1. In class, I have focused on the *fundamental multiplet* of the local $SU(N)$ symmetry, *i.e.*, a set of N fields (complex scalars or Dirac fermions) which transform as a complex N–vector,

$$\Psi'(x) \;=\; U(x)\Psi(x) \quad i.e. \quad \Psi'_i(x) \;= \sum_j U_i^{\;j}(x)\Psi_j(x), \quad i,j = 1,2,\ldots,N \tag{1}$$

where $U(x)$ is an x–dependent unitary $N\times N$ matrix, $\det U(x) \equiv 1$. Now consider N^2-1 real fields $\Phi^a(x)$ forming an *adjoint multiplet*: In matrix form

$$\Phi(x) \;=\; \sum_a \Phi^a(x) \times \frac{\lambda^a}{2} \tag{2}$$

is a traceless hermitian $N\times N$ matrix which transforms under the local $SU(N)$ symmetry as

$$\Phi'(x) \;=\; U(x)\Phi(x)U^\dagger(x). \tag{3}$$

Note that this transformation law preserves the $\Phi^\dagger = \Phi$ and $\operatorname{tr}(\Phi) = 0$ conditions.

The covariant derivatives D_μ act on an adjoint multiplet of fields as

$$D_\mu\Phi(x) \;=\; \partial_\mu\Phi(x) \;+\; i[\mathcal{A}_\mu(x),\Phi(x)] \;\equiv\; \partial_\mu\Phi(x) \;+\; i\mathcal{A}_\mu(x)\Phi(x) \;-\; i\Phi(x)\mathcal{A}_\mu(x), \tag{4}$$

or in components

$$D_\mu\Phi^a(x) = \;\partial_\mu\Phi_a(x) \;-\; f^{abc}\mathcal{A}^b_\mu(x)\Phi^c(x). \tag{5}$$

(a) Verify that these derivatives are indeed covariant — the $D_\mu\Phi(x)$ transforms under the local $SU(N)$ symmetry exactly like the $\Phi(x)$ itself.

(b) Verify the Leibniz rule for covariant derivatives of matrix products. Let $\Phi(x)$ and $\Xi(x)$ be two adjoint multiplets while $\Psi(x)$ is a fundamental multiplet and $\Psi^\dagger(x)$ is its hermitian conjugate (row vector of Ψ_i^*). Show that

$$\begin{aligned} D_\mu(\Phi\Xi) &= (D_\mu\Phi)\Xi + \Phi(D_\mu\Xi), \\ D_\mu(\Phi\Psi) &= (D_\mu\Phi)\Psi + \Phi(D_\mu\Psi), \\ D_\mu(\Psi^\dagger\Xi) &= (D_\mu\Psi^\dagger)\Xi + \Psi^\dagger(D_\mu\Xi). \end{aligned} \tag{6}$$

(c) Show that for an adjoint multiplet $\Phi(x)$,

$$[D_\mu, D_\nu]\Phi(x) = i[\mathcal{F}_{\mu\nu}(x), \Phi(x)] = ig[F_{\mu\nu}(x), \Phi(x)] \tag{7}$$

or in components $[D_\mu, D_\nu]\Phi^a(x) = -gf^{abc}F^b_{\mu\nu}(x)\Phi^c(x)$.

- In my notations A_μ and $F_{\mu\nu}$ are canonically normalized fields while $\mathcal{A}_\mu = gA_\mu$ and $\mathcal{F}_{\mu\nu} = gF_{\mu\nu}$ are normalized by the symmetry action.

In class, I have argued (using covariant derivatives) that the tension fields $\mathcal{F}_{\mu\nu}(x)$ themselves transform according to eq. (3). In other words, the $\mathcal{F}^a_{\mu\nu}(x)$ form an adjoint multiplet of the $SU(N)$ symmetry group.

(d) Verify the $\mathcal{F}'_{\mu\nu}(x) = U(x)\mathcal{F}_{\mu\nu}(x)U^\dagger(x)$ transformation law directly from the definition $\mathcal{F}_{\mu\nu} \stackrel{\text{def}}{=} \partial_\mu\mathcal{A}_\nu - \partial_\mu\mathcal{A}_\nu + i[\mathcal{A}_\mu, \mathcal{A}_\nu]$ and the non-abelian gauge transform of the $\mathcal{A}_\mu$ fields.

(e) Verify the Bianchi identity for the non-abelian tension fields $\mathcal{F}_{\mu\nu}(x)$:

$$D_\lambda\mathcal{F}_{\mu\nu} + D_\mu\mathcal{F}_{\nu\lambda} + D_\nu\mathcal{F}_{\lambda\mu} = 0. \tag{8}$$

Note the covariant derivatives in this equation.

Finally, consider the $SU(N)$ Yang–Mills theory — the non-abelian gauge theory that does not have any fields except $\mathcal{A}^a(x)$ and $\mathcal{F}^a(x)$; its Lagrangian is

$$\mathcal{L}_{\text{YM}} = -\frac{1}{2g^2}\,\text{tr}\big(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\big) = \sum_a \tfrac{-1}{4}F^a_{\mu\nu}F^{a\mu\nu}. \tag{9}$$

(f) Show that the Euler–Lagrange field equations for the Yang–Mills theory can be written in covariant form as $D_\mu\mathcal{F}^{\mu\nu} = 0$.
Hint: first show that for an infinitesimal variation $\delta\mathcal{A}_\mu(x)$ of the non-abelian gauge fields, the tension fields vary according to $\delta\mathcal{F}_{\mu\nu}(x) = D_\mu\delta\mathcal{A}_\nu(x) - D_\nu\delta\mathcal{A}_\mu(x)$.

2. Continuing the previous problem, consider an $SU(N)$ gauge theory in which N^2-1 vector fields $A^a_\mu(x)$ interact with some "matter" fields $\phi_\alpha(x)$,

$$\mathcal{L} \;=\; -\frac{1}{2g^2}\,\mathrm{tr}\big(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\big) \;+\; \mathcal{L}_{\mathrm{mat}}(\phi, D_\mu\phi). \tag{10}$$

For the moment, let me keep the matter fields completely generic — they can be scalars, or vectors, or spinors, or whatever, and form any kind of a multiplet of the local $SU(N)$ symmetry as long as such multiplet is complete and non-trivial. All we need to know right now is that there are well-defined covariant derivatives $D_\mu\phi$ that depend on the gauge fields A^a_μ, which give rise to the currents

$$J^{a\mu} \;=\; -\frac{\partial\mathcal{L}_{\mathrm{mat}}}{\partial A^a_\mu}\,. \tag{11}$$

Collectively, these N^2-1 currents should form an adjoint multiplet $J^\mu = \sum_a(\frac{1}{2}\lambda^a)J^{a\mu}$ of the $SU(N)$ symmetry.

(a) Show that in this theory the equation of motion for the A^a_μ fields are $D_\mu F^{a\mu\nu} = J^{a\nu}$ and that consistency of these equations requires the currents to be *covariantly conserved,*

$$D_\mu J^\mu \;=\; \partial_\mu J^\mu \;+\; i[\mathcal{A}_\mu, J^\mu] \;=\; 0, \tag{12}$$

or in components, $\partial_\mu J^{a\mu} - f^{abc}\mathcal{A}^b_\mu J^{c\mu} = 0$.

Note: a covariantly conserved current does *not* lead to a conserved charge, $(d/dt)\int d^3\mathbf{x}\, J^{a0}(\mathbf{x},t) \neq 0$!

Now consider a simple example of matter fields — a fundamental multiplet $\Psi(x)$ of N Dirac fermions $\Psi_i(x)$, with a Lagrangian

$$\mathcal{L}_{\mathrm{mat}} \;=\; \overline{\Psi}\big(i\gamma^\mu D_\mu \;-\; m\big)\Psi, \qquad \mathcal{L}_{\mathrm{net}} \;=\; \mathcal{L}_{\mathrm{mat}} \;-\; \frac{1}{2g^2}\,\mathrm{tr}\big(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\big). \tag{13}$$

(b) Derive the $SU(N)$ currents $J^{a\mu}$ for these fermions and verify that under $SU(N)$ symmetries the currents transform covariantly into each other as members of the adjoint multiplet. That is, the $N \times N$ matrix $J^\mu = \sum_a (\frac{1}{2}\lambda^a) J^{a\mu}$ transforms according to eq. (3).

Hint: for any complex N–vectors ξ_i and η_j,

$$\sum_a \bigl(\eta^\dagger \lambda^a \xi\bigr) \times \bigl(\lambda^a\bigr)_i^{\ j} \;=\; 2\,\eta^{*j}\xi_i \;-\; \frac{2}{N}\bigl(\eta^\dagger \xi\bigr) \times \delta_i^j \,.. \tag{14}$$

(c) Finally, verify the covariant conservation $D_\mu J^{a\mu} = 0$ of these currents when the fermionic fields $\Psi_i(x)$ and $\overline{\Psi}^i(x)$ obey their equations of motion.

3. This problem is about general multiplets of general gauge groups. Consider a Lie group G with generators $\hat{T}^a$ obeying commutation relations $[\hat{T}^a, \hat{T}^b] = i f^{abc} \hat{T}^c$. Under an infinitesimal local symmetry

$$\mathcal{G}(x) \;=\; 1 \;+\; i\Lambda^a(x)\hat{T}^a \;+\; \cdots, \quad \text{infinitesimal } \Lambda^a(x), \tag{15}$$

the gauge fields $\mathcal{A}_\mu^a(x)$ transform as

$$\mathcal{A}_\mu^a(x) \;\to\; \mathcal{A}_\mu^a(x) \;-\; D_\mu \Lambda^a(x) \;=\; \mathcal{A}_\mu^a(x) \;-\; \partial_\mu \Lambda^a(x) \;-\; f^{abc}\Lambda^b(x)\mathcal{A}_\mu^c(x). \tag{16}$$

Other fields of the gauge theory (scalar, spinor, or whatever) must form complete multiplets of the gauge group G. In any such multiplet (m), the generators $\hat{T}^a$ are represented by size$(m) \times$ size(m) matrices $(T^a_{(m)})_\alpha^{\ \beta}$ satisfying similar commutation relations, $[T^a_{(m)}, T^b_{(m)}] = i f^{abc} T^c_{(m)}$. The fields $\Psi_\alpha(x)$ belonging to such multiplet transform under infinitesimal gauge transforms (15) as

$$\Psi_\alpha(x) \;\to\; \Psi_\alpha(x) \;+\; i\Lambda^a(x)(T^a_{(m)})_\alpha^{\ \beta}\Psi_\beta(x) \tag{17}$$

and the covariant derivatives D_μ act on these fields as

$$D_\mu \Psi_\alpha(x) \;=\; \partial_\mu \Psi_\alpha(x) \;+\; i\mathcal{A}_\mu^a(x)(T^a_{(m)})_\alpha^{\ \beta}\Psi_\beta(x). \tag{18}$$

- Verify covariance of these derivatives under infinitesimal gauge transforms (15).

⋆ For extra challenge, only for the students familiar with the basic theory of Lie groups: Prove covariance of the derivatives (18) under finite gauge transforms.
Hint: use Lemma on the next page.

Lemma: For any finite symmetry $\mathcal{G} \in G$, the matrix $\bigl(R_{(m)}(\mathcal{G})\bigr)_\alpha^{\ \beta}$ representing this symmetry in the multiplet (m) satisfies

$$\bigl(R_{(m)}(\mathcal{G})\bigr)_\alpha^{\ \beta}\bigl(T^a_{(m)}\bigr)_\beta^{\ \gamma}\bigl(R^{-1}_{(m)}(\mathcal{G})\bigr)_\gamma^{\ \delta} \;=\; \bigl(T^b_{(m)}\bigr)_\alpha^{\ \delta} R^{ba}_{\rm adj}(\mathcal{G}) \tag{19}$$

where $R^{ba}_{\rm adj}(\mathcal{G})$ represents $\mathcal{G}$ in the adjoint multiplet. Note that the same $R^{ba}_{\rm adj}(\mathcal{G})$ appears on right hand sides of eqs. (19) for all multiplets (m) of G — and that's what allows us to use the same gauge fields $\mathcal{A}^a_\mu(x)$ to make covariant derivatives (18) for all multiplets of the gauge group G.

4. In the previous homework (set#11, problem#4), we had continuous global symmetry $G = SU(N)_L \times SU(N)_R \times U(1)$ spontaneously broken down to $H = SU(N)_V$. Now let's gauge the entire $SU(N)_L\times SU(N)_R\times U(1)$ symmetry and work out the Higgs mechanism.

   The present theory comprises N^2 complex scalar fields $\Phi_i^{\ j}(x)$ organized into an $N \times N$ matrix, and $2N^2 - 1$ real vector fields $B_\mu(x)$, $L^a_\mu(x)$, and $R^a_\mu(x)$, the latter organized into traceless hermitian matrices $L_\mu(x) = \sum_a L^a_\mu(x) \times \frac{1}{2}\lambda^a$ and $R_\mu(x) = \sum_a R^a_\mu(x) \times \frac{1}{2}\lambda^a$, where $a = 1, \ldots, (N^2-1)$ and λ^a are the Gell-Mann matrices of $SU(N)$. The Lagrangian is

$$\mathcal{L} \;=\; -\tfrac{1}{4}B_{\mu\nu}B^{\mu\nu} \;-\; \tfrac{1}{2}\,{\rm tr}\,(L_{\mu\nu}L^{\mu\nu}) \;-\; \tfrac{1}{2}\,{\rm tr}\,(R_{\mu\nu}R^{\mu\nu}) \;+\; {\rm tr}\,\Bigl(D^\mu\Phi^\dagger\, D_\mu\Phi\Bigr) - V(\Phi^\dagger\Phi), \tag{20}$$

   where

$$\begin{aligned} B_{\mu\nu} &= \partial_\mu B_\nu \;-\; \partial_\nu B_\mu\,, \\ L_{\mu\nu} &= \partial_\mu L_\nu \;-\; \partial_\nu L_\mu \;+\; ig[L_\mu, L_\nu], \\ R_{\mu\nu} &= \partial_\mu R_\nu \;-\; \partial_\nu R_\mu \;+\; ig[R_\mu, R_\nu], \\ D_\mu\Phi &= \partial_\mu\Phi \;+\; ig'B_\mu\Phi \;+\; igL_\mu\Phi \;-\; ig\Phi R_\mu\,, \\ D_\mu\Phi^\dagger &= (D_\mu\Phi)^\dagger \;=\; \partial_\mu\Phi^\dagger \;-\; ig'B_\mu\Phi^\dagger \;+\; igR_\mu\Phi^\dagger \;-\; ig\Phi^\dagger L_\mu\,. \end{aligned} \tag{21}$$

   For simplicity, I assume equal gauge couplings $g_L = g_R = g$ for the two $SU(N)$ factors of the gauge group, but the abelian coupling g' is different.

The scalar potential V is precisely as in the previous homework,

$$V = \frac{\alpha}{2}\,\mathrm{tr}\big(\Phi^\dagger\Phi\Phi^\dagger\Phi\big) + \frac{\beta}{2}\,\mathrm{tr}^2\big(\Phi^\dagger\Phi\big) + m^2\,\mathrm{tr}\big(\Phi^\dagger\Phi\big), \qquad \alpha,\beta>0, \quad m^2<0, \tag{22}$$

hence similar VEVs of the scalar fields: up to a gauge symmetry,

$$\langle\Phi\rangle = C\times\mathbf{1}_{N\times N} \quad \text{where} \quad C = \sqrt{\frac{-m^2}{\alpha+N\beta}}\,, \tag{23}$$

which breaks the $G = SU(N)_L\times SU(N)_R\times U(1)$ down to $SU(N)_V$.

(a) The Higgs mechanism makes N^2 out of $2N^2-1$ vector fields massive. Calculate their masses by plugging $\langle\Phi\rangle$ for the $\Phi(x)$ into the $\mathrm{tr}(D_\mu\Phi^\dagger D^\mu\Phi)$ term of the Lagrangian. In particular, show that the abelian gauge field B_μ and the $X^a_\mu = \frac{1}{\sqrt{2}}(L^a_\mu - R^a_\mu)$ combinations of the $SU(N)$ gauge fields become massive, while the $V^a_\mu = \frac{1}{\sqrt{2}}(L^a_\mu + R^a_\mu)$ combinations remain massless.

(b) Find the effective Lagrangian for the massless vector fields $V^a_\mu(x)$ by freezing all the other fields, *i.e.* setting $\Phi(x)\equiv\langle\Phi\rangle$, $B_\mu(x)\equiv 0$ and $X^a_\mu(x)\equiv 0$. Show that this Lagrangian describes a Yang–Mills theory with gauge group $SU(N)_V$ and gauge coupling $g_V = g/\sqrt{2}$.

⋆ For extra challenge, allow for un-equal gauge coulings $g_L\neq g_R$. Find which combinations of the $L^a_\mu(x)$ and $R^a_\mu(x)$ fields remain massless in this case, then derive the effective Lagrangian for these massless fields by freezing everything else. As in part (b), you should get an $SU(N)$ Yang–Mills theory, but this time the gauge coupling is

$$g_v = \frac{g_L g_R}{\sqrt{g_L^2+g_R^2}}\,. \tag{24}$$