1. Consider the elastic scattering $e^-e^+ \to e^-e^+$ of ultra-relativistic electrons and positrons. This process is called the *Bhabha scattering* after Homi Bhabha who has calculated the cross-section back in 1935. His calculation was the leading order in perturbation theory; in modern terms, it corresponds to the three-level of QED. Today, the Bhabha cross-section is known to very high precision, so the observed rate of Bhabha scatterings at electron-positron colliders is used to monitor the collider's luminosity.

   At the tree level of QED, there are two diagrams contributing to the Bhabha scattering, namely

$$\oplus \tag{1}$$
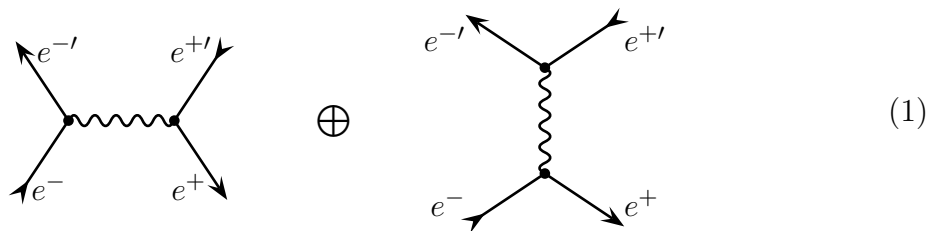
   (a) Evaluate the two diagrams and write down the amplitude $\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2$. Mind the sign rules for the fermions.

   Now comes the real work: calculating the un-polarized partial cross-section

$$\left(\frac{d\sigma}{d\Omega} \right)_{\text{c.m.}} = \frac{\overline{|\mathcal{M}|^2}}{64\pi^2 s} \tag{2}$$

   where $\overline{|\mathcal{M}|^2}$ stands for $|\mathcal{M}|^2$ summed over final particle spins and averaged over the spins of the initial particles. Note the two diagrams (1) must be added together before squaring the amplitude, because

$$|\mathcal{M}_1 + \mathcal{M}_2|^2 = |\mathcal{M}_1|^2 + |\mathcal{M}_2|^2 + 2\operatorname{Re}(\mathcal{M}_1^*\mathcal{M}_2) \neq |\mathcal{M}_1|^2 + |\mathcal{M}_2|^2. \tag{3}$$

   For simplicity, assume $E \gg m_e$ and neglect the electron's mass throughout your calculation. You may find it convenient to express products of momenta in terms of Mandelstam's variables s, t, and u. In the $m_e \approx 0$ approximation, $p_1^2 = p_2^2 = p_1'^2 = p_2'^2 = m_e^2 \approx 0$ while

$$(p_1 p_2) = (p_1' p_2') \approx \tfrac{1}{2}s, \quad (p_1 p_1') = (p_2 p_2') \approx -\tfrac{1}{2}t, \quad (p_1 p_2') = (p_2 p_1') \approx -\tfrac{1}{2}u. \tag{4}$$

(b) Let's start with the second diagram's amplitude $\mathcal{M}_2$. Sum / average the $|\mathcal{M}_2|^2$ over all spins and show that

$$\frac{1}{4}\sum_{\text{all spins}}|\mathcal{M}_2|^2 = 2e^4 \times \frac{t^2 + u^2}{s^2}. \tag{5}$$

(c) Similarly, show that for the first diagram

$$\frac{1}{4}\sum_{\text{all spins}}|\mathcal{M}_1|^2 = 2e^4 \times \frac{s^2 + u^2}{t^2}. \tag{6}$$

(d) Now consider the interference $\mathcal{M}_1^* \times \mathcal{M}_2$ between the two diagrams. Show that

$$\frac{1}{4}\sum_{\text{all spins}}\mathcal{M}_1^* \times \mathcal{M}_2 = 2e^4 \times \frac{u^2}{st}. \tag{7}$$

(e) Finally assemble all the terms together and show that for the Bhabha scattering

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{c.m.}} = \frac{\alpha^2}{2s} \times \frac{s^4 + t^4 + u^4}{s^2 \times t^2} = \frac{\alpha^2}{4s} \times \left(\frac{3 + \cos^2\theta}{1 - \cos\theta}\right)^2. \tag{8}$$

2. Next, consider the $e^+e^- \to \mu^+\mu^-$ pair production in the Standard Model rather than in the pure QED. At the tree level of the Standard Model, there are 3 diagrams contributing to the pair production: one diagram with a virtual photon in the s channel, one with a virtual Z vector, and one with a virtual Higgs scalar,
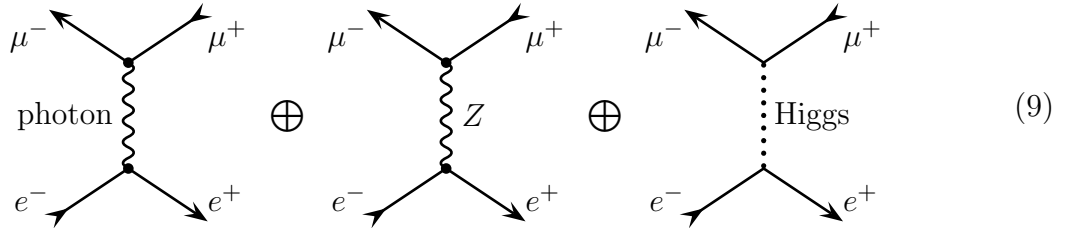
Fortunately, the Higgs field has very weak couplings to electrons and muons, so to a good approximation the third diagram may be neglected relative to the first two diagrams,

$$\mathcal{M}_{\text{tree}} = \mathcal{M}_\gamma + \mathcal{M}_Z + \mathcal{M}_H \approx \mathcal{M}_\gamma + \mathcal{M}_Z. \tag{10}$$

Of the two remaining terms here, the virtual photon dominates at low energies $E \ll M_Z \approx 91$ GeV, but at higher energies both diagrams become equally important.

The diagram with the virtual photon was studied in class in great detail, and also in problem 4 of the previous homework. In this problem, we focus on the virtual–Z diagram and its interference with the virtual-photon diagram.

Later in class I shall explain the Z field and its couplings in great detail, but for now all we need to know is that it's a massive neutral vector field Z^μ with propagator

$$\mu \!\!\sim\!\!\overset{Z}{\sim\!\sim\!\sim}\!\!\nu \;=\; \frac{i}{q^2 - M_Z^2 + i\epsilon} \times \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{M_Z^2}\right), \qquad (11)$$

and that it couples to the charged leptons (e, μ, τ) according to

$$\mathcal{L} \supset g' Z_\mu \times \sum_{\ell=e,\mu,\tau} \overline{\Psi}_\ell \gamma^\mu \left(\sin^2\theta_w - \frac{1-\gamma^5}{4}\right) \Psi_\ell. \qquad (12)$$

In this formula

$$g' = \frac{e}{\sin\theta_w \cos\theta_w} \qquad (13)$$

and θ_w is the Weinberg's weak mixing angle; experimentally $\sin^2\theta_w = 0.232$.

(a) Write down the Feynman rules for the Zee and $Z\mu\mu$ vertices and evaluate the virtual–Z diagram for the muon pair production.

(b) Now let's go the the center-of-mass frame and assume both the electrons and the muons to be ultra-relativistic ($E_{\text{c.m.}} = O(M_Z) \gg m_\mu, m_e$). Evaluate the virtual–$Z$ amplitude $\mathcal{M}_Z$ for all possible helicities of the e^+e^- and $\mu^+\mu^-$.

Use similar techniques to problem 4 of the previous homework and make good use of the chirality of the ultrarelativistic spinors $u(p, \lambda)$ and $v(p, \lambda)$. Note that in the Weyl basis

$$\gamma^0 \gamma^\mu \left(\sin^2\theta_w - \frac{1-\gamma^5}{4}\right) = \begin{pmatrix} g_L \bar{\sigma}^\mu & 0 \\ 0 & g_R \sigma^\mu \end{pmatrix} \qquad \text{where} \quad \begin{aligned} g_L &= \sin^2\theta_w - \tfrac{1}{4}, \\ g_R &= \sin^2\theta_w. \end{aligned}$$

(14)

(c) Combine the amplitudes $\mathcal{M}_Z$ from part (b) with the virtual-photon amplitudes $\mathcal{M}_\gamma$ from problem 4 of the previous homework, and use them to calculate the total cross

section $\sigma(e^+e^- \to \mu^+\mu^-)$ and the forward-backward asymmetry

$$A = \frac{\sigma(\theta < \pi/2) - \sigma(\theta > \pi/2)}{\sigma(\theta < \pi/2) + \sigma(\theta > \pi/2)} \tag{15}$$

as functions of the total energy $E_{\text{c.m.}}$. For simplicity, approximate $\sin^2\theta_w \approx \frac{1}{4}$ and hence $g_R \approx -g_L \approx \frac{1}{4}$.

Note that in QED the tree-level pair production is symmetric with respect to $\theta \to \pi-\theta$; the asymmetry in the Standard Model arises from the interference between the virtual-photon and virtual-Z diagrams.

3. Finally, consider the muon decay. Most of the time, a muon decays into an electron, an electron-flavored antineutrino, and a muon-flavored neutrino, $\mu^- \to e^-\bar{\nu}_e\nu_\mu$. At the tree level of the Standard model, this decay proceeds through a single Feynman diagram

$$\tag{16}$$
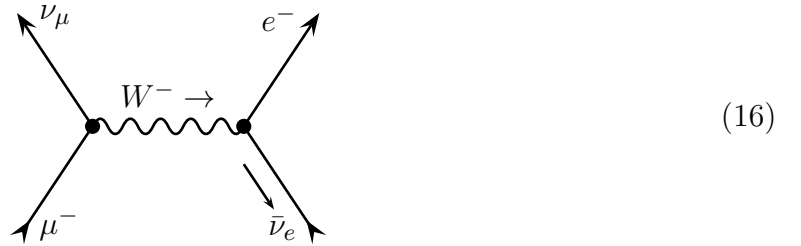
Since all the momenta in this diagram are much smaller than M_W, me may approximate the W propagator as simply $ig^{\kappa\lambda}/M_W^2$. Consequently, the decay amplitude is

$$\left\langle e^-, \bar{\nu}_e, \nu_\mu \right| \mathcal{M} \left| \mu^- \right\rangle \approx \frac{ig^{\kappa\lambda}}{M_W^2} \times \bar{u}(\nu_\mu)\left(-ig_2\gamma_\kappa \frac{1-\gamma^5}{2}\right)u(\mu^-) \times$$

$$\times \bar{u}(e^-)\left(-ig_2\gamma_\lambda \frac{1-\gamma^5}{2}\right)v(\bar{\nu}_e) \tag{17}$$

$$= \frac{G_F}{\sqrt{2}}\left[\bar{u}(\nu_\mu)\gamma^\lambda(1-\gamma^5)u(\mu^-)\right] \times \left[\bar{u}(e^-)\gamma_\lambda(1-\gamma^5)v(\bar{\nu}_e)\right]$$

where $G_F \approx 1.17 \cdot 10^{-5}\,\text{GeV}^{-2}$ is the Fermi constant. In this exercise, you will use this amplitude to calculate the muon's net decay rate Γ and the energy spectrum $d\Gamma/dE_e$ of the final state electrons.

(a) Sum the absolute square of the amplitude (17) over the final particle spins and average over the initial muon's spin. Show that altogether,

$$\tfrac{1}{2}\sum_{\substack{\text{all}\\\text{spins}}}\left|\left\langle e^-,\bar{\nu}_e,\nu_\mu\right|\mathcal{M}\left|\mu^-\right\rangle\right|^2 \;=\; 64G_F^2\,(p_\mu\cdot p_{\bar\nu})\,(p_e\cdot p_\nu). \qquad (18)$$

The rest of this problem is the phase space calculation. The following lemma is very useful for three-body decays like $\mu^- \to e^- + \nu_\mu + \bar{\nu}_e$:

For a decay of initial particle of mass M_0 into three final particles of respective masses m_1, m_2, and m_3, the partial decay rate in the rest frame of the original particle is

$$d\Gamma \;=\; \frac{1}{2M_0}\times\overline{|\mathcal{M}|^2}\times\frac{d^3\Omega}{256\pi^5}\times dE_1\,dE_2\,dE_3\,\delta(E_1+E_2+E_3-M_0), \qquad (19)$$

where $d^3\Omega$ comprises three angular variables parametrizing the directions of the three final-state particles relative to some external frame, but not affecting the angles between the three momenta. For example, one may use two angles to describe the orientation of the decay plane (the three momenta are coplanar, $\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3=0$) and one more angle to fix the direction of e.g., $\mathbf{p}_1$ in that plane. Altogether, $\int d^3\Omega = 4\pi\times 2\pi = 8\pi^2$.

(b) Prove this lemma.

(c) Show that when $m_1 = m_2 = m_3 = 0$, the kinematically allowed range of the final particles' energies is given by

$$0 \;\le\; E_1, E_2, E_3 \;\le\; \tfrac{1}{2}M_0 \qquad\text{while}\quad E_1 + E_2 + E_3 \;=\; M_0, \qquad (20)$$

but for the non-zero masses $m_{1,2,3}$ this range is much more complicated.

Note that the electron and the neutrinos are much lighter then the muon, so in most decay events all three final-state particles are ultra-relativistic. This allows us to approximate $m_e \approx m_\nu \approx m_{\bar\nu} \approx 0$, which gives us the limits (20) for the final particles' energies.

Experimentally, the neutrinos and the antineutrinos are hard to detect. But it is easy to measure the muon's net decay rate $\Gamma = 1/\tau_\mu$ and the energy distribution $d\Gamma/dE_e$ of the electrons produced by decaying muons.

(d) Integrate the muon's partial decay rate over the final particle energies and derive first the $d\Gamma/dE_e$ and then the total decay rate.