Phase Space Factors

For quantum transitions to un-bound states — for example, an atom emitting a photon, or a radioactive decay, or scattering, which is a kind of unbound $\to$ unbound transition, — the transition rate is given by the Fermi's golden rule:

$$\Gamma \stackrel{\text{def}}{=} \frac{d\,\text{probability}}{d\,\text{time}} = \frac{2\pi\rho}{\hbar} \times \left|\langle\text{final}|\,\hat{T}\,|\text{initial}\rangle\right|^2 \tag{1}$$

where $\hat{T} = \hat{H}_{\text{perturbation}} + \text{higher order corrections}$, and ρ is the *density of final states,*

$$\rho = \frac{dN_{\text{final states}}}{dE_{\text{final}}}\,. \tag{2}$$

For example, for an atom emitting a photon, and using the large-box normalization for the photon's states, we have

$$dN_{\text{final}} = 2_{\text{polarizations}} \times \left(\frac{L}{2\pi}\right)^3 d^3\mathbf{k}_\gamma = \frac{2L^3}{(2\pi)^3} \times k_\gamma^2\,dk_\gamma\,d^2\Omega_\gamma \tag{3}$$

while $dE_{\text{final}} = dE_\gamma = \hbar c dk_\gamma$, hence

$$\rho = L^3 \times \frac{2k_\gamma^2}{(2\pi)^3\hbar c} \times d^2\Omega_\gamma\,, \tag{4}$$

where the L^3 factor cancels against the $L^{-3/2}$ factor in the matrix element $\langle\text{atom}' + \gamma|\,\hat{T}\,|\text{atom}\rangle$ due to the photon's wave function. As to the remaining $d^2\Omega_\gamma$ factor, we should integrate over it to get the total decay rate, or divide by it to get the partial emission rate $d\Gamma/d\Omega$ for photons going into a particular direction.

In relativistic normalization of quantum states and matrix elements, there are no $L^{-3/2}$ factors but instead there are $\sqrt{2E}$ factors for each final-state or initial state particle, and they must be compensated by dividing the density of states ρ by the $\prod_i(2E_i)$. Also, we must allow for motion of all the final-state particles (*i.e.*, both the photon and the recoiled atom) but impose the momentum conservation as a constraint. Thus, for a decay of 1 initial particle into n final particles,

$$\Gamma = \frac{1}{2E_{\rm in}}\int\!\frac{d^3\mathbf{p}'_1}{(2\pi)^3\,2E'_1}\cdots\int\!\frac{d^3\mathbf{p}'_n}{(2\pi)^3\,2E'_n}\,\left|\langle p'_1,\ldots,p'_n|\,\mathcal{M}\,|p_{\rm in}\rangle\right|^2\times(2\pi^4)\delta^{(4)}(p'_1+\cdots+p'_n-p_{\rm in}), \tag{5}$$

where the δ function takes care of both momentum conservation and of the denominator dE_f in the density-of-states factor (2). Likewise, the transition rate for a generic $2\to n$ scattering process is given by

$$\begin{aligned}\Gamma = \frac{1}{2E_1\times 2E_2}\int\!\frac{d^3\mathbf{p}'_1}{(2\pi)^3\,2E'_1}\cdots\int\!\frac{d^3\mathbf{p}'_n}{(2\pi)^3\,2E'_n}\,&\left|\langle p'_1,\ldots,p'_n|\,\mathcal{M}\,|p_1,p_2\rangle\right|^2\times\\ &\times(2\pi^4)\delta^{(4)}(p'_1+\cdots+p'_n-p_1-p_2).\end{aligned} \tag{6}$$

In terms of the scattering cross-section σ, the rate $\Gamma=\sigma\times$ flux of initial particles. In the large-box normalization, the flux is $L^{-3}|\mathbf{v}_1-\mathbf{v}_2|$, so in the continuum normalization it's simply the relative speed $|\mathbf{v}_1-\mathbf{v}_2|$. Consequently, the total scattering cross-section is given by

$$\begin{aligned}\sigma_{\rm tot} = \frac{1}{4E_1E_2|\mathbf{v}_1-\mathbf{v}_2|}\int\!\frac{d^3\mathbf{p}'_1}{(2\pi)^3\,2E'_1}\cdots\int\!\frac{d^3\mathbf{p}'_n}{(2\pi)^3\,2E'_n}\,&\left|\langle p'_1,\ldots,p'_n|\,\mathcal{M}\,|p_1,p_2\rangle\right|^2\times\\ &\times(2\pi^4)\delta^{(4)}(p'_1+\cdots+p'_n-p_1-p_2).\end{aligned} \tag{7}$$

In particle physics, all the factors in eqs (5) or (7) besides the matrix elements — as well as the integrals over such factors — are collectively called the *phase space* factors.

A note on Lorentz invariance of decay rates or cross-sections. The matrix elements $\langle{\rm final}|\,\mathcal{M}\,|{\rm initial}\rangle$ are Lorentz invariant, and so are all the integrals over the final-particles' momenta and the δ-functions. The only non-invariant factor in the decay-rate formula (5) is the pre-integral $1/E_{\rm init}$, hence the decay rate of a moving particle is

$$\Gamma({\rm moving}) = \Gamma({\rm rest\,frame})\times\frac{M}{E} \tag{8}$$

where M/E is precisely the time dilation factor in the moving frame.

As to the scattering cross-section, it should be invariant under Lorentz boosts along the initial axis of scattering, thus the same cross-section in any frame where $\mathbf{p}_1\parallel\mathbf{p}_2$. This includes the *lab frame* where one of the two particles is initially at rest, the *center-of-mass frame* where $\mathbf{p}_1 + \mathbf{p}_2 = 0$, and any other frame where the two particles collide head-on. And indeed, the pre-integral factor is

$$\frac{1}{4E_1E_2|\mathbf{v}_1 - \mathbf{v}_2|} \;=\; \frac{1}{4|E_1\mathbf{p}_2 - E_2\mathbf{p}_1|} \tag{9}$$

in eq. (7) for the cross-section is invariant under Lorentz boosts along the scattering axis.

Let's simplify eq. (7) for a 2 particle → 2 particle scattering process in the center-of-mass frame where $\mathbf{p}_1 + \mathbf{p}_2 = 0$. In this frame, the pre-exponential factor (9) becomes

$$\frac{1}{4|\mathbf{p}| \times (E_1 + E_2)} \tag{10}$$

while the remaining phase space factors amount to

$$\begin{aligned}
\mathcal{P}_{\text{int}} \;&=\; \int\!\frac{d^3\mathbf{p}'_1}{(2\pi)^3\,2E'_1}\int\!\frac{d^3\mathbf{p}'_2}{(2\pi)^3\,2E'_2}\,(2\pi)^4\delta^{(3)}(\mathbf{p}'_1+\mathbf{p}'_2)\delta(E'_1+E'_2-E_{\text{net}})\\
&=\; \int\!\frac{d^3\mathbf{p}'_1}{(2\pi)^3\times 2E'_1\times 2E'_2}\,(2\pi)\delta(E'_1(\mathbf{p}'_1)+E'_2(-\mathbf{p}'_1)-E_{\text{net}})\\
&=\; \int\! d^2\Omega_{\mathbf{p}'}\times\int\limits_0^\infty\! dp'\,\frac{p'^2}{16\pi^2E'_1E'_2}\times\delta(E'_1+E'_2-E_{\text{tot}})\\
&=\; \int\! d^2\Omega_{\mathbf{p}'}\left[\frac{p'^2}{16\pi^2E'_1E'_2}\Bigg/\frac{d(E'_1+E'_2)}{dp'}\right]^{\text{when}}_{E'_1+E'_2=E_{\text{tot}}}.
\end{aligned} \tag{11}$$

On the last 3 lines here $E'_1 = E'_1(\mathbf{p}'_1) = \sqrt{p'^2 + m_1'^2}$ while $E'_2 = E'_2(\mathbf{p}'_2 = -\mathbf{p}'_1) = \sqrt{p'^2 + m_2'^2}$. Consequently,

$$\frac{dE'_1}{dp'} \;=\; \frac{p'}{E'_1}\,,\quad \frac{dE'_2}{dp'} \;=\; \frac{p'}{E'_2}\,, \tag{12}$$

hence

$$\frac{d(E'_1+E'_2)}{dp'} \;=\; \frac{p'}{E'_1} + \frac{p'}{E'_2} \;=\; \frac{p'}{E'_1E'_2}\times(E'_2+E'_1 \;=\; E_{\text{tot}}), \tag{13}$$

and therefore

$$\mathcal{P}_{\text{int}} \;=\; \frac{1}{16\pi^2}\times\frac{p'}{E_{\text{tot}}}\times\int\! d^2\Omega_{\mathbf{p}'}\,. \tag{14}$$

Including the pre-integral factor (10), we arrive at the net phase space factor

$$\mathcal{P} \;=\; \frac{p'}{p} \times \frac{1}{64\pi^2 E_{\text{tot}}^2} \times \int d^2\Omega_{\mathbf{p}'} \,. \tag{15}$$

The matrix element $\mathcal{M}$ for the scattering should be put inside the direction-angle integral in this phase-space formula. Thus, the total scattering cross-section is

$$\sigma_{\text{tot}}(1+2\to 1'+2') \;=\; \frac{p'}{p} \times \frac{1}{64\pi^2 E_{\text{cm}}^2} \times \int d^2\Omega\, \left|\langle p_1'+p_2'|\,\mathcal{M}\,|p_1+p_2\rangle\right|^2 , \tag{16}$$

while the partial cross-section for scattering in a particular direction is

$$\frac{d\sigma(1+2\to 1'+2')}{d\Omega_{\text{cm}}} \;=\; \frac{p'}{p} \times \frac{1}{64\pi^2 E_{\text{cm}}^2} \times \left|\langle p_1'+p_2'|\,\mathcal{M}\,|p_1+p_2\rangle\right|^2 . \tag{17}$$

Note: the total cross-section is the same in frames where the initial momenta are collinear, but in the partial cross-section, $d\Omega$ depends on the frame of reference, so eq. (17) applies only in the center-of mass frame. Also, the E_{cm} factor in denominators of both formulae stands for the net energy in the center-of-mass frame. In frame-independent terms,

$$E_{\text{cm}}^2 \;=\; (p_1+p_2)^2 \;=\; (p_1'+p_2')^2 \tag{18}$$

Finally, let me write down the phase-space factor for a 2-body decay (1 particle $\to$ 2 particles) in the rest frame of the initial particle. The under-the-integral factors for such a decay are the same as in eq. (14) for a $2\to 2$ scattering, but the pre-integral factor is $1/2M_{\text{init}}$ instead of the (10), thus

$$\mathcal{P} \;=\; \frac{p'}{32\pi^2 M^2} \,, \tag{19}$$

meaning

$$\frac{d\Gamma(0\to 1'+2')}{d\Omega} \;=\; \frac{p'}{32\pi^2 M^2} \times \left|\langle p_1'+p_2'|\,\mathcal{M}\,|p_0\rangle\right|^2 , \tag{20}$$

$$\Gamma(0\to 1'+2') \;=\; \frac{p'}{32\pi^2 M^2} \times \int d^2\Omega\, \left|\langle p_1'+p_2'|\,\mathcal{M}\,|p_0\rangle\right|^2 . \tag{21}$$