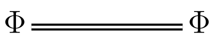
Problem **1**:

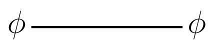
In the Lagrangian

$$\mathcal{L} \;=\; \frac{1}{2}(\partial_\mu \Phi)^2 \;-\; \frac{M^2}{2}\,\Phi^2 \;+\; \frac{1}{2}(\partial_\mu \phi)^2 \;-\; \frac{m^2}{2}\,\phi^2 \;-\; \frac{\mu}{2}\,\Phi\phi^2, \tag{1}$$

the first 4 terms on the RHS describe two free scalar fields $\Phi(x)$ and $\phi(x)$, while the fifth term is the interaction that we treat as a perturbation. In Feynman rules, the propagators follow from the free part of the Lagrangian, so for the theory at hand there are two distinct propagators,



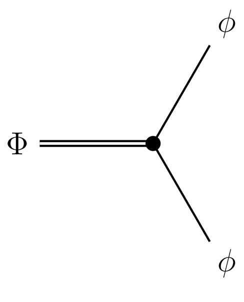
$$=\; \frac{i}{q^2 - m^2 + i0} \qquad \text{and}$$



$$=\; \frac{i}{q^2 - M^2 + i0}\,. \tag{S.1}$$

Likewise, there are two kinds of external lines according to the species of the incoming or outgoing particles for the process in question.

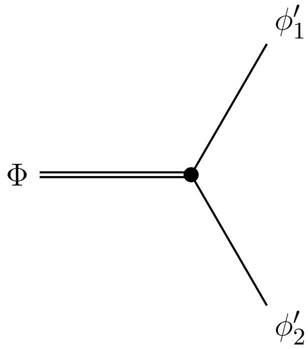
The Feynman vertices follow from the interaction part of the Lagrangian, which for the theory at hand is the cubic potential term $V_3 = \frac{\mu}{2}\,\Phi\phi^2$. Consequently, all vertices should be connected to three lines (net valence = 3), one double line for the one $\hat{\Phi}$ field, and two single lines for the two $\hat{\phi}$ fields,



$$=\; -i\frac{\mu}{2} \times 2! \;=\; -i\mu \tag{S.2}$$

where the 2! factor comes from the interchangeability of two identical $\hat{\phi}$ fields in the vertex.

Now consider the decay process $\Phi \to \phi + \phi$. To the lowest order of the perturbation theory, the decay amplitude follows from a single tree diagrams



(S.3)

This diagram has one vertex, one incoming double line, two outgoing single lines and no internal lines of either kind, hence

$$\left\langle \phi'_1 + \phi'_2 \right| i\hat{T} \left| \Phi \right\rangle \;\equiv\; i\mathcal{M} \times (2\pi)^4 \delta^{(4)}(p - p'_1 - p'_2) \;=\; -i\mu \times (2\pi)^4 \delta^{(4)}(p - p'_1 - p'_2), \quad \text{(S.4)}$$

or in other words

$$\mathcal{M}(\Phi \to \phi'_1 + \phi'_2) \;=\; -\mu. \quad \text{(S.5)}$$

This amplitude is related to the $\Phi \to \phi\phi$ decay rate as

$$\Gamma \;=\; \int |\mathcal{M}|^2 \, d\mathcal{P} \quad \text{(S.6)}$$

where the phase space factor for 1 particle $\to$ 2 particles decays is

$$\begin{aligned} d\mathcal{P} \;&=\; \frac{1}{2E} \times \frac{d^3\mathbf{p}'_1}{(2\pi)^3\, 2E'_1} \times \frac{d^3\mathbf{p}'_2}{(2\pi)^3\, 2E'_2} \times (2\pi)^4 \delta^{(4)}(p - p'_1 - p'_2) \\ &=\; \frac{1}{32\pi^2\, EE'_1E'_2} \times d^3\mathbf{p}'_1\, \delta(E - E'_1 - E'_2) \quad \text{for} \quad \mathbf{p}'_2 \;=\; \mathbf{p} - \mathbf{p}'_1 \quad \text{and on-shell energies,} \\ &=\; \frac{|\mathbf{p}'|}{32\pi^2 M^2} \times d\Omega_{\mathbf{p}'_1} \qquad \text{in the rest frame of the decaying particle,} \end{aligned} \quad \text{(S.7)}$$

cf. my notes on phase space. For decays to two particles of equal masses $m < \frac{M}{2}$,

$$E_1' = E_2' = \frac{M}{2} \quad\Longrightarrow\quad |\mathbf{p}'| = \sqrt{E_1'^2 - m^2} = \frac{M}{2} \times \sqrt{1 - \frac{4m^2}{M^2}}\,, \tag{S.8}$$

hence

$$d\mathcal{P} = \sqrt{1 - \frac{4m^2}{M^2}} \times \frac{1}{64\pi^2 M} \times d\Omega, \tag{S.9}$$

and therefore the partial decay rate is

$$\frac{d\Gamma}{d^2\Omega} = \sqrt{1 - \frac{4m^2}{M^2}} \times \frac{|\mathcal{M}|^2}{64\pi^2 M}\,. \tag{S.10}$$

For the problem at hand, $\mathcal{M}_{\text{tree}} = -\mu$ regardless of directions of final particles, hence

$$\frac{d\Gamma_{\text{tree}}}{d^2\Omega} = \sqrt{1 - \frac{4m^2}{M^2}} \times \frac{\mu^2}{64\pi^2 M}\,. \tag{S.11}$$

Integrating this partial decay rate over the directions of $\mathbf{p}'$ we must remember that the two final particles are identical bosons, so we cannot tell $\mathbf{p}_1'$ from $\mathbf{p}_2' = -\mathbf{p}_1'$. Consequently, $\int d^2\Omega = 4\pi/2$ and therefore

$$\Gamma = \sqrt{1 - \frac{4m^2}{M^2}} \times \frac{\mu^2}{32\pi M}\,. \tag{S.12}$$

Problem **2**:

Similar to the previous problem, the Feynman propagators of a theory follow from the free part of its Lagrangian. This time, we have N scalar fields $\phi^i(x)$ of similar mass m, hence in momentum space

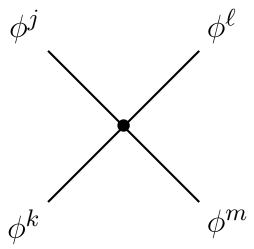
$$\phi^j \;\text{———————}\; \phi^k = \frac{i\delta^{jk}}{q^2 - m^2 + i0}\,. \tag{S.13}$$

Note the δ^{jk} factor — the two fields connected by a propagator must be of the same species. Graphically, this means that both ends of the propagator carry the same species label $j = k$. Likewise, the external lines should also carry a species label of the incoming or outgoing particle in question. For the external lines, these labels are fixed (for a particular process), while for the internal lines we sum over $j = 1, 2, \ldots, N$.

The Feynman vertices follow from the interactions between the fields; for the theory in question, they come form the quartic potential

$$V_4 \;=\; \frac{\lambda}{8}\Bigl(\phi\cdot\phi = \sum_j \phi^j\phi^j\Bigr)^2 \;=\; \sum_j \frac{\lambda}{8}\,\bigl(\hat\phi^j\bigr)^4 \;+\; \sum_{j<k}\frac{\lambda}{4}\,\bigl(\hat\phi^j\bigr)^2\bigl(\hat\phi^k\bigr)^2. \qquad \text{(S.14)}$$

Consequently, all vertices have net valence = 4, but there are two vertex types with different indexologies: (1) a vertex involving 4 lines of the same field species ϕ^j, with the vertex factor $-i(\lambda/8)\times 4! = -3i\lambda$; and (2) a vertex involving 2 lines of one field species ϕ^j and 2 lines of a different species ϕ^k, with the vertex factor $-i(\lambda/4)\times(2!)^2 = -i\lambda$. (The combinatorial factors arise from the interchanges of the identical fields in the same vertex, thus 4! for the first vertex type and $(2!)^2$ for the second type.) Equivalently, we may use a single vertex type involving 4 fields of whatever species, with the species-dependent vertex factor



$$= \; -i\lambda\bigl(\delta^{jk}\delta^{\ell m} + \delta^{j\ell}\delta^{km} + \delta^{jm}\delta^{k\ell}\bigr). \qquad \text{(S.15)}$$

Now consider the scattering process $\phi^j + \phi^k \to \phi^\ell + \phi^m$. At the lowest order of the perturbation theory, there is just one Feynman diagram for this process; it has one vertex, 4 external legs and no internal lines. Consequently, at the lowest order,

$$\mathcal{M}(\phi^j + \phi^k \to \phi^\ell + \phi^m) \;=\; -\lambda\bigl(\delta^{jk}\delta^{\ell m} + \delta^{j\ell}\delta^{km} + \delta^{jm}\delta^{k\ell}\bigr) \qquad \text{(S.16)}$$

independent of the particles' momenta. Specifically,

$$\begin{aligned} \mathcal{M}(\phi^1 + \phi^2 \to \phi^1 + \phi^2) \; &= -\lambda, \\ \mathcal{M}(\phi^1 + \phi^1 \to \phi^2 + \phi^2) \; &= -\lambda, \\ \mathcal{M}(\phi^1 + \phi^1 \to \phi^1 + \phi^1) \; &= -3\lambda. \end{aligned} \qquad \text{(S.17)}$$

The *partial* cross sections in the CM frame follow from these amplitudes via eq. (4.85) of the textbook or eq. (17) of my notes on phase space: For elastic scattering,

$$\frac{d\sigma}{d\Omega_{\text{c.m.}}} = \frac{|\mathcal{M}|^2}{64\pi^2 E_{\text{c.m.}}^2}], \tag{S.18}$$

hence

$$\begin{aligned}
\frac{d\sigma(\phi^1+\phi^2\to\phi^1+\phi^2)}{d\Omega_{\text{c.m.}}} &= \frac{\lambda^2}{64\pi^2 E_{\text{c.m.}}^2}\,, \\
\frac{d\sigma(\phi^1+\phi^1\to\phi^2+\phi^2)}{d\Omega_{\text{c.m.}}} &= \frac{\lambda^2}{64\pi^2 E_{\text{c.m.}}^2}\,, \\
\frac{d\sigma(\phi^1+\phi^1\to\phi^1+\phi^1)}{d\Omega_{\text{c.m.}}} &= \frac{9\lambda^2}{64\pi^2 E_{\text{c.m.}}^2}\,.
\end{aligned} \tag{S.19}$$

To calculate the total cross sections, we integrate over $d\Omega$, which gives the factor of 4π when the two final particles are of distinct species, but for the same species, we only get 2π because of Bose statistics. Thus,

$$\begin{aligned}
\sigma_{\text{tot}}(\phi^1+\phi^2\to\phi^1+\phi^2) &= \frac{\lambda^2}{16\pi E_{\text{c.m.}}^2}\,, \\
\sigma_{\text{tot}}(\phi^1+\phi^1\to\phi^2+\phi^2) &= \frac{\lambda^2}{32\pi E_{\text{c.m.}}^2}\,, \\
\sigma_{\text{tot}}(\phi^1+\phi^1\to\phi^1+\phi^1) &= \frac{9\lambda^2}{32\pi E_{\text{c.m.}}^2}\,.
\end{aligned} \tag{S.20}$$

Problem **3**(a):

In perturbation theory, the Feynman propagators follow from the quadratic part of the Lagrangian (and hence free Hamiltonian), while the vertices follow from the cubic, quartic, *etc.*, terms treated as perturbation. For the linear sigma model's Lagrangian (3),

$$\mathcal{L} = \mathcal{L}_{\text{free}} - V_{\text{pert}}\,, \tag{S.21}$$

$$\mathcal{L}_{\text{free}} = \frac{1}{2}\sum\nolimits_i (\partial_\mu\phi_i)^2 + \frac{1}{2}(\partial_\mu\sigma)^2 - \frac{M_\sigma^2 = \lambda f^2}{2}\times\sigma^2, \tag{S.22}$$

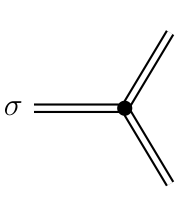
$$V_{\rm pert} \;=\; \frac{3\lambda f}{6}\times\sigma^3 \;+\; \frac{\lambda f}{2}\times\sum\nolimits_i \sigma\phi_i^2 \;+\; \frac{3\lambda}{24}\times\sigma^4 \;+\; \frac{\lambda}{4}\times\sum\nolimits_i \sigma^2\phi_i^2 \;+\; \frac{\lambda}{8}\times\Bigl(\sum\nolimits_i \phi_i^2\Bigr)^2 . \tag{S.23}$$

The free Lagrangian (S.22) describes one massive field σ plus N massless fields π_i, hence two types of scalar propagators,

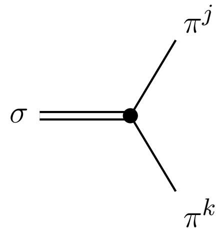
$$\sigma \;=\!=\!=\!=\; \sigma \;=\; \frac{i}{q^2-2\mu^2+i0} \qquad \text{and} \qquad \pi^j \;\text{———}\; \pi^k \;=\; \frac{i\delta^{jk}}{q^2+i0}\,, \tag{S.24}$$

and the $\pi\pi$ propagator carries a label $j=k=1,2\ldots,N$ specifying a particular species of the pion field.

As to the perturbation (S.23), it has two cubic terms and 3 quartic terms, hence two types of valence $=3$ vertices and three types of valence $=4$ vertices: the $\sigma\sigma\sigma$ and $\sigma\pi\pi$ vertices

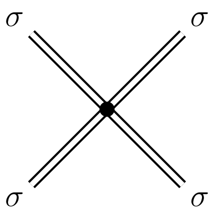


$$= -3i\lambda f \qquad \text{and}$$

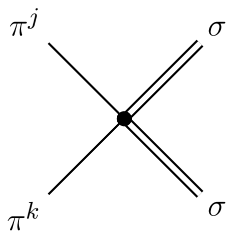


$$= -i\lambda f\delta^{jk}\,, \tag{S.25}$$

the $\sigma\sigma\sigma\sigma$ and $\sigma\sigma\pi\pi$ vertices

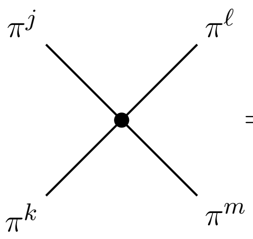


$$= -3i\lambda \qquad \text{and}$$



$$= -i\lambda\delta^{ik} \tag{S.26}$$

and finally the $\pi\pi\pi\pi$ vertex similar to what we had in problem (2),



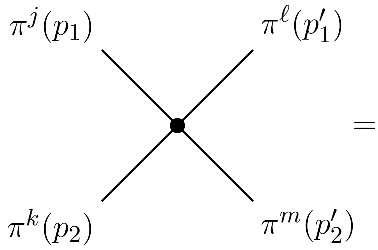
$$= -i\lambda\bigl(\delta^{jk}\delta^{\ell m}+\delta^{j\ell}\delta^{km}+\delta^{jm}\delta^{k\ell}\bigr). \tag{S.27}$$

This completes the Feynman rules of the linear sigma model.

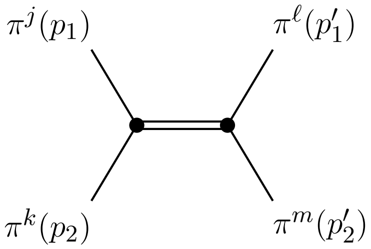
Problem **3**(b):

As explained in class, a tree diagram ($L = 0$) with $E = 4$ external lines has either (A) one valence $= 4$ vertex and no propagators, or else (B) two valence $= 3$ vertices and one propagator. Topologically, there are three diagrams of type (B) with different arrangements of incoming versus outgoing external lines, so altogether there are 4 tree diagrams.

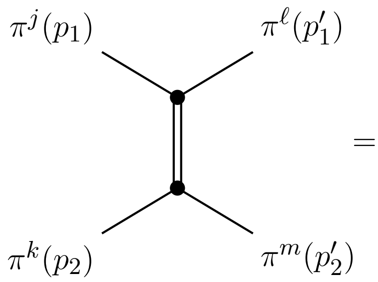
Specifically for the $\pi^j + \pi^k \to \pi^\ell + \pi^m$ scattering, the diagrams are



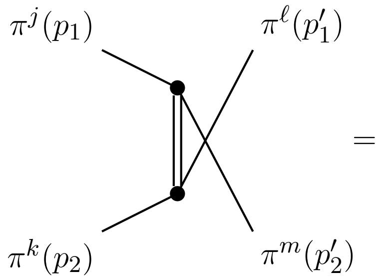
$$= -i\lambda\big(\delta^{jk}\delta^{\ell m} + \delta^{j\ell}\delta^{km} + \delta^{jm}\delta^{k\ell}\big), \tag{S.28}$$



$$= (-i\lambda f\delta^{jk})\,\frac{i}{s - M_\sigma^2}\,(-i\lambda f\delta^{\ell m}), \tag{S.29}$$



$$= (-i\lambda f\delta^{j\ell})\,\frac{i}{t - M_\sigma^2}\,(-i\lambda f\delta^{km}), \tag{S.30}$$



$$= (-i\lambda f\delta^{jm})\,\frac{i}{u - M_\sigma^2}\,(-i\lambda f\delta^{k\ell}), \tag{S.31}$$

where s, t, u are the Mandelstam variables

$$\begin{aligned} s &\stackrel{\text{def}}{=} (p_1 + p_2)^2 \equiv (p_1' + p_2')^2, \\ t &\stackrel{\text{def}}{=} (p_1' - p_1)^2 \equiv (p_2' - p_2)^2, \\ u &\stackrel{\text{def}}{=} (p_1' - p_2)^2 \equiv (p_2' - p_1')^2. \end{aligned} \tag{S.32}$$

Note that in the diagrams (S.29), (S.30), and (S.31), the internal line belongs to the σ field rather than to any π^i fields since there are no $\pi\pi\pi$ vertices but only $\pi\pi\sigma$.

Also, each of the diagrams (S.29), (S.30), and (S.31) yields a different combination of Kronecker $\delta\delta$ for the j, k, ℓ, m indices of the four pions, while the first diagram (S.28) yields all three combinations. So when we total up the four tree diagrams' amplitudes, it's convenient to reorganize the net tree amplitude by the j, k, ℓ, m indexology, thus

$$\begin{aligned} \mathcal{M}\big(\pi^j(p_1) + \pi^k(p_2) \to \pi^\ell(p_1') + \pi^m(p_2')\big) = &-\delta^{jk}\delta^{\ell m}\left(\lambda + \frac{\lambda^2 f^2}{s - M_\sigma^2}\right) \\ &- \delta^{j\ell}\delta^{km}\left(\lambda + \frac{\lambda^2 f^2}{t - M_\sigma^2}\right) \\ &- \delta^{jm}\delta^{k\ell}\left(\lambda + \frac{\lambda^2 f^2}{u - M_\sigma^2}\right). \end{aligned} \tag{S.33}$$

Problem **3**(c):

The Lagrangian (3) of the linear sigma models has a very important relation between the quartic coupling λ, the cubic coupling $\mu = \lambda f$, and the σ particle's mass $M_\sigma^2 = \lambda f^2$, thus

$$\big(M_\sigma^2 = \lambda f^2\big) \times \lambda = \big(\mu = \lambda f\big)^2. \tag{S.34}$$

Thanks to this relation,

$$\lambda + \frac{(\lambda f)^2}{s - M_\sigma^2} = \frac{\lambda s - \cancel{\lambda M_\sigma^2} + \cancel{(\lambda f)^2}}{s - M_\sigma^2} = \frac{\lambda s}{s - M_\sigma^2} \tag{S.35}$$

and likewise

$$\lambda + \frac{(\lambda f)^2}{t - M_\sigma^2} = \frac{\lambda t}{t - M_\sigma^2} \quad \text{and} \quad \lambda + \frac{(\lambda f)^2}{u - M_\sigma^2} = \frac{\lambda u}{u - M_\sigma^2}. \tag{S.36}$$

Thanks to these formulae, the scattering amplitude (S.33) simplifies to

$$\mathcal{M} \;=\; -\lambda\Bigl(\delta^{jk}\delta^{\ell m}\times\frac{s}{s-M_\sigma^2}\;+\;\delta^{j\ell}\delta^{km}\times\frac{t}{t-M_\sigma^2}\;+\;\delta^{jm}\delta^{k\ell}\times\frac{u}{u-M_\sigma^2}\Bigr). \tag{S.37}$$

Now consider the low-energy limit of this amplitude. In the CM frame, all 4 pions have the same energy E, hence

$$s \;=\; (E_{\rm cm}^{\rm tot})^2 \;=\; 4E^2,\quad t \;=\; -4E^2\times\sin^2(\theta/2),\quad u \;=\; -4E^2\times\cos^2(\theta/2), \tag{S.38}$$

and therefore

$$s,t,u \;=\; O(E^2). \tag{S.39}$$

Consequently, when the pion's energies are much smaller than the σ particle's mass, the denominators in the amplitude (S.37) may be approximated as

$$\frac{1}{s-M_\sigma^2} \;\approx\; \frac{1}{t-M_\sigma^2} \;\approx\; \frac{1}{u-M_\sigma^2} \;\approx\; \frac{-1}{m_\sigma^2} \;=\; \frac{-1}{\lambda f^2}\,. \tag{S.40}$$

Consequently, the scattering amplitude (S.37) simplifies to

$$\mathcal{M} \;=\; \left(\frac{+\lambda}{M_\sigma^2}\;=\;\frac{+1}{f^2}\right)\times\left(\delta^{jk}\delta^{\ell m}\times s\;+\;\delta^{j\ell}\delta^{km}\times t\;+\;\delta^{jm}\delta^{k\ell}\times u\;+\;O\left(\frac{E^4}{M_\sigma^2}\right)\right). \tag{S.41}$$

The magnitude of this amplitude is generally $O(E^2/v^2)$, so in the low-energy limit it becomes quite small.

Now consider the $\pi\pi\to\pi\pi$ scattering in a completely general frame of reference. Since the pions are massless, Mandelstam's s,t,u variables may be written as

$$\begin{aligned} s &\stackrel{\rm def}{=} (p_1+p_2)^2 \;\equiv\; (p_1'+p_2')^2 \;=\; +2(p_1p_2) \;=\; +2(p_1'p_2'),\\ t &\stackrel{\rm def}{=} (p_1'-p_1)^2 \;\equiv\; (p_2'-p_2)^2 \;=\; -2(p_1'p_1) \;=\; -2(p_2'p_2),\\ u &\stackrel{\rm def}{=} (p_1'-p_2)^2 \;\equiv\; (p_2'-p_1')^2 \;=\; -2(p_1p_2') \;=\; -2(p_1'p_2), \end{aligned} \tag{S.42}$$

so whenever any one of the four momenta becomes small, all 3 of the s,t,u become small. In particular, when 3 of the momenta are $O(M_\sigma)$ or smaller while the fourth momentum becomes much smaller, we have

$$s, t, u \;=\; O(M_\sigma \times p_{\rm smallest}) \;\ll\; M_\sigma^2. \tag{S.43}$$

Consequently, the scattering amplitude becomes as in eq. (S.41), and its magnitude is generally

$$\mathcal{M} \;\sim\; (s,t,u) \times \frac{\lambda}{M_\sigma^2} \;\lesssim\; P_{\rm smallest} \times \frac{\lambda}{M_\sigma}\,. \tag{S.44}$$

The physical reason for this behavior is the Goldstone theorem: Among other things, it says that *all scattering amplitudes involving Goldstone particles — such as the pions in this problem — become small as* $O(p_\pi)$ *when* **any** *Goldstone particle's momentum* p_π *becomes small.* A few lines above we saw how this works for the tree-level $\langle\pi,\pi|\,\mathcal{M}\,|\pi,\pi\rangle$ amplitude (S.37); the same behavior persists at all the higher orders of the perturbation theory, but seeing how *that* works is waaay beyond the scope of this exercise.

Problem **3**(d):

In the low-energy limit $E \ll M_\sigma$, the tree-level $\pi\pi \to \pi\pi$ amplitudes may be approximated as in eq. (S.41). In particular,

$$\begin{aligned}
\mathcal{M}(\pi^1+\pi^2\to\pi^1+\pi^2) \;&=\; \frac{\lambda t}{M_\sigma^2} \;+\; O\left(\frac{\lambda E^4}{M_\sigma^4}\right) \;\approx\; \frac{t}{f^2}\,,\\
\mathcal{M}(\pi^1+\pi^1\to\pi^2+\pi^2) \;&=\; \frac{\lambda s}{M_\sigma^2} \;+\; O\left(\frac{\lambda E^4}{M_\sigma^4}\right) \;\approx\; \frac{s}{f^2}\,,\\
\mathcal{M}(\pi^1+\pi^1\to\pi^1+\pi^1) \;&=\; \frac{\lambda(s+t+u)}{M_\sigma^2} \;+\; O\left(\frac{\lambda E^4}{M_\sigma^4}\right) \;=\; O\left(\frac{\lambda E^4}{M_\sigma^4}\right)\\
&\qquad\langle\!\langle\,\text{since } s+t+u=4m_\pi^2=0\,\rangle\!\rangle
\end{aligned} \tag{S.45}$$

Translating these amplitudes into the partial and total scattering cross-sections, we obtain

$$\begin{aligned}
\frac{d\sigma(\pi^1+\pi^2\to\pi^1+\pi^2)}{d\Omega_{\text{c.m.}}} &= \frac{1}{64\pi^2 s}\times\frac{t^2}{f^4} = \frac{E^2_{\text{c.m.}}}{64\pi^2 f^4}\times\sin^4\frac{\theta_{\text{c.m.}}}{2}\,,\\
\sigma_{\text{tot}}(\pi^1+\pi^2\to\pi^1+\pi^2) &= \frac{E^2_{\text{c.m.}}}{48\pi f^4}\,,\\
\frac{d\sigma(\pi^1+\pi^1\to\pi^2+\pi^2)}{d\Omega_{\text{c.m.}}} &= \frac{1}{64\pi^2 s}\times\frac{s^2}{f^4} = \frac{E^2_{\text{c.m.}}}{64\pi^2 f^4}\,,\\
\sigma_{\text{tot}}(\pi^1+\pi^1\to\pi^2+\pi^2) &= \frac{E^2_{\text{c.m.}}}{32\pi f^4}\,,\\
\frac{d\sigma(\pi^1+\pi^1\to\pi^1+\pi^1)}{d\Omega_{\text{c.m.}}} &= \frac{1}{64\pi^2 s}\times O\left(\frac{\lambda^2 E^8}{M_\sigma^8}\right) = O\left(\frac{E^6_{\text{c.m.}}}{f^4 M_\sigma^4}\right),\\
\sigma(\pi^1+\pi^1\to\pi^1+\pi^1) &= O\left(\frac{E^6_{\text{c.m.}}}{f^4 M_\sigma^4}\right) \ll \frac{E^2_{\text{c.m.}}}{f^4}\,.
\end{aligned} \tag{S.46}$$

For a more accurate approximation to the same-species scattering like $\pi^1+\pi^2\to\pi^1+\pi^1$, we need to go back to the amplitude (S.37) and expand it to second powers in s,t,u. Thus,

$$\begin{aligned}
-\frac{\lambda s}{s-M_\sigma^2} &\approx \frac{\lambda s}{M_\sigma^2} + \frac{\lambda s^2}{M_\sigma^4}\,,\\
-\frac{\lambda t}{t-M_\sigma^2} &\approx \frac{\lambda t}{M_\sigma^2} + \frac{\lambda t^2}{M_\sigma^4}\,,\\
-\frac{\lambda u}{u-M_\sigma^2} &\approx \frac{\lambda u}{M_\sigma^2} + \frac{\lambda u^2}{M_\sigma^4}\,,
\end{aligned} \tag{S.47}$$

and therefore

$$\begin{aligned}
\mathcal{M}(\pi^1+\pi^2\to\pi^1+\pi^1) &= -\frac{\lambda s}{s-M_\sigma^2} - \frac{\lambda t}{t-M_\sigma^2} - \frac{\lambda u}{u-M_\sigma^2}\\
&\approx \frac{\lambda}{M_\sigma^2}\times(s+t+u=0) + \frac{\lambda}{M_\sigma^4}\times(s^2+t^2+u^2).
\end{aligned} \tag{S.48}$$

In the center of mass frame,

$$\begin{aligned}
s^2+t^2+u^2 &= 16E^4 + 16E^4\times\sin^4(\theta/2) + 16E^4\times\cos^4(\theta/2)\\
&= 8E^4\times(3+\cos^2\theta) = \left(E^{\text{tot}}_{\text{cm}}\right)^4\times\frac{3+\cos^2\theta}{2}\,,
\end{aligned} \tag{S.49}$$

hence

$$\frac{d\sigma(\pi^1+\pi^1\to\pi^1+\pi^1)}{d\Omega_{\mathrm{c.m.}}}\ \approx\ \frac{\lambda^2E_{\mathrm{cm}}^6}{256\pi^2M_\sigma^8}\times(3+\cos^2\theta)^2\,, \tag{S.50}$$

and therefore

$$\sigma(\pi^1+\pi^1\to\pi^1+\pi^1)\ \approx\ \frac{7\lambda^2E_{\mathrm{cm}}^6}{80\pi M_\sigma^8}\,. \tag{S.51}$$

Problem **4**(a):

For the ultra-relativistic electrons and positrons, the spinors $u(p,s)$ and $v(p,s)$ become chiral. Indeed, by inspection of eqs. (5), u has chirality matching the electron's helicity — left for $\lambda=-\frac12$ and right for $\lambda=+\frac12$ — while v has chirality opposite to the positron's helicity — left for $\lambda=+\frac12$ and right for $\lambda=-\frac12$. At the same time, the amplitude (4) depends on the electron's and positron's spin states via the 'Dirac sandwich' $\bar v(e^+)\gamma_\nu u(e^-)$ which does not mix helicities. Indeed,

$$\bar v\gamma_\nu u\ =\ v^\dagger\gamma^0\gamma_\nu u\ =\ v^\dagger\begin{pmatrix}\sigma_\nu & 0\\ 0 & \bar\sigma_\nu\end{pmatrix}u\ =\ v_L^\dagger\sigma_\nu u_L\ +\ v_R^\dagger\bar\sigma_\nu u_R\,, \tag{S.52}$$

so if u and v are chiral, then they should have the same chirality — both left or both right — or else $\bar v\gamma_\nu u=0$. In terms of helicities, $u(e^-)$ and $v(e^+)$ being both left-handed means $\lambda(e^-)=-\frac12$ while $\lambda(e^+)=+\frac12$; likewise, $u(e^-)$ and $v(e^+)$ being both right-handed means $\lambda(e^-)=+\frac12$ while $\lambda(e^+)=-\frac12$. Thus, a non-zero amplitude (4) requires the electron and the positron to have opposite helicities, $\lambda(e^+)=-\lambda(e^-)$; for similar helicities of the two initial particles, the amplitude vanishes. $\mathcal{Q.E.D.}$

For future reference, let me calculate the explicit Dirac sandwiches (S.52) for the ultra-relativistic electron and positron spinors (5):

$$\bar v(e_L^+)\gamma_\nu u(e_L^-)\ =\ -2E\begin{pmatrix}0\\ \eta_L\end{pmatrix}^\dagger\begin{pmatrix}\sigma_\nu & 0\\ 0 & \bar\sigma_\nu\end{pmatrix}\begin{pmatrix}\xi_L\\ 0\end{pmatrix}\ =\ 0, \tag{6}$$

$$\bar v(e_L^+)\gamma_\nu u(e_R^-)\ =\ -2E\begin{pmatrix}0\\ \eta_L\end{pmatrix}^\dagger\begin{pmatrix}\sigma_\nu & 0\\ 0 & \bar\sigma_\nu\end{pmatrix}\begin{pmatrix}0\\ \xi_R\end{pmatrix}\ =\ -2E\times\eta_L^\dagger\bar\sigma_\nu\xi_R\,, \tag{S.53}$$

$$\bar{v}(e^+_R)\gamma_\nu u(e^-_L) \;=\; +2E\begin{pmatrix}\eta_R\\ 0\end{pmatrix}^\dagger\begin{pmatrix}\sigma_\nu & 0\\ 0 & \bar{\sigma}_\nu\end{pmatrix}\begin{pmatrix}\xi_L\\ 0\end{pmatrix} \;=\; +2E\times\eta^\dagger_R\sigma_\nu\xi_L\,, \tag{S.54}$$

$$\bar{v}(e^+_R)\gamma_\nu u(e^-_R) \;=\; +2E\begin{pmatrix}\eta_R\\ 0\end{pmatrix}^\dagger\begin{pmatrix}\sigma_\nu & 0\\ 0 & \bar{\sigma}_\nu\end{pmatrix}\begin{pmatrix}0\\ \xi_R\end{pmatrix} \;=\; 0. \tag{6}$$

As promised, the sandwich — and hence the amplitude (4) vanishes for the same helicities.

This fact is of practical importance for the electron-positron colliders. Any kind of fermion pair production — $\mu^-\mu^+$, or $\tau^-\tau^+$, or $q\bar{q}$ — which proceeds through a virtual vector particle — a photon, or Z^0, or even something not yet discovered — would have the $\bar{v}(e^+)\gamma_\nu u(e^-)$ factor in the amplitude, so the electron and the positron must have have opposite helicities, or they would not annihilate each other and make pairs.

Now suppose we have a longitudinally polarized electron beam — say $\lambda = +\frac{1}{2}$ only — but the positron beam is un-polarized. Because of eq. (3), only the left-handed positrons would collide with the right-handed electrons and produce pairs, while the left-handed positrons would do something else. Likewise, if the electron beam has the $\lambda = -\frac{1}{2}$ polarization, then only the right-handed positrons would collide with our left-handed electrons and make pairs, while the left-handed positrons would do something else. Thus, as far as the pair-production is concerned, the positron beam could just as well be longitudinally polarized with $\lambda(e^+) = -\lambda(e^-)$.

In other words, if we want to study polarization effects in fermion pair production, it's enough to longitudinally polarize just the electron beam. We do not need to polarize the positron beam — which is much harder to do — because the electrons of a definite helicity would automatically select positrons of the opposite helicity.

Problem **4**(b):

For the ultra-relativistic muons, the $u(\mu-)$ and $v(\mu^+)$ are chiral, and the chiralities behave exactly similar to the electron and the positron in part (a): the Dirac sandwich $\bar{u}(\mu^-)\gamma^\nu v(\mu^+)$ vanishes unless u and v have the same chirality, which requires the μ^- and the μ^+ to have opposite helicities. Specifically,

$$\begin{aligned}
\bar{u}(\mu_L^-)\gamma_\nu v(\mu_L^+) &= -2E\begin{pmatrix}\xi_L\\ 0\end{pmatrix}^\dagger\begin{pmatrix}\sigma_\nu & 0\\ 0 & \bar{\sigma}_\nu\end{pmatrix}\begin{pmatrix}0\\ \eta_L\end{pmatrix} = 0, & (7)\\
\bar{u}(\mu_L^-)\gamma_\nu v(\mu_R^+) &= -2E\begin{pmatrix}\xi_L\\ 0\end{pmatrix}^\dagger\begin{pmatrix}\sigma_\nu & 0\\ 0 & \bar{\sigma}_\nu\end{pmatrix}\begin{pmatrix}\eta_R\\ 0\end{pmatrix} = +2E\times\xi_L^\dagger\sigma_\nu\eta_R\,, & \text{(S.55)}\\
\bar{u}(\mu_R^-)\gamma_\nu v(\mu_L^+) &= -2E\begin{pmatrix}0\\ \xi_L\end{pmatrix}^\dagger\begin{pmatrix}\sigma_\nu & 0\\ 0 & \bar{\sigma}_\nu\end{pmatrix}\begin{pmatrix}0\\ \eta_L\end{pmatrix} = -2E\times\xi_R^\dagger\bar{\sigma}_\nu\eta_L\,, & \text{(S.56)}\\
\bar{u}(\mu_R^-)\gamma_\nu v(\mu_L^+) &= -2E\begin{pmatrix}0\\ \xi_L\end{pmatrix}^\dagger\begin{pmatrix}\sigma_\nu & 0\\ 0 & \bar{\sigma}_\nu\end{pmatrix}\begin{pmatrix}\eta_R\\ 0\end{pmatrix} = 0. & (7)
\end{aligned}$$

Eqs. (7) — and similar formulae for the other fermion-antifermion pairs produced with ultra-relativistic speeds in electron-positron collisions — assure that the fermion and the antifermion always have opposite helicities. Experimentally, this means that if for some event we are able to determine the helicity of one final particle, then we may infer the second final particle's helicity without any further experimental effort.

Problem **4**(c):

The electron moves in the positive z direction, so its helicity λ is the same as its S_z — the z component of its spin. Hence, the ξ spinors corresponding to the 2 helicities are

$$\xi(e_L^-) = \begin{pmatrix}0\\ 1\end{pmatrix} \quad\text{and}\quad \xi(e_R^-) = \begin{pmatrix}1\\ 0\end{pmatrix}. \tag{S.57}$$

The positron moves in the negative z direction, so its helicity is opposite from S_z, hence

$$\xi(e_L^+) = \begin{pmatrix}1\\ 0\end{pmatrix} \quad\text{and}\quad \xi(e_R^+) = \begin{pmatrix}0\\ 1\end{pmatrix}. \tag{S.58}$$

However, the η spinors in eqs. (12.2) for the positrons have opposite spins from these ξ

spinors, specifically $\eta = \sigma_2 \xi^*$, thus

$$\eta(e^+_L) \;=\; \begin{pmatrix} 0 \\ +i \end{pmatrix} \quad \text{and} \quad \eta(e^+_R) \;=\; \begin{pmatrix} -i \\ 0 \end{pmatrix}. \tag{S.59}$$

Substituting these 2–component spinors into eqs. (S.53) and (S.54), we obtain

$$\begin{aligned}
\bar v(e^+_L)\gamma^\nu u(e^-_R) &= -2E \times (\,0 \quad -i\,)\,\bar\sigma^\nu \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= +2iE \times (\bar\sigma^\nu)_{21} \\
&= 2E \times (0, +i, +1, 0)^\nu, \\
\bar v(e^+_R)\gamma^\nu u(e^-_L) &= +2E \times (\,+i \quad 0\,)\,\sigma^\nu \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= +2iE \times (\sigma^\nu)_{12} \\
&= 2E \times (0, -i, +1, 0)^\nu.
\end{aligned} \tag{8}$$

When taking the 21 and 12 matrix elements of the $\bar\sigma^\nu$ and σ^ν matrices, please remember that $\bar\sigma^\nu = (1, \sigma^x, \sigma^y, \sigma^z)$ while $\sigma^\nu = (1, -\sigma^x, -\sigma^y, -\sigma^z)$.

Problem **4**(d):

Suppose for a moment $\theta = 0$ and the $\mu^\mp$ move in the same directions as $e^\mp$. Then the muons have exactly the same spinors u and v as the $e^\mp$ of the same charge and helicity, and similarly to eqs. (12.5) we have

$$\bar v(\mu^+_L)\gamma^\nu u(\mu^-_R) \;=\; 2E\times(0, +i, +1, 0)^\nu \quad \text{and} \quad \bar v(\mu^+_R)\gamma^\nu u(\mu^-_L) \;=\; 2E\times(0, -i, +1, 0)^\nu. \tag{S.60}$$

For the muons, the amplitude (4) involves the $\bar u(\mu^-)\gamma^\nu v(\mu^+)$ Dirac sandwich rather than the $\bar v(\mu^+)\gamma^\nu u(\mu^-)$, but these two sandwiches are related by complex conjugation,

$$\big(\bar v \gamma^\nu u\big)^* \;=\; \bar u \overline{\gamma^\nu} v = \; \bar u \gamma^\nu v, \tag{S.61}$$

hence

$$\begin{aligned}
\bar u(\mu^-_R)\gamma^\nu v(\mu^+_L) &= 2E \times (0, +i, +1, 0)^{\nu *} \;=\; 2E \times (0, -i, +1, 0)^\nu, \\
\bar u(\mu^-_L)\gamma^\nu v(\mu^+_R) &= 2E \times (0, -i, +1, 0)^{\nu *} \;=\; 2E \times (0, +i, +1, 0)^\nu.
\end{aligned} \tag{S.62}$$

Eqs. (S.62) apply for $\theta = 0$. For other muon directions, we may simply rotate the

4–vectors (S.62) through angle θ in the xz plane, thus

$$\begin{aligned}\bar{u}(\mu_R^-)\gamma^\nu v(\mu_L^+) &= 2E\times(0,-i\cos\theta,+1,+i\sin\theta)^\nu,\\ \bar{u}(\mu_L^-)\gamma^\nu v(\mu_R^+) &= 2E\times(0,+i\cos\theta,+1,-i\sin\theta)^\nu.\end{aligned} \tag{10}$$

Problem **4**(e):

Substituting the Dirac sandwiches (8) and (10) into the pair production amplitude (4), we obtain

$$\begin{aligned}\langle\mu_L^-,\mu_R^+|\,\mathcal{M}\,|e_L^-,e_R^+\rangle &= \langle\mu_R^-,\mu_L^+|\,\mathcal{M}\,|e_R^-,e_L^+\rangle = -e^2\times(1+\cos\theta),\\ \langle\mu_R^-,\mu_L^+|\,\mathcal{M}\,|e_L^-,e_R^+\rangle &= \langle\mu_L^-,\mu_R^+|\,\mathcal{M}\,|e_R^-,e_L^+\rangle = -e^2\times(1-\cos\theta),\end{aligned} \tag{S.63}$$

while all the other polarized amplitudes vanish by eqs. (6) and (7):

$$\begin{aligned}\langle\mu_{\rm any}^-,\mu_{\rm any}^+|\,\mathcal{M}\,|e_L^-,e_L^+\rangle &= \langle\mu_{\rm any}^-,\mu_{\rm any}^+|\,\mathcal{M}\,|e_R^-,e_R^+\rangle = 0,\\ \langle\mu_L^-,\mu_L^+|\,\mathcal{M}\,|e_{\rm any}^-,e_{\rm any}^+\rangle &= \langle\mu_R^-,\mu_R^+|\,\mathcal{M}\,|e_{\rm any}^-,e_{\rm any}^+\rangle = 0.\end{aligned} \tag{S.64}$$

The partial cross-sections (11) follow from these amplitudes according to

$$\frac{d\sigma}{d\Omega_{\rm c.m.}} = \frac{|\mathcal{M}|^2}{64\pi^2 s}\times\left(\frac{|\mathbf{p}'|}{|\mathbf{p}|}=1\right). \tag{S.65}$$

Problem **4**(f):

Summing the polarized cross-sections (11) over the muons' helicities, we get

$$\begin{aligned}\frac{d\sigma(e_L^-+e_R^+\to\mu_{\rm any}^-+\mu_{\rm any}^+)}{d\Omega_{\rm c.m.}} &= \frac{d\sigma(e_R^-+e_L^+\to\mu_{\rm any}^-+\mu_{\rm any}^+)}{d\Omega_{\rm c.m.}}\\ &= \frac{\alpha^2}{4s}\times(1+\cos\theta)^2 + \frac{\alpha^2}{4s}\times(1-\cos\theta)^2 + 0 + 0\\ &= \frac{\alpha^2}{2s}\times(1+\cos^2\theta)\end{aligned} \tag{S.66}$$

while

$$\frac{d\sigma(e_L^-+e_L^+\to\mu_{\rm any}^-+\mu_{\rm any}^+)}{d\Omega_{\rm c.m.}} = \frac{d\sigma(e_R^-+e_R^+\to\mu_{\rm any}^-+\mu_{\rm any}^+)}{d\Omega_{\rm c.m.}} = 0. \tag{S.67}$$

Averaging these cross-sections over the electron's and positron's helicities gives

$$\frac{d\sigma(e^-_{\rm avg}+e^+_{\rm avg}\to\mu^-_{\rm any}+\mu^+_{\rm any})}{d\Omega_{\rm c.m.}} = \frac{1}{4}\left(\frac{\alpha^2}{2s}\times(1+\cos^2\theta)\ +\ \frac{\alpha^2}{2s}\times(1+\cos^2\theta)\ +\ 0\ +\ 0\right)$$
$$= \frac{\alpha^2}{4s}\times(1+\cos^2\theta), \tag{S.68}$$

which is exactly what we found in class for the un-polarized cross-section when $E\gg M_\mu$.