QCD Feynman Rules

Quarks are Dirac fermion fields Ψ_{if} which come in 3 colors $i = 1, 2, 3$ and 6 flavors $f = u, d, s, c, b, t$. The $SU(3)$ symmetry of the 3 colors is an exact local symmetry, so it comes with $3^2 - 1 = 8$ vector fields $A^a_\mu(x)$ — the gluons. The semi-classical Lagrangian of the theory is quite simple, especially in the color-matrix notations:

$$\mathcal{L} \;=\; -\frac{1}{2g^2}\,\mathrm{tr}\left(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\right) \;+\; \sum_f \overline{\Psi}_f\big(i\gamma^\mu D_\mu \;-\; m_f\big)\Psi_f \tag{1}$$

where $D_\mu\Psi = \partial_\mu\Psi + i\mathcal{A}_\mu\Psi$ and $\mathcal{F}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu + i[\mathcal{A}_\mu, \mathcal{A}_\nu]$. Rescaling $\mathcal{A}_\mu = gA_\mu$, expanding it into individual gluon fields $A^a_\mu(x)$ as $\mathcal{A} = (g/2)A^a_\mu\lambda^a$ — where λ^a are the Gell–Mann matrices — and keeping all colors explicit makes for more complicated formulae:

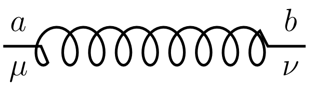
$$\begin{aligned} D_\mu\Psi_i(x) \;&=\; \partial_\mu\Psi_i(x) \;+\; ig\,A_\mu(x)\left(\frac{\lambda^a}{2}\right)_i^{\;j}\Psi_j(x), \\ \mathcal{F}^a_{\mu\nu}(x) \;&=\; g\partial_\mu A^a_\nu(x) \;-\; g\partial_\nu A^a_\mu(x) \;-\; g^2 f^{abc}A^b_\mu(x)A^c_\nu(x), \end{aligned} \tag{2}$$

and therefore, after expanding in powers of g,

$$\begin{aligned} \mathcal{L} \;=\; & -\tfrac{1}{4}\big(\partial_\mu A^a_\nu \;-\; \partial_\nu A^a_\mu\big)^2 + \sum_f \overline{\Psi}{}^i_f(i\gamma^\mu\partial_\mu - m_f)\Psi_{if} \\ & + \; gf^{abc}(\partial_\mu A^a_\nu)A^b_\mu A_\nu c \;-\; \frac{g^2}{4}\,f^{abc}f^{ade}\,A^b_\mu A^c_\nu A^d_\mu A^e_\nu \;+\; ig\left(\frac{\lambda^a}{2}\right)_i^{\;j} A^a_\mu \times \sum_f \overline{\Psi}{}^i_f\gamma^\mu\Psi_{jf}\,. \end{aligned} \tag{3}$$

In perturbation theory, the top line here describes the free gluon and quark fields, while the bottom line describes the 3–gluon, 4–gluon, and gluon-quark-antiquark interactions. The Feynman rules of QCD follows from this expansion in powers of g:

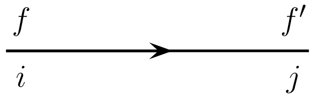
- Gluon propagator



$$= \frac{-i\delta^{ab}}{k^2 + i0}\left(g^{\mu\nu} \;+\; (\xi - 1)\frac{k^\mu k^\nu}{k^2 + i0}\right) \tag{4}$$

where ξ is the gauge-fixing parameter. For $\xi = 0$ we have the Landau gauge while for $\xi = 1$ — the Feynman gauge.

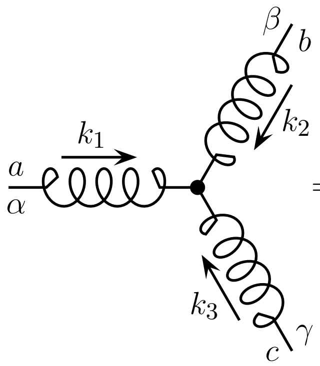
- Quark propagator



$$= \frac{i\delta^i_j \delta_{ff'}}{\not{p} - m_f + i0}\,. \tag{5}$$

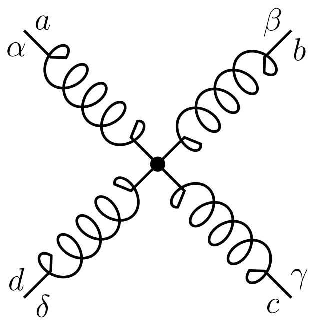
Note: the colors and the flavors must be the same at both ends of the propagator.

- Three-gluon vertex



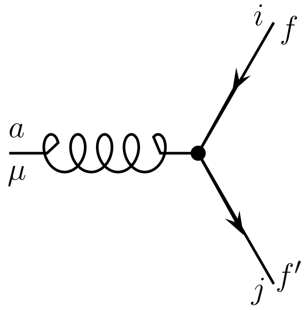
$$= -gf^{abc}\left[g^{\alpha\beta}(k_1 - k_2)^\gamma + g^{\beta\gamma}(k_2 - k_3)^\alpha + g^{\gamma\alpha}(k_3 - k_1)^\beta\right]. \tag{6}$$

- Four-gluon vertex



$$= -ig^2 \begin{bmatrix} f^{abe}f^{cde}(g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\delta}g^{\beta\gamma}) \\ + f^{ace}f^{bde}(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\delta}g^{\gamma\beta}) \\ + f^{ade}f^{bce}(g^{\alpha\beta}g^{\delta\gamma} - g^{\alpha\gamma}g^{\delta\beta}) \end{bmatrix}. \tag{7}$$

- Quark-gluon vertex



$$= -ig\gamma^\mu \times \delta_{ff'} \times \left(\frac{\lambda^a}{2}\right)^j_i . \tag{8}$$

Note: the quark lines connected to the vertex must have the same flavors $f' = f$ but they may have different colors $j \neq i$.

$\star$ The external line factors and the sign rules of QCD are exactly the same as in QED.

Actually, the above Feynman rules are OK at the tree level but insufficient for the loop calculations. The problem stems from the gauge fixing the gluon fields in order to quantize them canonically and get the gluon propagator. In QED, simple linear constraints like $\nabla \cdot \mathbf{A} = 0$ or $\partial_\mu A^\mu = 0$ on the abelian gauge field were harmless, but for the non-abelian gluon fields of QCD such linear constraints create all kinds of problems at the loop level. In the path integral formalism, such constraints screw up the measure of the path integral, but we can un-screw it by introducing additional un-physical fields called the *ghosts*. In Feynman rules, there are ghost propagators and ghost-ghost-gluon vertices, but no external ghost lines — the ghost may run in loops but never as external particles. We shall deal with the "ghostly" aspects of QCD Feynman rules in the QFT (II) class in Spring 2017.