix}. \end{aligned} \tag{5}$$

cf. homework set#7, eqs. (11–13).

(a) Show that in the approximation (5),

$$\bar{v}(e^+_L)\gamma_\nu u(e^-_L) \;=\; \bar{v}(e^+_R)\gamma_\nu u(e^-_R) \;=\; 0, \tag{6}$$

which means there is no muon pairs production unless the initial electron and positron have *opposite helicities*.

(b) Show that the μ^- and the μ^+ must also have *opposite helicities* because

$$\bar{u}(\mu^-_L)\gamma^\nu v(\mu^+_L) \;=\; \bar{u}(\mu^-_R)\gamma^\nu v(\mu^+_R) \;=\; 0. \tag{7}$$

(c) Let's work in the center-of-mass frame where the initial e^- and e^+ collide along the z axis, $p_1^\nu = (E, 0, 0, +E)$, $p_2^\nu = (E, 0, 0, -E)$. Calculate the 4–vector $\bar{v}(e^+)\gamma^\nu u(e^-)$ in this frame and show that

$$\bar{v}(e^+_L)\gamma_\nu u(e^-_R) \;=\; 2E \times (0, +i, +1, 0), \qquad \bar{v}(e^+_R)\gamma_\nu u(e^-_L) \;=\; 2E \times (0, -i, +1, 0). \tag{8}$$

(d) In the CM frame the muons fly away in opposite directions at some angle θ to the electron / positron directions. Without loss of generality we may assume the muons' momenta being in the xz plane, thus

$$p_1'^\nu \;=\; (E, +E\sin\theta, 0, +E\cos\theta), \qquad p_1'^\nu \;=\; (E, -E\sin\theta, 0, -E\cos\theta) \tag{9}$$

Calculate the 4–vector $\bar{u}(\mu^-)\gamma_\nu v(\mu^+)$ for the muons and show that

$$\begin{aligned} \bar{u}(\mu^-_R)\gamma^\nu v(\mu^+_L) &= 2E \times (0, -i\cos\theta, +1, +i\sin\theta), \\ \bar{u}(\mu^-_L)\gamma^\nu v(\mu^+_R) &= 2E \times (0, +i\cos\theta, +1, -i\sin\theta). \end{aligned} \tag{10}$$

(e) Now calculate the amplitudes (4) for all possible combinations of particles' helicities, calculate the partial cross-sections, and show that

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\begin{aligned}
\frac{d\sigma(e_L^- + e_R^+ \to \mu_L^- + \mu_R^+)}{d\Omega_{\text{c.m.}}} &= \frac{d\sigma(e_R^- + e_L^+ \to \mu_R^- + \mu_L^+)}{d\Omega_{\text{c.m.}}} = \frac{\alpha^2}{4s} \times (1+\cos\theta)^2, \\
\frac{d\sigma(e_L^- + e_R^+ \to \mu_R^- + \mu_L^+)}{d\Omega_{\text{c.m.}}} &= \frac{d\sigma(e_R^- + e_L^+ \to \mu_L^- + \mu_R^+)}{d\Omega_{\text{c.m.}}} = \frac{\alpha^2}{4s} \times (1-\cos\theta)^2, \\
\frac{d\sigma(e_L^- + e_L^+ \to \mu_{\text{any}}^- + \mu_{\text{any}}^+)}{d\Omega_{\text{c.m.}}} &= \frac{d\sigma(e_R^- + e_R^+ \to \mu_{\text{any}}^- + \mu_{\text{any}}^+)}{d\Omega_{\text{c.m.}}} = 0, \\
\frac{d\sigma(e_{\text{any}}^- + e_{\text{any}}^+ \to \mu_L^- + \mu_L^+)}{d\Omega_{\text{c.m.}}} &= \frac{d\sigma(e_{\text{any}}^- + e_{\text{any}}^+ \to \mu_R^- + \mu_R^+)}{d\Omega_{\text{c.m.}}} = 0.
\end{aligned}
\tag{11}
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(f) Finally, sum / average over the helicities and calculate the un-polarized cross-section for the muon pair production.