1. First, read *carefully* my notes on annihilation and Compton scattering and pay attention to the algebra. Make sure you understand and can follow all the calculations.

2. Consider a QFT where heavy (*i.e.*, $M_s \gg m_e$) neutral scalar particles have Yukawa–like coupling to electrons, which in turn couple to photons according to the usual QED rules, thus

$$\mathcal{L} \;=\; -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \;+\; \overline{\Psi}(i\not{D} - m_e)\Psi + \;\left[\tfrac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi \;-\; \tfrac{1}{2}M_s^2\varphi^2\right] \;+\; g\varphi\times\overline{\Psi}\Psi. \tag{1}$$

In this theory, an electron and a positron colliding with energy $E_{\text{c.m.}} > M_s$ may annihilate into one photon and one scalar particle, $e^- + e^+ \to \gamma + S$.

(a) Draw tree diagrams for the $e^- + e^+ \to \gamma + S$ process and write down the tree-level matrix element $\langle\gamma S|\,\mathcal{M}\,\big|e^-e^+\big\rangle$.

(b) Verify the Ward identity for the photon.

(c) Sum $|\mathcal{M}|^2$ over the photon's polarizations and average over the fermion's spins. For simplicity, neglect the electron's mass. But don't neglect the mass of the scalar.

Note: because of the scalar's mass, the kinematic relations between Mandelstam's $s$, $t$, and $u$ and between momentum products such as $(k_\gamma p_\mp)$, *etc.*, are different from the $e^+e^- \to \gamma\gamma$ annihilation.

(d) Finally, work out the kinematics in the CM frame and calculate the partial cross-section

$$\frac{d\sigma(e^-e^+ \to \gamma S)}{d\Omega_{\text{c.m.}}}\,.$$

3. When an *exact* symmetry of a quantum field theory is spontaneously broken down, it gives rise to exactly massless Goldstone bosons. But when the spontaneously broken symmetry was only approximate to begin with, the would-be Goldstone bosons are no longer exactly massless but only relatively light. The best-known examples of such pseudo-Goldstone bosons are the pi-mesons $\pi^\pm$ and $\pi^0$, which are indeed much lighter then other hadrons. The Quantum ChromoDynamics theory (QCD) of strong interactions has an approximate chiral isospin symmetry $SU(2)_L \times SU(2)_R \cong \mathrm{Spin}(4)$. This symmetry would be exact if the two lightest quark flavors $u$ and $d$ were massless; in real life, the masses $m_u$ and $m_d$ are small but non quite zero, and the symmetry is only approximate. Somehow (and people are still arguing how), the chiral isospin symmetry is spontaneously broken down to the ordinary isospin symmetry $SU(2) \cong \mathrm{Spin}(3)$, and the 3 generators of the broken $\mathrm{Spin}(4)/\mathrm{Spin}(3)$ give rise to 3 (pseudo) Goldstone bosons $\pi^\pm$ and $\pi^0$.

   As a toy model of *approximate* $SO(N+1)$ symmetry spontaneously broken down to $SO(N)$, consider the *linear sigma model* of $N+1$ scalar fields $\phi_i$ with the Lagrangian

$$\mathcal{L} \;=\; \sum_i \tfrac{1}{2}(\partial_\mu\phi_i)^2 \;-\; \frac{\lambda}{8}\left(\sum_i \phi_i^2 \,-\, f^2\right)^2 \;+\; \beta\lambda f^2 \times \phi_{N+1}\,. \tag{2}$$

   For $\beta = 0$ this Lagrangian has exact $O(N+1)$ symmetry, which would be spontaneously broken down to $O(N)$ by non-zero vacuum expectation values of the scalar fields. For a non-zero $\beta$, the last term in the Lagrangian (2) *explicitly* breaks the $O(N+1)$ symmetry, but for $\beta \ll f$ we may treat the $O(N+1)$ as *approximate* symmetry.

   (a) Assume $\beta > 0$ and $\beta \ll f$. Show that the scalar potential of the linear sigma model has a unique minimum at

$$\langle\phi_1\rangle \;=\; \cdots\; \langle\phi_N\rangle \;=\; 0, \quad \left\langle\phi_{N+1}\right\rangle \;=\; f \;+\; \beta \;+\; O(\beta^2/f). \tag{3}$$

   (b) Re-express the Lagrangian (2) in terms of the shifted fields

$$\sigma(x) \;=\; \phi_{N+1}(x) \;-\; \left\langle\phi_{N+1}\right\rangle, \qquad \pi^i(x) \;=\; \phi_i(x) \quad \text{for } i = 1,\ldots,N. \tag{4}$$

   and show that the $\pi^i$ fields are massive but much lighter than the $\sigma$ field. Specifically, $M_\pi^2 \approx \lambda f \times \beta$ while $M_\sigma^2 \approx \lambda f(f+3\beta) \approx \lambda f^2 \gg M_\pi^2$.

In QCD terms, $N = 3$, the three $\pi^{1,2,3}$ fields (or rather the $\pi^0 = \pi^3$ and the $\pi^\pm = (\pi^1 \pm i\pi^2)/\sqrt{2}$) correspond to the three pi-mesons of rather small mass $m_\pi \approx 140$ MeV, and the $\sigma$ corresponds to the very broad sigma resonance at about 500 MeV.

(c) Now consider the pion scattering $\pi\pi \to \pi\pi$ in the linear sigma model. Show that for $\beta = 0$, the quartic couplings, the cubic couplings, and the masses of the $\pi^i$ and $\sigma$ fields are precisely as in problem 3 of homework set#9 (eq. (3)). Therefore — as we saw in that homework — for low-energy pions with $E \ll M_\sigma$, the scattering amplitudes $\mathcal{M}(\pi^j + \pi^k \to \pi^\ell + \pi^m)$ become small as $O(\lambda E_{\rm cm}^2/M_\sigma^2)$ or smaller.

(d) For $\beta \neq 0$, the cubic coupling and the $M_\sigma^2$ are a bit different from what we had in homework#9, so different (tree) diagrams contributing to the scattering of low-energy pions do not quite cancel each other.

Show that to the leading order in $\beta$, for $s, t, u \ll M_\sigma$,

$$\mathcal{M}(\pi^j + \pi^k \to \pi^\ell + \pi^m) \;\approx\; \frac{1}{f^2}\begin{pmatrix} (s - m_\pi^2) \times \delta^{jk}\delta^{\ell m} \;+\; (t - m_\pi^2) \times \delta^{j\ell}\delta^{km} \\ \qquad\qquad + \; (u - m_\pi^2) \times \delta^{jm}\delta^{k\ell} \end{pmatrix}, \tag{5}$$

which does not vanish when any of the pion's momenta becomes small. Instead, for *slow* pions with $|\mathbf{p}| \ll m_\pi$, this amplitude becomes

$$\mathcal{M}(\pi^j + \pi^k \to \pi^\ell + \pi^m) \;\approx\; \left(3\delta^{jk}\delta^{\ell m} - \delta^{j\ell}\delta^{km} - \delta^{jm}\delta^{k\ell}\right) \times \left(\frac{m_\pi^2}{f^2} \approx \frac{\lambda\beta}{f}\right) \neq 0. \tag{6}$$

4. In this problem, the spontaneously broken symmetry is exact but more complicated. Consider an $N \times N$ matrix $\Phi(x)$ of complex scalar fields $\Phi^i{}_j(x)$, $i, j = 1, \ldots, N$. In matrix notations, the Lagrangian is

$$\mathcal{L} \;=\; \mathrm{tr}\left(\partial^\mu \Phi^\dagger\, \partial_\mu \Phi\right) \;-\; V(\Phi^\dagger \Phi) \tag{7}$$

where the potential is

$$V \;=\; \frac{\alpha}{2}\,\mathrm{tr}\left(\Phi^\dagger\Phi\Phi^\dagger\Phi\right) \;+\; \frac{\beta}{2}\left(\mathrm{tr}\left(\Phi^\dagger\Phi\right)\right)^2 \;+\; m^2\,\mathrm{tr}\left(\Phi^\dagger\Phi\right). \tag{8}$$

(a) Show that this theory has global symmetry group $G = SU(N)_L \times SU(N)_R \times U(1)$ acting as

$$\Phi(x) \;\to\; e^{i\theta} U_L \Phi(x) U_R^\dagger, \qquad U_L, U_R \in SU(N). \tag{9}$$

($\star$) *Optional exercise, only for experts in group theory:*
Show that the theory has no other continuous symmetries besides $G$ and Poincare (Lorentz and translations of spacetime).

From now on, we take $\alpha, \beta > 0$ but $m^2 < 0$. In this regime, $V$ is minimized for non-zero vacuum expectation values $\langle\Phi\rangle \neq 0$ of the scalar fields.

(b) Let $(\kappa_1, \ldots, \kappa_N)$ be eigenvalues of the hermitian matrix $\Phi^\dagger\Phi$. Express the potential (8) in terms of these eigenvalues and show that the minimum lies at

$$\kappa_1 \;=\; \kappa_2 \;=\; \cdots \;=\; \kappa_N \;=\; C^2 \;=\; \frac{-m^2}{\alpha + N\beta} \;>\; 0. \tag{10}$$

In terms of the matrix $\Phi$, eq. (10) means $\Phi = C \times$ a unitary matrix. All such minima are related by symmetries (9) to $\Phi = C \times$ the unit matrix, so without loss of generality we may assume that the vacuum lies at

$$\langle\Phi\rangle \;=\; C \times \mathbf{1}_{N\times N} \quad i.e. \quad \left\langle \Phi^i{}_j \right\rangle \;=\; C \times \delta^i_j\,. \tag{11}$$

(c) Show that the symmetries (9) preserving these VEVs are limited to the $U_L = U_R \in SU(N)$ and $\theta = 0$. In other words, the $SU(N) \times SU(N) \times U(1)$ symmetry of the theory is spontaneously broken down to $SU(N)$.

Let's expand the theory around the vacuum (11). For convenience, let's also decompose the complex matrix $\Phi$ into its hermitian and anti-hermitian parts,

$$\Phi(x) \;=\; C \times \mathbf{1}_{N\times N} \;+\; \frac{\varphi_1(x) \,+\, i\,\varphi_2(x)}{\sqrt{2}} \qquad \text{where } \varphi_1^\dagger \equiv \varphi_1 \text{ and } \varphi_2^\dagger \equiv \varphi_2\,. \tag{12}$$

(d) Expand the Lagrangian in powers of $\varphi_1$ and $\varphi_2$ and use the quadratic part $\mathcal{L}_2$ to determine the particle spectrum of the theory.

(e) Altogether, the $N^2$ complex scalar fields give rise to $2N^2$ particle species. Organize these particles into multiplets of the unbroken $SU(N)$ symmetry and make sure that all members of each multiple have the same mass.

Also, check the Nambu–Goldstone theorem for this model — verify that for each *spontaneously broken* generator of the symmetry (9) there is a massless particle with similar quantum numbers (WRT the unbroken $SU(N)$).