

1. In class, I have focused on the *fundamental multiplet* of the local  $SU(N)$  symmetry, *i.e.*, a set of  $N$  fields (complex scalars or Dirac fermions) which transform as a complex  $N$ -vector,

$$\Psi'(x) = U(x)\Psi(x) \quad \text{i.e.} \quad \Psi'_i(x) = \sum_j U_i^j(x)\Psi_j(x), \quad i, j = 1, 2, \dots, N \quad (1)$$

where  $U(x)$  is an  $x$ -dependent unitary  $N \times N$  matrix,  $\det U(x) \equiv 1$ . Now consider  $N^2 - 1$  real fields  $\Phi^a(x)$  forming an *adjoint multiplet*: In matrix form

$$\Phi(x) = \sum_a \Phi^a(x) \times \frac{\lambda^a}{2} \quad (2)$$

is a traceless hermitian  $N \times N$  matrix which transforms under the local  $SU(N)$  symmetry as

$$\Phi'(x) = U(x)\Phi(x)U^\dagger(x). \quad (3)$$

Note that this transformation law preserves the  $\Phi^\dagger = \Phi$  and  $\text{tr}(\Phi) = 0$  conditions.

The covariant derivatives  $D_\mu$  act on an adjoint multiplet of fields as

$$D_\mu\Phi(x) = \partial_\mu\Phi(x) + i[\mathcal{A}_\mu(x), \Phi(x)] \equiv \partial_\mu\Phi(x) + i\mathcal{A}_\mu(x)\Phi(x) - i\Phi(x)\mathcal{A}_\mu(x), \quad (4)$$

or in components

$$D_\mu\Phi^a(x) = \partial_\mu\Phi^a(x) - f^{abc}\mathcal{A}_\mu^b(x)\Phi^c(x). \quad (5)$$

- (a) Verify that these derivatives are indeed covariant — the  $D_\mu\Phi(x)$  transforms under the local  $SU(N)$  symmetry exactly like the  $\Phi(x)$  itself.
- (b) Verify the Leibniz rule for covariant derivatives of matrix products. Let  $\Phi(x)$  and  $\Xi(x)$  be two adjoint multiplets while  $\Psi(x)$  is a fundamental multiplet and  $\Psi^\dagger(x)$  is

its hermitian conjugate (row vector of  $\Psi_i^*$ ). Show that

$$\begin{aligned} D_\mu(\Phi\xi) &= (D_\mu\Phi)\xi + \Phi(D_\mu\xi), \\ D_\mu(\Phi\Psi) &= (D_\mu\Phi)\Psi + \Phi(D_\mu\Psi), \\ D_\mu(\Psi^\dagger\xi) &= (D_\mu\Psi^\dagger)\xi + \Psi^\dagger(D_\mu\xi). \end{aligned} \tag{6}$$

(c) Show that for an adjoint multiplet  $\Phi(x)$ ,

$$[D_\mu, D_\nu]\Phi(x) = i[\mathcal{F}_{\mu\nu}(x), \Phi(x)] = ig[F_{\mu\nu}(x), \Phi(x)] \tag{7}$$

or in components  $[D_\mu, D_\nu]\Phi^a(x) = -gf^{abc}F_{\mu\nu}^b(x)\Phi^c(x)$ .

- In my notations  $A_\mu$  and  $F_{\mu\nu}$  are canonically normalized fields while  $\mathcal{A}_\mu = gA_\mu$  and  $\mathcal{F}_{\mu\nu} = gF_{\mu\nu}$  are normalized by the symmetry action.

In class, I have argued (using covariant derivatives) that the tension fields  $\mathcal{F}_{\mu\nu}(x)$  themselves transform according to eq. (3). In other words, the  $\mathcal{F}_{\mu\nu}^a(x)$  form an adjoint multiplet of the  $SU(N)$  symmetry group.

(d) Verify the  $\mathcal{F}'_{\mu\nu}(x) = U(x)\mathcal{F}_{\mu\nu}(x)U^\dagger(x)$  transformation law directly from the definition  $\mathcal{F}_{\mu\nu} \stackrel{\text{def}}{=} \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu + i[\mathcal{A}_\mu, \mathcal{A}_\nu]$  and the non-abelian gauge transform of the  $\mathcal{A}_\mu$  fields.

(e) Verify the Bianchi identity for the non-abelian tension fields  $\mathcal{F}_{\mu\nu}(x)$ :

$$D_\lambda\mathcal{F}_{\mu\nu} + D_\mu\mathcal{F}_{\nu\lambda} + D_\nu\mathcal{F}_{\lambda\mu} = 0. \tag{8}$$

Note the covariant derivatives in this equation.

Finally, consider the  $SU(N)$  Yang–Mills theory — the non-abelian gauge theory that does not have any fields except  $\mathcal{A}^a(x)$  and  $\mathcal{F}^a(x)$ ; its Lagrangian is

$$\mathcal{L}_{\text{YM}} = -\frac{1}{2g^2} \text{tr}(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}) = \sum_a \frac{-1}{4} F_{\mu\nu}^a F^{a\mu\nu}. \tag{9}$$

(f) Show that the Euler–Lagrange field equations for the Yang–Mills theory can be written in covariant form as  $D_\mu\mathcal{F}^{\mu\nu} = 0$ .

Hint: first show that for an infinitesimal variation  $\delta\mathcal{A}_\mu(x)$  of the non-abelian gauge fields, the tension fields vary according to  $\delta\mathcal{F}_{\mu\nu}(x) = D_\mu\delta\mathcal{A}_\nu(x) - D_\nu\delta\mathcal{A}_\mu(x)$ .

2. Continuing the previous problem, consider an  $SU(N)$  gauge theory in which  $N^2 - 1$  vector fields  $A_\mu^a(x)$  interact with some “matter” fields  $\phi_\alpha(x)$ ,

$$\mathcal{L} = -\frac{1}{2g^2} \text{tr}(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}) + \mathcal{L}_{\text{mat}}(\phi, D_\mu\phi). \quad (10)$$

For the moment, let me keep the matter fields completely generic — they can be scalars, or vectors, or spinors, or whatever, and form any kind of a multiplet of the local  $SU(N)$  symmetry as long as such multiplet is complete and non-trivial. All we need to know right now is that there are well-defined covariant derivatives  $D_\mu\phi$  that depend on the gauge fields  $A_\mu^a$ , which give rise to the currents

$$J^{a\mu} = -\frac{\partial\mathcal{L}_{\text{mat}}}{\partial A_\mu^a}. \quad (11)$$

Collectively, these  $N^2 - 1$  currents should form an adjoint multiplet  $J^\mu = \sum_a (\frac{1}{2}\lambda^a) J^{a\mu}$  of the  $SU(N)$  symmetry.

- (a) Show that in this theory the equation of motion for the  $A_\mu^a$  fields are  $D_\mu F^{a\mu\nu} = J^{a\nu}$  and that consistency of these equations requires the currents to be *covariantly conserved*,

$$D_\mu J^\mu = \partial_\mu J^\mu + i[\mathcal{A}_\mu, J^\mu] = 0, \quad (12)$$

or in components,  $\partial_\mu J^{a\mu} - f^{abc} A_\mu^b J^{c\mu} = 0$ .

Note: a covariantly conserved current does *not* lead to a conserved charge,  $(d/dt) \int d^3\mathbf{x} J^{a0}(\mathbf{x}, t) \neq 0!$

Now consider a simple example of matter fields — a fundamental multiplet  $\Psi(x)$  of  $N$  Dirac fermions  $\Psi_i(x)$ , with a Lagrangian

$$\mathcal{L}_{\text{mat}} = \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi, \quad \mathcal{L}_{\text{net}} = \mathcal{L}_{\text{mat}} - \frac{1}{2g^2} \text{tr}(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}). \quad (13)$$

- (b) Derive the  $SU(N)$  currents  $J^{a\mu}$  for these fermions and verify that under  $SU(N)$  symmetries the currents transform covariantly into each other as members of the

adjoint multiplet. That is, the  $N \times N$  matrix  $J^\mu = \sum_a (\frac{1}{2}\lambda^a) J^{a\mu}$  transforms according to eq. (3).

Hint: for any complex  $N$ -vectors  $\xi_i$  and  $\eta_j$ ,

$$\sum_a (\eta^\dagger \lambda^a \xi) \times (\lambda^a)_i^j = 2\eta^{*j}\xi_i - \frac{2}{N}(\eta^\dagger \xi) \times \delta_i^j \dots \quad (14)$$

(c) Finally, verify the covariant conservation  $D_\mu J^{a\mu} = 0$  of these currents when the fermionic fields  $\Psi_i(x)$  and  $\bar{\Psi}^i(x)$  obey their equations of motion.

3. This problem is about general multiplets of general gauge groups. Consider a Lie group  $G$  with generators  $\hat{T}^a$  obeying commutation relations  $[\hat{T}^a, \hat{T}^b] = if^{abc}\hat{T}^c$ . Under an infinitesimal local symmetry

$$\mathcal{G}(x) = 1 + i\Lambda^a(x)\hat{T}^a + \dots, \quad \text{infinitesimal } \Lambda^a(x), \quad (15)$$

the gauge fields  $\mathcal{A}_\mu^a(x)$  transform as

$$\mathcal{A}_\mu^a(x) \rightarrow \mathcal{A}_\mu^a(x) - D_\mu \Lambda^a(x) = \mathcal{A}_\mu^a(x) - \partial_\mu \Lambda^a(x) - f^{abc}\Lambda^b(x)\mathcal{A}_\mu^c(x). \quad (16)$$

Other fields of the gauge theory (scalar, spinor, or whatever) must form complete multiplets of the gauge group  $G$ . In any such multiplet ( $m$ ), the generators  $\hat{T}^a$  are represented by  $\text{size}(m) \times \text{size}(m)$  matrices  $(T_{(m)}^a)_\alpha^\beta$  satisfying similar commutation relations,  $[T_{(m)}^a, T_{(m)}^b] = if^{abc}T_{(m)}^c$ . The fields  $\Psi_\alpha(x)$  belonging to such multiplet transform under infinitesimal gauge transforms (15) as

$$\Psi_\alpha(x) \rightarrow \Psi_\alpha(x) + i\Lambda^a(x)(T_{(m)}^a)_\alpha^\beta \Psi_\beta(x) \quad (17)$$

and the covariant derivatives  $D_\mu$  act on these fields as

$$D_\mu \Psi_\alpha(x) = \partial_\mu \Psi_\alpha(x) + i\mathcal{A}_\mu^a(x)(T_{(m)}^a)_\alpha^\beta \Psi_\beta(x). \quad (18)$$

- Verify covariance of these derivatives under infinitesimal gauge transforms (15).
- ★ For extra challenge, only for the students familiar with the basic theory of Lie groups: Prove covariance of the derivatives (18) under finite gauge transforms.  
Hint: use Lemma on the next page.

**Lemma:** For any finite symmetry  $\mathcal{G} \in G$ , the matrix  $(R_{(m)}(\mathcal{G}))_\alpha^\beta$  representing this symmetry in the multiplet  $(m)$  satisfies

$$(R_{(m)}(\mathcal{G}))_\alpha^\beta (T_{(m)}^a)_\beta^\gamma (R_{(m)}^{-1}(\mathcal{G}))_\gamma^\delta = (T_{(m)}^b)_\alpha^\delta R_{\text{adj}}^{ba}(\mathcal{G}) \quad (19)$$

where  $R_{\text{adj}}^{ba}(\mathcal{G})$  represents  $\mathcal{G}$  in the adjoint multiplet. Note that the same  $R_{\text{adj}}^{ba}(\mathcal{G})$  appears on right hand sides of eqs. (19) for all multiplets  $(m)$  of  $G$  — and that's what allows us to use the same gauge fields  $\mathcal{A}_\mu^a(x)$  to make covariant derivatives (18) for all multiplets of the gauge group  $G$ .

4. In the [previous homework](#) (set#11, problem#4), we had continuous global symmetry  $G = SU(N)_L \times SU(N)_R \times U(1)$  spontaneously broken down to  $H = SU(N)_V$ . Now let's gauge the entire  $SU(N)_L \times SU(N)_R \times U(1)$  symmetry and work out the Higgs mechanism.

The present theory comprises  $N^2$  complex scalar fields  $\Phi_i^j(x)$  organized into an  $N \times N$  matrix, and  $2N^2 - 1$  real vector fields  $B_\mu(x)$ ,  $L_\mu^a(x)$ , and  $R_\mu^a(x)$ , the latter organized into traceless hermitian matrices  $L_\mu(x) = \sum_a L_\mu^a(x) \times \frac{1}{2} \lambda^a$  and  $R_\mu(x) = \sum_a R_\mu^a(x) \times \frac{1}{2} \lambda^a$ , where  $a = 1, \dots, (N^2 - 1)$  and  $\lambda^a$  are the Gell-Mann matrices of  $SU(N)$ . The Lagrangian is

$$\mathcal{L} = -\frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{2} \text{tr}(L_{\mu\nu} L^{\mu\nu}) - \frac{1}{2} \text{tr}(R_{\mu\nu} R^{\mu\nu}) + \text{tr}(D^\mu \Phi^\dagger D_\mu \Phi) - V(\Phi^\dagger \Phi), \quad (20)$$

where

$$\begin{aligned} B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu, \\ L_{\mu\nu} &= \partial_\mu L_\nu - \partial_\nu L_\mu + ig[L_\mu, L_\nu], \\ R_{\mu\nu} &= \partial_\mu R_\nu - \partial_\nu R_\mu + ig[R_\mu, R_\nu], \\ D_\mu \Phi &= \partial_\mu \Phi + ig' B_\mu \Phi + ig L_\mu \Phi - ig \Phi R_\mu, \\ D_\mu \Phi^\dagger &= (D_\mu \Phi)^\dagger = \partial_\mu \Phi^\dagger - ig' B_\mu \Phi^\dagger + ig R_\mu \Phi^\dagger - ig \Phi^\dagger L_\mu. \end{aligned} \quad (21)$$

For simplicity, I assume equal gauge couplings  $g_L = g_R = g$  for the two  $SU(N)$  factors of the gauge group, but the abelian coupling  $g'$  is different.

The scalar potential  $V$  is precisely as in the [previous homework](#),

$$V = \frac{\alpha}{2} \text{tr}(\Phi^\dagger \Phi \Phi^\dagger \Phi) + \frac{\beta}{2} \text{tr}^2(\Phi^\dagger \Phi) + m^2 \text{tr}(\Phi^\dagger \Phi), \quad \alpha, \beta > 0, \quad m^2 < 0, \quad (22)$$

hence similar VEVs of the scalar fields: up to a gauge symmetry,

$$\langle \Phi \rangle = C \times \mathbf{1}_{N \times N} \quad \text{where} \quad C = \sqrt{\frac{-m^2}{\alpha + N\beta}}, \quad (23)$$

which breaks the  $G = SU(N)_L \times SU(N)_R \times U(1)$  down to  $SU(N)_V$ .

- (a) The Higgs mechanism makes  $N^2$  out of  $2N^2 - 1$  vector fields massive. Calculate their masses by plugging  $\langle \Phi \rangle$  for the  $\Phi(x)$  into the  $\text{tr}(D_\mu \Phi^\dagger D^\mu \Phi)$  term of the Lagrangian. In particular, show that the abelian gauge field  $B_\mu$  and the  $X_\mu^a = \frac{1}{\sqrt{2}}(L_\mu^a - R_\mu^a)$  combinations of the  $SU(N)$  gauge fields become massive, while the  $V_\mu^a = \frac{1}{\sqrt{2}}(L_\mu^a + R_\mu^a)$  combinations remain massless.
- (b) Find the effective Lagrangian for the massless vector fields  $V_\mu^a(x)$  by freezing all the other fields, *i.e.* setting  $\Phi(x) \equiv \langle \Phi \rangle$ ,  $B_\mu(x) \equiv 0$  and  $X_\mu^a(x) \equiv 0$ . Show that this Lagrangian describes a Yang–Mills theory with gauge group  $SU(N)_V$  and gauge coupling  $g_V = g/\sqrt{2}$ .
- ★ For extra challenge, allow for un-equal gauge couplings  $g_L \neq g_R$ . Find which combinations of the  $L_\mu^a(x)$  and  $R_\mu^a(x)$  fields remain massless in this case, then derive the effective Lagrangian for these massless fields by freezing everything else. As in part (b), you should get an  $SU(N)$  Yang–Mills theory, but this time the gauge coupling is

$$g_v = \frac{g_L g_R}{\sqrt{g_L^2 + g_R^2}}. \quad (24)$$