Problem **1**(a):

In the first diagram (1), the virtual photon has momentum $q = p'_1 - p_1 = p_2 - p'_2$, hence $q^2 = t$. In the second diagram, the virtual photon's momentum is $\tilde{q} = p_1 + p_2 = p'_1 + p'_2$, hence $\tilde{q}^2 = s$. Accordingly, the two diagrams are called the s-channel diagram and the t-channel diagram.

The t-channel diagram evaluates to

$$\begin{aligned} i\mathcal{M}_1 &= -\Big(\bar{v}(e^+)(ie\gamma_\mu)v(e^{+\prime})\Big) \times \Big(\bar{u}(e^{-\prime})(ie\gamma_\nu)u(e^-)\Big) \times \frac{-ig^{\mu\nu}}{q^2} \\ &= \frac{-ie^2}{t} \times \bar{v}(e^+)\gamma_\mu v(e^{+\prime}) \times \bar{u}(e^{-\prime})\gamma^\mu u(e^-) \end{aligned} \tag{S.1}$$

where the overall minus sign is due to the positron-out to positron-in fermionic line. And the s-channel diagram evaluates to

$$\begin{aligned} i\mathcal{M}_2 &= +\Big(\bar{v}(e^+)(ie\gamma_\mu)u(e^-)\Big) \times \Big(\bar{u}(e^{-\prime})(ie\gamma_\nu)v(e^{+\prime}\Big) \times \frac{-ig^{\mu\nu}}{\tilde{q}^2} \\ &= \frac{+ie^2}{s} \times \bar{v}(e^+)\gamma_\mu u(e^-) \times \bar{u}(e^{-\prime})\gamma^\mu v(e^{+\prime}) \end{aligned} \tag{S.2}$$

where the overall sign is plus because both fermionic lines have an incoming or outgoing electron at one end.

Problem **1**(b):

Summing /averaging the $|\mathcal{M}_2|^2$ over spins works exactly as for the muon pair production discussed in class:

$$\begin{aligned} \sum_{\text{spins}} |\mathcal{M}_2|^2 &= \left(\frac{e^2}{s}\right)^2 \sum_{\text{spins}} \Big[\bar{v}(e^+)\gamma_\mu u(e^-) \times \bar{u}(e^-)\gamma_\nu v(e^+)\Big] \times \Big[\bar{u}(e^{-\prime})\gamma^\mu v(e^{+\prime}) \times \bar{v}(e^{+\prime})\gamma^\nu u(e^{-\prime})\Big] \\ &= \left(\frac{e^2}{s}\right)^2 \operatorname{tr}\left[(\not{p}_2 - m)\gamma_\mu(\not{p}_1 + m)\gamma_\nu\right] \times \operatorname{tr}\left[(\not{p}'_1 - m)\gamma^\mu(\not{p}'_2 - m)\gamma^\nu\right] \\ &\qquad \langle\!\langle \text{ neglecting the mass relative to the momenta } \rangle\!\rangle \\ &\approx \left(\frac{e^2}{s}\right)^2 \operatorname{tr}\left[\not{p}_2\gamma_\mu \not{p}_1\gamma_\nu\right] \times \operatorname{tr}\left[\not{p}'_1\gamma^\mu \not{p}'_2\gamma^\nu\right] \end{aligned} \tag{S.3}$$

$$\begin{aligned}
&= \left(\frac{e^2}{s}\right)^2 \times 4\left[p_{2\mu}p_{1\nu} + p_{2\nu}p_{1\mu} - g_{\mu\nu}(p_2p_1)\right] \times 4\left[p_2'^\mu p_1'^\nu + p_2'^\nu p_1'^\mu - g^{\mu\nu}(p_2'p_1')\right] \\
&= 16\left(\frac{e^2}{s}\right)^2 \Big[2(p_2'p_2)(p_1'p_1) + 2(p_2'p_1)(p_1'p_2). \\
&\qquad - 2(p_2'p_1')(p_2p_1) - 2(p_2'p_1')(p_2p_1) + 4(p_2'p_1')(p_2p_1)\Big] \\
&= 32\left(\frac{e^2}{s}\right)^2 \left[(p_2'p_2)(p_1'p_1) + (p_2'p_1)(p_1'p_2)\right] \\
&= 8\left(\frac{e^2}{s}\right)^2 \left[t^2 + u^2\right] \qquad \text{(S.3)}
\end{aligned}$$

where the last equality follows from the kinematic relations (4). Altogether,

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_2|^2 = 2e^4 \times \frac{t^2+u^2}{s^2}\,. \qquad (5)$$

Problem **1**(c):

The two diagrams for Bhabha scattering are related by the *crossing symmetry*, so the amplitudes $\mathcal{M}_1$ and $\mathcal{M}_2$ are related to each other via analytic continuation of particle's momenta. In terms of the spin-summed $|\mathcal{M}|^2$ and Mandelstam variables,

$$\sum_{\text{spins}} |\mathcal{M}_1(s,t,u)|^2 = \sum_{\text{spins}} |\mathcal{M}_2(t,s,u)|^2, \qquad \text{(S.4)}$$

hence eq. (5) for the second amplitude implies a similar equation for the first amplitude, but with s and t exchanged with each other — *i.e.*, eq. (6).

Alternatively, we may sum the $|\mathcal{M}_1|^2$ over all the spins in the same way as we summed the $|\mathcal{M}_2|^2$ in part (b):

$$\begin{aligned}
\sum_{\text{spins}} |\mathcal{M}_1|^2 &= \left(\frac{e^2}{t}\right)^2 \sum_{\text{spins}} \left[\bar{u}(e^{-\prime})\gamma^\mu u(e^-) \times \bar{u}(e^-)\gamma^\nu u(e^{-\prime})\right] \times \left[\bar{v}(e^+)\gamma_\mu v(e^{+\prime}) \times \bar{v}(e^{+\prime})\gamma_\nu v(e^+)\right] \\
&= \left(\frac{e^2}{t}\right)^2 \operatorname{tr}\left[(\not{p}_1' + m)\gamma^\mu(\not{p}_1 + m)\gamma^\nu\right] \times \operatorname{tr}\left[(\not{p}_2 - m)\gamma_\mu(\not{p}_2' - m)\gamma_\nu\right] \\
&\approx \left(\frac{e^2}{t}\right)^2 \operatorname{tr}\left[\not{p}_1'\gamma^\mu \not{p}_1\gamma^\nu\right] \times \operatorname{tr}\left[\not{p}_2\gamma_\mu \not{p}_2'\gamma_\nu\right] \qquad \text{(S.5)}
\end{aligned}$$

$$= \left(\frac{e^2}{t}\right)^2 \times 4\left[p_1'^{\mu}p_1^{\nu} + p_1'^{\nu}p_1^{\mu} - g^{\mu\nu}(p_1'p_1)\right] \times 4\left[p'_{2\mu}p_{2\nu} + p'_{2\nu}p_{2\mu} - g_{\mu\nu}(p_2'p_2)\right]$$

$$= 16\left(\frac{e^2}{t}\right)^2 \Big[2(p_1'p_2')(p_1p_2) + 2(p_1'p_2)(p_1p_2') - 2(p_1'p_1)(p_2'p_2) - 2(p_1'p_1)(p_2'p_2) + 4(p_1'p_1)(p_2'p_2)\Big]$$

$$= 32\left(\frac{e^2}{t}\right)^2 \left[(p_1'p_2')(p_1p_2) + (p_1'p_2)(p_1p_2')\right]$$

$$= 8\left(\frac{e^2}{t}\right)^2 \left[s^2 + u^2\right] \tag{S.5}$$

and hence

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_1|^2 = 2e^4 \times \frac{s^2+u^2}{t^2}\,. \tag{6}$$

Problem **1**(d):

The interference term between the two diagrams is more complicated:

$$\mathcal{M}_1^* \times \mathcal{M}_2 = -\frac{e^2}{t}\Big(\bar{u}(e^-)\gamma^\nu u(e^{-\prime}) \times \bar{v}(e^{+\prime})\gamma_\nu v(e^+)\Big) \times \times \frac{e^2}{s}\Big(\bar{v}(e^+)\gamma_\mu u(e^-) \times \bar{u}(e^{-\prime})\gamma^\mu v(e^{+\prime})\Big)$$
$$= -\frac{e^4}{st} \times \bar{u}(e^-)\gamma^\nu u(e^{-\prime}) \times \bar{u}(e^{-\prime})\gamma^\mu v(e^{+\prime}) \times \bar{v}(e^{+\prime})\gamma_\nu v(e^+) \times \bar{v}(e^+)\gamma_\mu u(e^-) \tag{S.6}$$

where on the last line I have re-ordered the factors so that each $\bar{u}$ is followed by u of the same electron and each $\bar{v}$ is followed by v for the same positron. After summing over all the spins, each $u \times \bar{u}$ becomes $(\not{p} + m)$, each $v \times \bar{v}$ becomes $(\not{p} - m)$, and the whole product becomes a single big trace rather than a product of two traces,

$$\sum_{\text{spins}} \mathcal{M}_1^* \times \mathcal{M}_2 = -\frac{e^4}{st} \times \operatorname{tr}\Big[(\not{p}_1 + m)\gamma^\nu(\not{p}_1' + m)\gamma^\mu(\not{p}_2' - m)\gamma_\nu(\not{p}_2 - m)\gamma_\mu\Big]$$
$$\approx -\frac{e^4}{st} \times \operatorname{tr}\Big[\not{p}_1\gamma^\nu\,\not{p}_1'\gamma^\mu\,\not{p}_2'\gamma_\nu\,\not{p}_2\gamma_\mu\Big]. \tag{S.7}$$

This trace looks more complicated than it is, and we may drastically simplify it by summing over ν and μ before taking the trace. Back in homework set#7 we saw that

$$\gamma^\alpha \not{a}\,\not{b}\,\not{c}\gamma_\alpha = -2\,\not{c}\,\not{b}\,\not{a} \quad \text{and} \quad \gamma^\alpha \not{a}\,\not{b}\gamma_\alpha = 4(ab). \tag{S.8}$$

For the problem at hand, this gives us $\gamma^\nu \not{p}_1'\gamma^\mu \not{p}_2'\gamma_\nu = -2\,\not{p}_2'\gamma^\mu \not{p}_1'$ and hence

$$\begin{aligned}
\operatorname{tr}\Big[\not{p}_1 \times \gamma^\nu \not{p}_1'\gamma^\mu \not{p}_2'\gamma_\nu \times \not{p}_2\gamma_\mu\Big] &= -2\operatorname{tr}\Big[\not{p}_1 \times \not{p}_2'\gamma^\mu \not{p}_1' \times \not{p}_2\gamma_\mu\Big] = -2\operatorname{tr}\Big[\not{p}_1\,\not{p}_2' \times \gamma^\mu \not{p}_1'\,\not{p}_2\gamma_\mu\Big] \\
&= -2\operatorname{tr}\Big[\not{p}_1\,\not{p}_2' \times 4(p_1'p_2)\Big] = -8(p_1'p_2) \times \operatorname{tr}\Big[\not{p}_1\,\not{p}_2'\Big] \\
&= -8(p_1'p_2) \times 4(p_1p_2') \\
&= -8u^2.
\end{aligned} \tag{S.9}$$

Plugging this trace back into eq. (S.6), we arrive at

$$\frac{1}{4}\sum_{\text{spins}} \mathcal{M}_1^* \times \mathcal{M}_2 = +2e^4 \times \frac{u^2}{st}\,. \tag{7}$$

Problem **1**(e):

Assembling the spin sums / averages (5–7) together according to eq. (3), we get

$$\begin{aligned}
\overline{|\mathcal{M}|^2} &\stackrel{\text{def}}{=} \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_1 + \mathcal{M}_2|^2 \\
&= \frac{1}{4}\sum_{\text{spins}} \Big(|\mathcal{M}_1|^2 + |\mathcal{M}_2|^2 + 2\operatorname{Re}\mathcal{M}_1^*\mathcal{M}_2\Big) \\
&= 2e^4 \times \frac{s^2+u^2}{t^2} + 2e^4 \times \frac{t^2+u^2}{s^2} + 4e^4 \times \frac{u^2}{st} \\
&= 2e^4\left(\frac{s^2}{t^2} + \frac{t^2}{s^2} + \frac{u^2}{s^2t^2} \times \Big(s^2+t^2+2st = (s+t)^2 = u^2\Big)\right) \\
&= 2e^4 \times \frac{s^4+t^4+u^4}{s^2 \times t^2}\,.
\end{aligned} \tag{S.10}$$

Consequently, the un-polarized partial cross-section for the Bhabha scattering is

$$\frac{d\sigma}{d\Omega_{\text{c.m.}}} = \frac{\overline{|\mathcal{M}|^2}}{64\pi^2 s} = \frac{\alpha^2}{2s} \times \frac{s^4+t^4+u^4}{s^2 \times t^2}\,. \tag{S.11}$$

To complete the problem, let's do the kinematics. In the center of mass frame

$$\begin{aligned} s &= 4E^2 \approx 4\mathbf{p}^2, \\ t &= -(\mathbf{p}'_1 - \mathbf{p}_1)^2 = -2\mathbf{p}^2(1-\cos\theta), \\ u &= -(\mathbf{p}'_2 - \mathbf{p}_1)^2 = -2\mathbf{p}^2(1+\cos\theta), \end{aligned} \tag{S.12}$$

hence

$$\begin{aligned} \frac{s^4+t^4+u^4}{s^2\,t^2} &= \frac{(4\mathbf{p}^2)^4 + (2\mathbf{p}^2)^4\times(1-\cos\theta)^4 + (2\mathbf{p}^2)^4\times(1+\cos\theta)^4}{(4\mathbf{p}^2)^2\times(2\mathbf{p}^2)^2(1-\cos\theta)^2} \\ &= \frac{16+(1-\cos\theta)^4+(1+\cos\theta)^4}{4\times(1-\cos\theta)^2} = \frac{18+12\cos^2\theta+2\cos^4\theta}{4\times(1-\cos\theta)^2} \\ &= \frac{(3+\cos^2\theta)^2}{2(1-\cos\theta)^2}\,. \end{aligned} \tag{S.13}$$

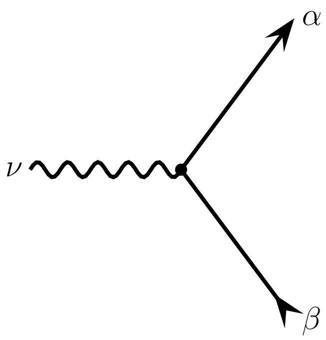
Plugging this formula into eq. (S.11), we finally obtain

$$\frac{d\sigma^{\text{Bhabha}}}{d\Omega_{\text{cm}}} = \frac{\alpha^2}{4s}\times\frac{(3+\cos^2\theta)^2}{(1-\cos\theta)^2}\,. \tag{8}$$

Quod erat demonstrandum.

Problem **2**(a):

Eq. (12) shows similar couplings of the Z^ν field to all charged leptons e,μ,τ, so in the Feynman rules for Z, the Zee, $Z\mu\mu$, and $Z\tau\tau$ vertices look completely similar:



$$= -ig\big(\gamma^\nu\Gamma\big)_{\alpha\beta} \tag{S.14}$$

where Γ is the short-hand for

$$\Gamma = \sin^2\theta_w - \frac{1-\gamma^5}{4}\,. \tag{S.15}$$

For future reference, let me also introduce

$$\overline{\Gamma} \;=\; \sin^2\theta_w \;-\; \frac{1+\gamma^5}{4} \tag{S.16}$$

so that $\gamma^\nu\Gamma = \overline{\Gamma}\gamma^\nu$.

Given the Z propagator (11) and the Zee and $Z\mu\mu$ vertices (S.14), the second diagram (9) — muon pair creation via a virtual Z in the s channel — evaluates to

$$i\mathcal{M}_Z \;=\; \frac{-i}{q^2-M_Z^2}\left(g^{\nu\lambda} \;-\; \frac{q^\nu q^\lambda}{M_Z^2}\right) \times \bar{v}(e^+)(-ig'\gamma_\nu\Gamma)u(e^-) \times \bar{u}(\mu^-)(-ig'\gamma_\lambda\Gamma)v(\mu^+) \tag{S.17}$$

where $q = p_1 + p_2 = p_1' + p_2'$, $q^2 = s$.

Let's simplify this amplitude a bit by suppressing the $q^\nu q^\lambda/M_Z^2$ term in the Z propagator. Were Z to have QED-like vertices ($ie\gamma^\nu$) we could have eliminated this term completely thanks to Gordon identities, but for the real-life vertices (S.14) the Gordon identities do not quite work. Indeed, let's multiply the q^ν by the electron line and q^λ by the muon line: Skipping the $-ig'$ factors, we have

$$\begin{aligned} q^\nu \times \bar{v}(e^+)(\gamma_\nu\Gamma)u(e^-) \;&=\; g'\,\bar{v}\,\not{q}\Gamma u \;=\; \bar{v}(\not{p}_1+\not{p}_2)\Gamma u \\ &=\; \bar{v}\,\not{p}_2\Gamma u \;-\; \bar{v}\overline{\Gamma}\,\not{p}_1 u \\ &=\; (-m_e\bar{v})\Gamma u \;-\; \overline{\Gamma}(+m_e u) \;=\; m_e \times \bar{v}(\overline{\Gamma}-\Gamma)\;u \\ &=\; \tfrac{1}{2}m_e \times \bar{v}\gamma^5 u, \end{aligned} \tag{S.18}$$

and similarly

$$q^\lambda \times \bar{u}(\mu^-)(\gamma_\nu\Gamma)v(\mu^+) \;=\; -\tfrac{1}{2}m_\mu \times \bar{u}\gamma^5 v. \tag{S.19}$$

We see that the right hand sides of these formulae do not quite vanish, but for the ultra-relativistic electrons and muons, they are suppressed by small factors of m_e or m_μ instead of much larger momentum factors q. In the context of the amplitude (S.17), these formulae give us

$$
\begin{aligned}
\mathcal{M}_Z \;=\; \frac{g'^2}{s-M_Z^2} \times \Bigl(& \bar{v}(e^+)\gamma^\nu \Gamma u(e^-) \times \bar{u}(\mu^-)\gamma_\nu \Gamma v(\mu^+) \\
& + \frac{m_e m_\mu}{4M_Z^2} \times \bar{v}(e^+)\gamma^5 u(e^-) \times \bar{u}(\mu^-)\gamma^5 v(\mu^+) \Bigr),
\end{aligned}
\tag{S.20}
$$

where the second term in () is suppressed relative to the first term by the very small factor

$$
\frac{m_e m_\mu}{4M_Z^2} \;=\; \frac{(511\ \text{keV}) \times (106\ \text{MeV})}{4(91\ \text{GeV})^2} \;\approx\; 1.6 \cdot 10^{-9}. \tag{S.21}
$$

Since our calculations are tree-level — *i.e.*, lowest order in perturbation theory — and we are ignoring higher-order corrections we expect to be smaller by powers of $\alpha/2\pi \sim 10^{-3}$, any terms suppressed by factors as small as (S.21) should be neglected, thus

$$
\mathcal{M}_Z \;=\; \frac{g'^2}{s-M_Z^2} \times \bar{v}(e^+)\gamma^\nu \Gamma u(e^-) \times \bar{u}(\mu^-)\gamma_\nu \Gamma v(\mu^+). \tag{S.22}
$$

By the way, the third diagram (9) — with the Higgs scalar in the s channel — is suppressed by the same small factor (S.21), and that's why we neglecting it in our calculations of the muon pair production. Indeed, as we shall learn later in class, the quarks and the leptons get their masses from their Yukawa couplings to the Higgs, so the light fermions like the electron and the muon have very small Yukawa couplings,

$$
Y(Hee) \;=\; \frac{m_e}{175\ \text{Gev}} \;\approx\; 3 \cdot 10^{-6}, \quad Y(H\mu\nu) \;=\; \frac{m_\mu}{175\ \text{Gev}} \;\approx\; 6 \cdot 10^{-4}. \tag{S.23}
$$

Consequently,

$$
\frac{\mathcal{M}_H}{\mathcal{M}_Z} \;\sim\; \frac{Y(Hee) \times Y(H\mu\mu)}{g'^2} \;=\; \frac{m_e \times m_\mu}{2M_Z^2} \;\approx\; 3.2 \cdot 10^{-9}, \tag{S.24}
$$

and that's why the third diagram may me safely neglected compared to the first two diagrams.

Problem **2**(b):

Now let's evaluate the amplitude (S.22) for the ultrarelativistic electrons and muons of specific helicities. Following the techniques of problem 4 of the previous homework, we start with the chiral ultrarelativistic spinors

$$u_L = \sqrt{2E}\begin{pmatrix}\xi_L\\ 0\end{pmatrix}, \quad u_R = \sqrt{2E}\begin{pmatrix}0\\ \xi_R\end{pmatrix}, \quad v_L = -\sqrt{2E}\begin{pmatrix}0\\ \eta_L\end{pmatrix}, \quad v_R = \sqrt{2E}\begin{pmatrix}\eta_R\\ 0\end{pmatrix}, \tag{S.25}$$

cf. eqs. (11-13) from homework set#7, and evaluate the Dirac sandwiches in the amplitude (S.22). Let's start with the electron-line sandwich

$$\bar{v}(e^+)\gamma^\nu\Gamma u(e^-) = v^\dagger(e^+)\gamma^0\gamma^\nu\Gamma u(e^-) = v^\dagger(e^+)\begin{pmatrix}g_L\bar{\sigma}^\nu & 0\\ 0 & g_R\sigma^\nu\end{pmatrix}u(e^-), \tag{S.26}$$

where the second equality follows from eq. (14). From this formula we immediately see that the electron and the positron must have the same chiralities and hence opposite helicities, otherwise there is no pair-production,

$$\bar{v}(e^+_L)\gamma^\nu\Gamma u(e^-_L) = \bar{v}(e^+_R)\gamma^\nu\Gamma u(e^-_R) = 0. \tag{S.27}$$

Also, for the similar helicities we have

$$\begin{aligned}\bar{v}(e^+_R)\gamma^\nu\Gamma u(e^-_L) &= 2Eg_L\times\eta^\dagger_R\bar{\sigma}^\nu\xi_L = 2Eg_L\times(0,-i,1,0)^\nu,\\ \bar{v}(e^+_L)\gamma^\nu\Gamma u(e^-_R) &= -2Eg_R\times\eta^\dagger_R\sigma^\nu\xi_L = 2Eg_R\times(0,+i,1,0)^\nu,\end{aligned} \tag{S.28}$$

where the two-component matrix products $\eta^\dagger_R\bar{\sigma}^\nu\xi_L$ and $\eta^\dagger_L\sigma^\nu\xi_R$ are evaluated exactly as in problem eq. (8) from the previous homework (problem 4(c)). Naturally, we make the same assumptions about the kinematics: center-of-mass frame of reference, and the collision along the z axis, $\mathbf{p}_1 = (0,0,+E)$ and $\mathbf{p}_2 = (0,0,-E)$.

Similarly, the Dirac sandwiches for the muons

$$\bar{u}(\mu^-)\gamma^\nu\Gamma v(\mu^+) = u^\dagger(\mu^-)\begin{pmatrix}g_L\bar{\sigma}^\nu & 0\\ 0 & g_R\sigma^\nu\end{pmatrix}v(\mu^+) \tag{S.29}$$

vanish for the $\mu^+\mu^-$ having similar helicities and hence opposite chiralities,

$$\bar{u}(\mu^-_L)\gamma^\nu\Gamma v(\mu^+_L) = \bar{u}(\mu^-_R)\gamma^\nu\Gamma v(\mu^+_R) = 0, \tag{S.30}$$

while for opposite helicities we have formulae similar to eqs. (10) from the previous homework:

Assuming without loss of generality that the muons move in the xz plane in the directions $\mathbf{p}'_1 = (+E\sin\theta, 0, +E\cos\theta)$, $\mathbf{p}'_2 = (-E\sin\theta, 0, -E\cos\theta)$, we have

$$\begin{aligned} \bar{u}(\mu_L^-)\gamma^\nu\Gamma v(\mu_R^+) &= 2Eg_L \times \xi_L^\dagger\bar{\sigma}^\nu\eta_R = 2Eg_L \times (0, +i\cos\theta, +1, -i\sin\theta)^\nu, \\ \bar{u}(\mu_R^-)\gamma^\nu\Gamma v(\mu_L^+) &= 2Eg_R \times \xi_L^\dagger\bar{\sigma}^\nu\eta_R = 2Eg_R \times (0, -i\cos\theta, +1, +i\sin\theta)^\nu. \end{aligned} \tag{S.31}$$

It remains to plug in the Dirac sandwiches (S.27) through (S.31) into eq. (S.22) to get the amplitudes:

$$\begin{aligned} \mathcal{M}_Z(e_L^- + e_R^+ \to \mu_L^- + \mu_R^+) &= \frac{g'^2 s}{s - M_Z^2} \times -g_L^2(1+\cos\theta), \\ \mathcal{M}_Z(e_L^- + e_R^+ \to \mu_R^- + \mu_L^+) &= \frac{g'^2 s}{s - M_Z^2} \times -g_L g_R(1-\cos\theta), \\ \mathcal{M}_Z(e_R^- + e_L^+ \to \mu_L^- + \mu_R^+) &= \frac{g'^2 s}{s - M_Z^2} \times -g_L G_R(1-\cos\theta), \\ \mathcal{M}_Z(e_R^- + e_L^+ \to \mu_R^- + \mu_L^+) &= \frac{g'^2 s}{s - M_Z^2} \times -g_R^2(1+\cos\theta), \\ \mathcal{M}_Z(\text{other helicities}) &= 0. \end{aligned} \tag{S.32}$$

Problem **2**(c):

Combining the amplitudes (S.32) due to the virtual Z particles with the amplitudes due to the virtual photon from the previous homework

$$\begin{aligned} \mathcal{M}_\gamma(e_L^- + e_R^+ \to \mu_L^- + \mu_R^+) &= \mathcal{M}_\gamma(e_R^- + e_L^+ \to \mu_R^- + \mu_L^+) = -e^2 \times (1+\cos\theta), \\ \mathcal{M}_\gamma(e_L^- + e_R^+ \to \mu_R^- + \mu_L^+) &= \mathcal{M}_\gamma(e_R^- + e_L^+ \to \mu_R^- + \mu_L^+) = -e^2 \times (1-\cos\theta), \\ \mathcal{M}_\gamma(\text{other helicities}) &= 0, \end{aligned} \tag{S.33}$$

we arrive at the net tree amplitudes

$$\begin{aligned} \mathcal{M}(e_L^- + e_R^+ \to \mu_L^- + \mu_R^+) &= -\left(e^2 + g'^2 g_L^2 \times \frac{s}{s - M_Z^2}\right) \times (1+\cos\theta), \\ \mathcal{M}(e_L^- + e_R^+ \to \mu_R^- + \mu_L^+) &= -\left(e^2 + g'^2 g_L g_R \times \frac{s}{s - M_Z^2}\right) \times (1-\cos\theta), \\ \mathcal{M}(e_R^- + e_L^+ \to \mu_L^- + \mu_R^+) &= -\left(e^2 + g'^2 g_L g_R \times \frac{s}{s - M_Z^2}\right) \times (1-\cos\theta), \\ \mathcal{M}(e_R^- + e_L^+ \to \mu_R^- + \mu_L^+) &= -\left(e^2 + g'^2 g_R^2 \times \frac{s}{s - M_Z^2}\right) \times (1+\cos\theta). \\ \mathcal{M}(\text{other helicities}) &= 0. \end{aligned} \tag{S.34}$$

Before we convert these amplitudes to the partial cross-sections, let's make explicit

$$g_L = x - \tfrac{1}{2}, \quad g_R = x \quad \text{for} \quad x \stackrel{\text{def}}{=} \sin^2\theta_w \tag{S.35}$$

as well as

$$g'^2 = \frac{e^2}{\sin^2\theta_w \cos^2\theta_2} = \frac{e^2}{x(1-x)} \tag{S.36}$$

and

$$\frac{e^4}{64\pi^2 E_{\text{cm}}^2} = \frac{\alpha_{\text{QED}}^2}{4s}\,. \tag{S.37}$$

With these preliminaries, plugging the amplitudes (S.34) into the partial cross-section formula

$$\frac{d\sigma}{d\Omega_{\text{c.m.}}} = \frac{|\mathcal{M}|^2}{64\pi^2 E_{\text{cm}}^2}\,, \tag{S.38}$$

we arrive at

$$\begin{aligned}
\frac{d\sigma(e_L^- + e_R^+ \to \mu_L^- + \mu_R^+)}{d\Omega_{\text{c.m.}}} &= \frac{\alpha_{\text{QED}}^2}{4s}\left(1 + \frac{(x-\frac{1}{2})^2}{x(1-x)}\cdot\frac{s}{s-M_Z^2}\right)^2 \times (1+\cos\theta)^2,\\
\frac{d\sigma(e_L^- + e_R^+ \to \mu_R^- + \mu_L^+)}{d\Omega_{\text{c.m.}}} &= \frac{\alpha_{\text{QED}}^2}{4s}\left(1 + \frac{x(x-\frac{1}{2})}{x(1-x)}\cdot\frac{s}{s-M_Z^2}\right)^2 \times (1-\cos\theta)^2,\\
\frac{d\sigma(e_R^- + e_L^+ \to \mu_L^- + \mu_R^+)}{d\Omega_{\text{c.m.}}} &= \frac{\alpha_{\text{QED}}^2}{4s}\left(1 + \frac{x(x-\frac{1}{2})}{x(1-x)}\cdot\frac{s}{s-M_Z^2}\right)^2 \times (1-\cos\theta)^2,\\
\frac{d\sigma(e_R^- + e_L^+ \to \mu_R^- + \mu_L^+)}{d\Omega_{\text{c.m.}}} &= \frac{\alpha_{\text{QED}}^2}{4s}\left(1 + \frac{x^2}{x(1-x)}\cdot\frac{s}{s-M_Z^2}\right)^2 \times (1+\cos\theta)^2,\\
\frac{d\sigma(\text{other helicities})}{d\Omega_{\text{c.m.}}} &= 0.
\end{aligned} \tag{S.39}$$

For the sake of calculational simplicity, we may approximate $x = 0.232 \approx \frac{1}{4}$, hence

$$\frac{(x-\frac{1}{2})^2}{x(1-x)} \approx \frac{x^2}{x(1-x)} \approx +\frac{1}{3} \qquad \frac{x(x-\frac{1}{2})}{x(1-x)} \approx -\frac{1}{3}\,,$$

which gives us

$$\begin{aligned}
\frac{d\sigma(e_L^- e_R^+ \to \mu_L^- \mu_R^+)}{d\Omega_{\text{c.m.}}} &= \frac{d\sigma(e_R^- e_L^+ \to \mu_R^- \mu_L^+)}{d\Omega_{\text{c.m.}}} = \frac{\alpha_{\text{QED}}^2}{4s}\left(\frac{4s - 3M_Z^2}{3(s - M_Z^2)}\right)^2 \times (1 + \cos\theta)^2, \\
\frac{d\sigma(e_L^- e_R^+ \to \mu_R^- \mu_L^+)}{d\Omega_{\text{c.m.}}} &= \frac{d\sigma(e_R^- e_L^+ \to \mu_L^- \mu_R^+)}{d\Omega_{\text{c.m.}}} = \frac{\alpha_{\text{QED}}^2}{4s}\left(\frac{2s - 3M_Z^2}{3(s - M_Z^2)}\right)^2 \times (1 - \cos\theta)^2,
\end{aligned} \tag{S.40}$$

Now let's integrate over the directions of the outgoing muons. We need to consider both the total cross-sections and the asymmetries (15) for two types of angular dependence. For $\frac{d\sigma}{d\Omega} \propto (1 + \cos\theta)^2$ we have

$$\begin{aligned}
\int_{\theta<\frac{\pi}{2}} d^2\Omega\,(1 + \cos\theta)^2 &= 2\pi \int_0^{+1} d\cos\theta\,(1 + \cos\theta)^2 = \frac{14\pi}{3}\,, \\
\int_{\theta>\frac{\pi}{2}} d^2\Omega\,(1 + \cos\theta)^2 &= 2\pi \int_{-1}^{0} d\cos\theta\,(1 + \cos\theta)^2 = \frac{2\pi}{3}\,,
\end{aligned} \tag{S.41}$$

and hence

$$\sigma_{\text{tot}} \propto \frac{16\pi}{3} \qquad \text{and} \quad A = +\frac{3}{4}\,. \tag{S.42}$$

For the other angular dependence, $\frac{d\sigma}{d\Omega} \propto (1 - \cos\theta)^2$, we use $\theta \to \pi - \theta$ symmetry and immediately obtain

$$\sigma_{\text{tot}} \propto \frac{16\pi}{3} \qquad \text{and} \quad A = -\frac{3}{4}\,. \tag{S.43}$$

Consequently, for the four allowed helicity combinations, we have

$$\begin{aligned}
A &= +\frac{3}{4}\,, \qquad \sigma_{\text{tot}} = \frac{4\pi\alpha^2}{3s}\left(\frac{4s - 3M_Z^2}{3(s - M_Z^2)}\right)^2 \qquad \text{for } e_L^- e_R^+ \to \mu_L^- \mu_R^+ \text{ or } e_R^- e_L^+ \to \mu_R^- \mu_L^+\,, \\
A &= -\frac{3}{4}\,, \qquad \sigma_{\text{tot}} = \frac{4\pi\alpha^2}{3s}\left(\frac{2s - 3M_Z^2}{3(s - M_Z^2)}\right)^2 \qquad \text{for } e_L^- e_R^+ \to \mu_R^- \mu_L^+ \text{ or } e_R^- e_L^+ \to \mu_L^- \mu_R^+\,.
\end{aligned} \tag{S.44}$$

It remains to sum over the muons' helicities and average over the electron's and positron's helicities. The net result is total cross-section

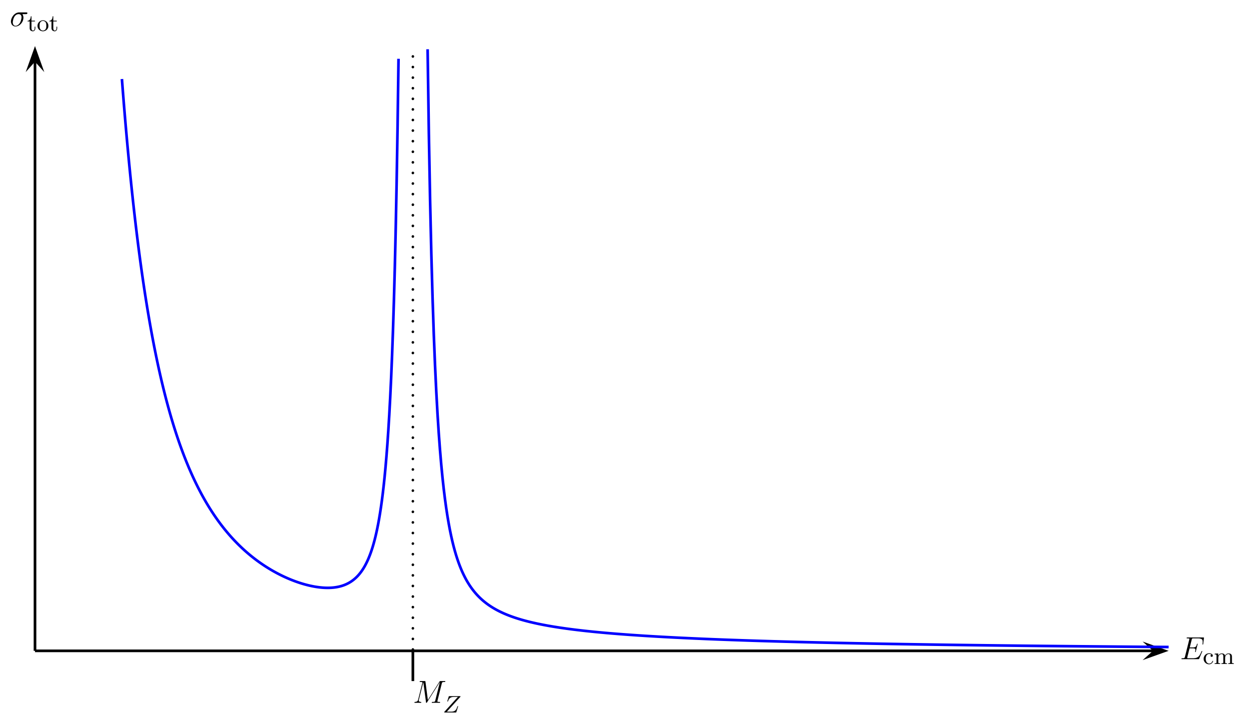
$$\begin{aligned}\sigma_{\rm tot}(e^-e^+ \to \mu^-\mu^+) &= \frac{4\pi\alpha^2}{3s} \times \frac{2}{4}\left(\left(\frac{4s - 3M_Z^2}{3(s - M_Z^2)}\right)^2 + \left(\frac{42 - 3M_Z^2}{3(s - M_Z^2)}\right)^2 + 0 + 0\right) \\ &= \frac{4\pi\alpha^2}{3s} \times \left(1 + \frac{s^2}{9(s - M_Z^2)^2}\right)\end{aligned} \tag{S.45}$$

and the asymmetry

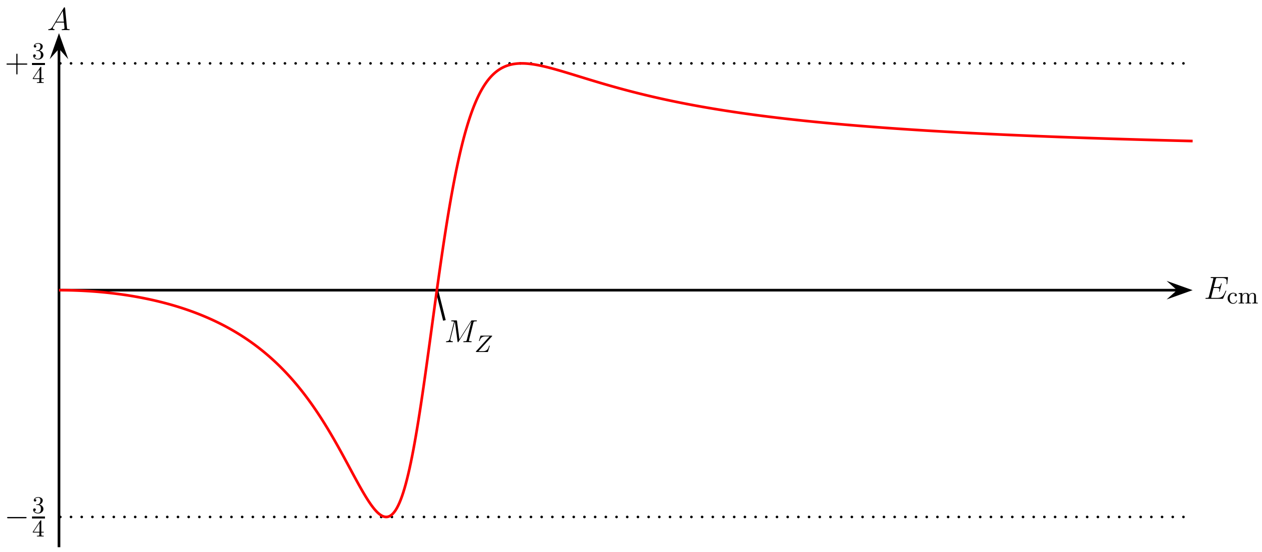
$$A = \frac{\frac{3}{4}(4s - 3M_Z^2)^2 - \frac{3}{4}(2s - 3M_Z^2)^2}{(4s - 3M_Z^2)^2 + (2s - 3M_Z^2)^2} = \frac{9s(s - M_Z^2)}{2s^2 + 18(s - M_Z^2)^2}. \tag{S.46}$$

Note the asymmetry is positive for $s > M_Z^2$ and negative for $s < M_Z^2$.

Graphically,



and



For your information, without the $x = \frac{1}{4}$ approximation, one has

$$\sigma_{\rm tot} = \frac{4\pi\alpha^2}{3s} \times \frac{1 + 2n + (r+n)^2}{r^2}\,,$$
$$A = \frac{3}{4} \times \frac{2(r+2n)}{1 + 2n + (r+n)^2}\,, \tag{S.47}$$

where I have denoted

$$n \stackrel{\rm def}{=} (1-4x)^2 \approx 0.0052 \qquad \text{and} \quad r \stackrel{\rm def}{=} 16x(1-x)\,\frac{s - M_Z^2}{s} \xrightarrow[x=\frac{1}{4}]{} 3\,\frac{s - M_Z^2}{s}\,. \tag{S.48}$$

Note that n is quite small, so eqs. (S.45) and (S.46) provide a very good approximation. In fact, the corrections due to $x \neq \frac{1}{4}$ are smaller than the leading loop corrections in the Standard Model, so we were quite justified in approximating $x \approx \frac{1}{4}$ at the tree level of analysis.

Problem **3**(a):

A point of notation: In the solutions to problem 3, the indices μ, e, $\nu \equiv \nu_\mu$, and $\bar{\nu} \equiv \bar{\nu}_e$ denote the particles. For the Lorentz indices, I shall use α, β, γ, δ, κ, λ, σ, ρ, but never μ or ν. Thus, $p_{\mu\alpha}$ denotes the α component of the muon's 4–momentum, *etc.*, *etc.*

Let's start with the muon decay amplitude

$$\mathcal{M}(\mu^- \to e^- \nu_\mu \bar{\nu}_e) \;=\; \frac{G_F}{\sqrt{2}} \left[\bar{u}(\nu_\mu)\gamma^\alpha(1-\gamma^5)u(\mu^-)\right] \times \left[\bar{u}(e^-)\gamma_\alpha(1-\gamma^5)v(\bar{\nu}_e)\right]. \tag{17}$$

Since the Dirac conjugate of the $\gamma^\alpha(1-\gamma^5)$ matrix is the same

$$\overline{\gamma^\alpha(1-\gamma^5)} \;=\; (1-\bar{\gamma}^5)\bar{\gamma}^\alpha \;=\; (1+\gamma^5)\gamma^\alpha \;=\; \gamma^\alpha(1-\gamma^5), \tag{S.49}$$

the complex conjugate of the amplitude (17) is

$$\mathcal{M}^* \;=\; \frac{G_F}{\sqrt{2}} \left[\bar{u}(\mu^-)\gamma^\alpha(1-\gamma^5)\gamma^\alpha u(\nu_\mu)\right] \times \left[\bar{v}(\bar{\nu}_e)\gamma_\alpha(1-\gamma^5)u(e^-)\right]. \tag{S.50}$$

Consequently,

$$\begin{aligned}
|\mathcal{M}|^2 \;&=\; \tfrac{1}{2}G_F^2 \times \left[\bar{u}(\nu_\mu)\gamma^\alpha(1-\gamma^5)u(\mu^-)\right] \times \left[\bar{u}(e^-)\gamma_\alpha(1-\gamma^5)v(\bar{\nu}_e)\right] \times \\
&\qquad \times \left[\bar{u}(\mu^-)\gamma^\beta(1-\gamma^5)u(\nu_\mu)\right] \times \left[\bar{v}(\bar{\nu}_e)\gamma_\beta(1-\gamma^5)u(e^-)\right] \\
&=\; \tfrac{1}{2}G_F^2 \times \left[\bar{u}(\nu_\mu)\gamma^\alpha(1-\gamma^5)u(\mu^-) \times \bar{u}(\mu^-)\gamma^\beta(1-\gamma^5)u(\nu_\mu)\right] \times \\
&\qquad \times \left[\bar{u}(e^-)\gamma_\alpha(1-\gamma^5)v(\bar{\nu}_e) \times \bar{v}(\bar{\nu}_e)\gamma_\beta(1-\gamma^5)u(e^-)\right],
\end{aligned}$$

and when this $|\mathcal{M}|^2$ is summed over the final fermions spins and averaged over the spin of the initial muon, it becomes a product of two traces,

$$\begin{aligned}
\overline{|\mathcal{M}|^2} \;\stackrel{\text{def}}{=}\; \tfrac{1}{2}\sum_{\substack{\text{all}\\\text{spins}}} |\mathcal{M}|^2 \;&=\; \tfrac{1}{4}G_F^2 \times \operatorname{tr}\Big(\gamma^\alpha(1-\gamma^5)(\not{p}_\mu + M_\mu)\gamma^\beta(1-\gamma^5)(\not{p}_\nu + m_\nu)\Big) \\
&\qquad \times \operatorname{tr}\Big(\gamma_\alpha(1-\gamma^5)(\not{p}_{\bar{\nu}} - m_{\bar{\nu}})\gamma_\beta(1-\gamma^5)(\not{p}_e + m_e)\Big).
\end{aligned} \tag{S.51}$$

Our next task is to evaluate these traces. For the first trace, we have

$$\begin{aligned}
&\operatorname{tr}\Big(\gamma^\alpha(1-\gamma^5)(\not{p}_\mu+M_\mu)\gamma^\beta(1-\gamma^5)(\not{p}_\nu+m_\nu)\Big) \;= \\
&\quad=\;\operatorname{tr}\Big(\gamma^\alpha(1-\gamma^5)\,\not{p}_\mu\gamma^\beta(1-\gamma^5)\,\not{p}_\nu\Big)\;+\;M_\mu m_\nu\times\operatorname{tr}\Big(\gamma^\alpha(1-\gamma^5)\gamma^\beta(1-\gamma^5)\Big) \\
&\qquad\quad+\;\text{vanishing traces of odd numbers of }\gamma^\lambda\text{ matrices} \\
&\qquad\quad\langle\!\langle\,\text{next, move the }(1-\gamma^5)\text{ factors left using }\gamma^\lambda(1\mp\gamma^5)=(1\pm\gamma^5)\gamma^\lambda\,\rangle\!\rangle \\
&\quad=\;\operatorname{tr}\Big((1+\gamma^5)\gamma^\alpha\,\not{p}_\mu\gamma^\beta\,\not{(}1-\gamma^5)p_\nu\Big)\;+\;M_\mu m_\nu\times\operatorname{tr}\Big((1+\gamma^5)\gamma^\alpha\gamma^\beta(1-\gamma^5)\Big) \\
&\quad=\;\operatorname{tr}\Big((1+\gamma^5)^2\gamma^\alpha\,\not{p}_\mu\gamma^\beta\,\not{p}_\nu\Big)\;+\;M_\mu m_e\times\operatorname{tr}\Big((1+\gamma^5)(1-\gamma^5)\gamma^\alpha\gamma^\beta\Big) \\
&\qquad\quad\langle\!\langle\,\text{use }(1+\gamma^5)^2=2(1+\gamma^5)\text{ while }(1+\gamma^5)(1-\gamma^5)=0\,\rangle\!\rangle \\
&\quad=\;2\operatorname{tr}\Big((1+\gamma^5)\gamma^\alpha\,\not{p}_\mu\gamma^\beta\,\not{p}_\nu\Big)\;+\;0 \\
&\quad=\;2\operatorname{tr}\Big(\gamma^\alpha\not{p}_\mu\gamma^\beta\not{p}_\nu\Big)\;+\;2\operatorname{tr}\Big(\gamma^5\gamma^\alpha\not{p}_\mu\gamma^\beta\not{p}_\nu\Big)\,.
\end{aligned}\tag{S.52}$$

The traces on the last line here were explicitly evaluated in my notes on Dirac traces:

$$\begin{aligned}
\operatorname{tr}\Big(\gamma^\alpha\not{p}_\mu\gamma^\beta\not{p}_\nu\Big)\;&=\;4p_\mu^\alpha p_\nu^\beta\;+\;4p_\mu^\beta p_\nu^\alpha\;-\;4g^{\alpha\beta}(p_\mu\cdot p_\nu), \\
\operatorname{tr}\Big(\gamma^5\gamma^\alpha\not{p}_\mu\gamma^\beta\not{p}_\nu\Big)\;&=\;-4i\epsilon^{\alpha\gamma\beta\delta}p_{\mu\gamma}p_{\nu\delta}\,,
\end{aligned}\tag{S.53}$$

hence

$$\begin{aligned}
&\operatorname{tr}\Big(\gamma^\alpha(1-\gamma^5)(\not{p}_\mu+M_\mu)\gamma^\beta(1-\gamma^5)(\not{p}_\nu+m_\nu)\Big) \;= \\
&\qquad=\;8\Big[p_\mu^\alpha p_\nu^\beta\;+\;p_\mu^\beta p_\nu^\alpha\;-\;g^{\alpha\beta}(p_\mu\cdot p_\nu)\Big]\;-\;8i\epsilon^{\alpha\gamma\beta\delta}p_{\mu\gamma}p_{\nu\delta}.
\end{aligned}\tag{S.54}$$

In exactly the same way, the second big trace in eq. (S.51) evaluates to

$$\begin{aligned}
&\operatorname{tr}\big(\gamma_\alpha(1-\gamma^5)(\not{p}_e+m_e)\gamma_\beta(1-\gamma^5)(\not{p}_{\bar\nu}-m_{\bar\nu})\big) \;= \\
&\qquad=\;8\big[(p_{e\alpha}p_{\bar\nu\beta}\;+\;p_{e\beta}p_{\bar\nu\alpha}\;-\;g_{\alpha\beta}(p_e\cdot p_{\bar\nu})\big]\;-\;8i\epsilon_{\alpha\rho\beta\sigma}p_{\bar\nu}^\rho p_e^\sigma\,.
\end{aligned}\tag{S.55}$$

Now let's plug the traces (S.54) and (S.55) back into eq. (S.51) and contract the Lorentz indices α and β,

$$\begin{aligned}\overline{|\mathcal{M}|^2} \;=\; 16G_F^2 \times \left(\left[p_\mu^\alpha p_\nu^\beta \,+\, p_\mu^\beta p_\nu^\alpha \,-\, g^{\alpha\beta}(p_\mu\cdot p_\nu)\right] \,-\, i\epsilon^{\alpha\gamma\beta\delta}p_{\mu\gamma}p_{\nu\delta}\right)\times \\ \times\left(\left[p_{e\alpha}p_{\bar\nu\beta} \,+\, p_{e\beta}p_{\bar\nu\alpha} \,-\, g_{\alpha\beta}(p_e\cdot p_{\bar\nu})\right] \,-\, i\epsilon_{\alpha\rho\beta\sigma}p_{\bar\nu}^\rho p_e^\sigma\right).\end{aligned} \tag{S.56}$$

Note that inside each pair of () here, the first term is symmetric WRT $\alpha \leftrightarrow \beta$ while the second term is antisymmetric, so when we multiply the two factors together and contract the indices, only the symmetric $\times$ symmetric and antisymmetric $\times$ antisymmetric products contribute to the

$$\overline{|\mathcal{M}|^2} \;=\; 16G_F^2\left(\begin{array}{c}\left[p_\mu^\alpha p_\nu^\beta \,+\, p_\mu^\beta p_\nu^\alpha \,-\, g^{\alpha\beta}(p_\mu\cdot p_\nu)\right]\times\left[p_{e\alpha}p_{\bar\nu\beta} \,+\, p_{e\beta}p_{\bar\nu\alpha} \,-\, g_{\alpha\beta}(p_e\cdot p_{\bar\nu})\right] \\ \qquad -\; i0 \;-\, i0 \;-\; \epsilon^{\alpha\gamma\beta\delta}p_{\mu\gamma}p_{\nu\delta}\times\epsilon_{\alpha\rho\beta\sigma}p_{\bar\nu}^\rho p_e^\sigma\end{array}\right). \tag{S.57}$$

Moreover, when we open the brackets on the top line here, several terms cancel each other:

$$\begin{aligned}\left[p_\mu^\alpha p_\nu^\beta \,+\, p_\mu^\beta p_\nu^\alpha \,-\, g^{\alpha\beta}(p_\mu\cdot p_\nu)\right]\times\left[p_{e\alpha}p_{\bar\nu\beta} \,+\, p_{e\beta}p_{\bar\nu\alpha} \,-\, g_{\alpha\beta}(p_e\cdot p_{\bar\nu})\right] \;=& \\ =\; 2(p_\mu\cdot p_e)(p_\nu\cdot p_{\bar\nu}) \;+\; 2(p_\mu\cdot p_{\bar\nu})(p_\nu\cdot p_e)& \\ -\;\cancel{2\times(p_\mu\cdot p_\nu)(p_e\cdot p_{\bar\nu})} \;-\; \cancel{2\times(p_\mu\cdot p_\nu)(p_e\cdot p_{\bar\nu})}& \\ +\;\cancel{(g^{\alpha\beta}g_{\alpha\beta}=4)\times(p_\mu\cdot p_\nu)(p_e\cdot p_{\bar\nu})}& \\ =\; 2(p_\mu\cdot p_e)(p_\nu\cdot p_{\bar\nu}) \;+\; 2(p_\mu\cdot p_{\bar\nu})(p_\nu\cdot p_e).&\end{aligned} \tag{S.58}$$

As to the second line of eq. (S.57), contracting the two Levi–Civita tensors gives us

$$\epsilon^{\alpha\gamma\beta\delta}\times\epsilon_{\alpha\rho\beta\sigma} \;=\; -2\delta^\gamma_\rho\delta^\delta_\sigma \;+\; 2\delta^\gamma_\sigma\delta^\delta_\rho\,, \tag{S.59}$$

hence

$$-\epsilon^{\alpha\gamma\beta\delta}p_{\mu\gamma}p_{\nu\delta}\times\epsilon_{\alpha\rho\beta\sigma}p_{\bar\nu}^\rho p_e^\sigma \;=\; +2(p_\mu\cdot p_{\bar\nu})(p_\nu\cdot p_e) \;-\; 2(p_\mu\cdot p_e)(p_\nu\cdot p_{\bar\nu}). \tag{S.60}$$

Finally, combining the two lines of eq. (S.57) gives us one more cancellation, thus

$$\overline{|\mathcal{M}|^2} \;=\; 16G_F^2\left(\begin{array}{c}\cancel{2(p_\mu\cdot p_e)(p_\nu\cdot p_{\bar\nu})} \;+\; 2(p_\mu\cdot p_{\bar\nu})(p_\nu\cdot p_e) \\ +\; 2(p_\mu\cdot p_{\bar\nu})(p_\nu\cdot p_e) \;-\; \cancel{2(p_\mu\cdot p_e)(p_\nu\cdot p_{\bar\nu})}\end{array}\right) \;=\; 64G_F^2\times(p_\mu\cdot p_{\bar\nu})(p_\nu\cdot p_e). \tag{18}$$

Quod erat demonstrandum.

Problem $\mathbf{3}$(b):

As explained in the *Peskin & Schroeder* textbook, partial rate of a decay process (in the rest frame of the initial particle) is given by

$$d\Gamma \;=\; \frac{1}{2M_0}\times\overline{|\mathcal{M}|^2}\times d\mathcal{P} \tag{S.61}$$

where $\mathcal{M}$ is the decay's amplitude, $\overline{|\mathcal{M}|^2}$ is $|\mathcal{M}|^2$ averaged over the unknown initial spins and summed over the unmeasured final spins, and $d\mathcal{P}$ is the infinitesimal phase space factor for the final particles. For three final particles,

$$d\mathcal{P} \;=\; \frac{d^3\mathbf{p}_1}{(2\pi)^3(2E_1)}\,\frac{d^3\mathbf{p}_2}{(2\pi)^3(2E_2)}\,\frac{d^3\mathbf{p}_3}{(2\pi)^3(2E_3)}\times(2\pi)^3\delta^{(3)}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\times(2\pi)\delta(E_1+E_2+E_3-M_0) \tag{S.62}$$

where the energy-momentum conservation laws apply in the rest frame, thus $\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3=\mathbf{p}_{\rm tot}=\mathbf{0}$ and $E_1+E_2+E_3=E_{\rm tot}=M_0$.

We start by using the momentum-conservation δ–function to eliminate the $\mathbf{p}_3$ as independent variable, thus

$$d\mathcal{P} \;=\; \frac{d^3\mathbf{p}_1\,d^3\mathbf{p}_2}{256\pi^5}\times\left.\frac{\delta(E_1+E_2+E_3-E_{\rm tot})}{E_1E_2E_3}\right|_{\mathbf{p}_3=-(\mathbf{p}_1+\mathbf{p}_2)}. \tag{S.63}$$

Next, we use spherical coordinates for the two remaining momenta,

$$d^3\mathbf{p}_1 \;=\; p_1^2\,dp_1\,d^2\Omega_1\,, \qquad d^3\mathbf{p}_2 \;=\; p_2^2\,dp_2\,d^2\Omega_2\,, \tag{S.64}$$

and then replace the $d^2\Omega_2$ describing the direction of the second particle's momentum relative to the fixed external frame with

$$d^2\Omega_2^{(1)} \;=\; d\theta_{12}\,\sin\theta_{12}\,d\phi_2^{(1)}$$

describing the same direction of $\mathbf{p}_2$ relative to the frame centered on the $\mathbf{p}_1$. Consequently,

$$d^2\Omega_1\,d^2\Omega_2 \;=\; d^2\Omega_1\,d^2\Omega_2^{(1)} \;=\; \left[d^2\Omega_1\,d\phi_2^{(1)}\right]\,d\theta_{12}\,\sin\theta_{12} \;\equiv\; d^3\Omega\times d(\cos\theta_{12}) \tag{S.65}$$

and hence

$$dP = \frac{d^3\Omega}{256\pi^5} \times \frac{p_1^2 p_2^2}{E_1 E_2 E_3} \, dp_1 \, dp_2 \times d(\cos\theta_{12}) \, \delta(E_1 + E_2 + E_3 - E_{\rm tot}) \Big|_{\mathbf{p}_3 = -(\mathbf{p}_1 + \mathbf{p}_2)} . \quad \text{(S.66)}$$

Next, we use the cosine theorem

$$p_3^2 = (\mathbf{p}_1 + \mathbf{p}_2)^2 = p_1^2 + p_2^2 + 2p_1 p_2 \cos\theta_{12}$$

which gives

$$d(\cos\theta_{12}) = \frac{p_3 \, dp_3}{p_1 \, p_2}$$

(for fixed p_1, p_2), and therefore

$$dP = \frac{d^3\Omega}{256\pi^5} \times \frac{p_1 p_2 p_3}{E_1 E_2 E_3} \times dp_1 \, dp_2 \, dp_3 \times \delta(E_1 + E_2 + E_3 - E_{\rm tot}). \quad \text{(S.67)}$$

Finally, we notice that for a relativistic particle of any mass, $pdp = EdE$ and hence

$$dP = \frac{d^3\Omega}{256\pi^5} \times dE_1 \, dE_2 \, dE_3 \, \delta(E_1 + E_2 + E_3 - E_{\rm tot}). \quad \text{(S.68)}$$

Substituting this formula into eq. (S.61) gives eq. (19) for the partial decay rate. $\mathcal{Q.E.D.}$

Problem **3**(c):

The kinematic limits on the final particles' energies follow from the triangle inequalities for the magnitudes of three momentum vectors which add up to zero:

$$\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 = \mathbf{0} \implies p_1 \le p_2 + p_3 \quad \textbf{and} \quad p_2 \le p_1 + p_3 \quad \textbf{and} \quad p_3 \le p_1 + p_2 \, . \quad \text{(S.69)}$$

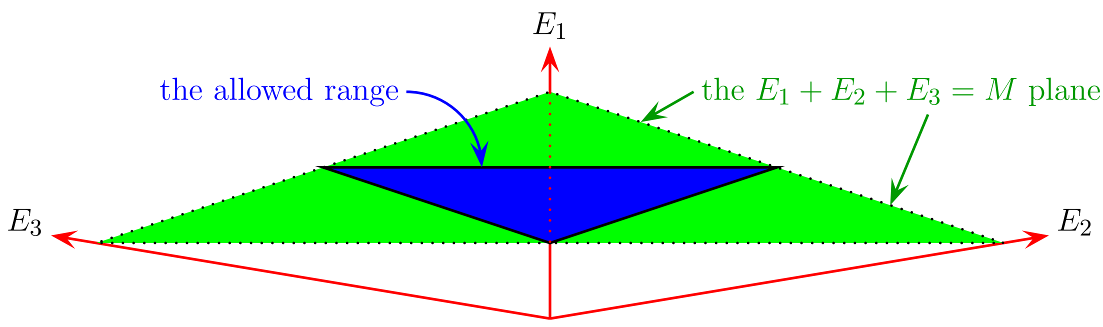
These inequalities look simple in terms of momenta but generally produce rather complicated inequalities for the energies $E_1 = \sqrt{p_1^2 + m_1^2}$, $E_2 = \sqrt{p_2^2 + m_2^2}$, and $E_3 = \sqrt{p_3^2 + m_3^2}$. However, when all three final particles are massless, the kinematic restrictions become simply

$$\begin{aligned} E_1 &\le E_2 + E_3 = M - E_1 \, , \\ E_2 &\le E_1 + E_3 = M - E_2 \, , \\ E_3 &\le E_1 + E_2 = M - E_3 \, , \end{aligned} \quad \text{(S.70)}$$

where the second expression on each right hand side follows from the net energy conservation $E_1 + E_2 + E_3 = M$. In other words, the kinematically allowed energies of the three final particles' range over

$$0 \;\le\; E_1, E_2, E_3 \;\le\; \tfrac{1}{2}M_0\,, \qquad \text{while} \quad E_1 + E_2 + E_3 \;=\; M_0\,. \tag{21}$$

The picture below shows this range in the (E_1, E_2, E_3) space:



(S.71)

Problem **3**(d):

In the muon's rest frame

$$(p_\mu \cdot p_{\bar\nu}) \;=\; M_\mu E_{\bar\nu} \tag{S.72}$$

while

$$\begin{aligned}
(p_e \cdot p_e) \;&=\; E_e E_\nu \;-\; p_e p_\nu \cos\theta_{e\nu} \\
&=\; E_e E_\nu \;+\; \tfrac{1}{2}p_e^2 \;+\; \tfrac{1}{2}p_\nu^2 \;-\; \tfrac{1}{2}p_{\bar\nu}^2 \\
&\qquad \langle\!\langle \text{ neglecting } m_e, m_\nu, m_{\bar\nu} \rangle\!\rangle \\
&\approx\; E_e E_\nu \;+\; \tfrac{1}{2}E_e^2 \;+\; \tfrac{1}{2}E_\nu^2 \;-\; \tfrac{1}{2}E_{\bar\nu}^2 \\
&=\; \tfrac{1}{2}(E_e + E_\nu)^2 \;-\; \tfrac{1}{2}E_{\bar\nu}^2 \\
&\qquad \langle\!\langle \text{ using } E_e + E_\nu = M_\mu - E_{\bar\nu} \rangle\!\rangle \\
&=\; \tfrac{1}{2}(M_\mu - E_{\bar\nu})^2 \;-\; \tfrac{1}{2}E_{\bar\nu}^2 \\
&=\; \tfrac{1}{2}M_\mu(M_\mu - 2E_{\bar\nu}).
\end{aligned} \tag{S.73}$$

Consequently, the spin-averaged muon decay amplitude^2 (18) becomes

$$\overline{|\mathcal{M}|^2} \;=\; 32 G_F^2 M_\mu^2 E_{\bar\nu}(M_\mu - 2E_{\bar\nu}). \tag{S.74}$$

Plugging this formula into eq. (19) for the decay rate gives us

$$d\Gamma(\mu^- \to e^- \nu_\mu \bar{\nu}_e) \;=\; \frac{G_F^2}{16\pi^5}\, M_\mu E_{\bar\nu}(M_\mu - 2E_{\bar\nu}) \times dE_e\, dE_\nu\, dE_{\bar\nu}\, d^3\Omega\, \delta(E_e + E_\nu + E_{\bar\nu} - M_\mu), \tag{S.75}$$

and all we need to do now is to integrate this formula over the final-state variables.

The integration variables comprise 3 angles $d^3\Omega$ — which integrate to $\int d^3\Omega = 8\pi^2$ — and 3 particles' energies subject to the constraint $E_e + E_\nu + E_{\bar\nu} = M_\mu$ and the kinematic limits (20). Integrating the decay rate (S.75) over these variables, we have

$$\begin{aligned}
\Gamma \;&=\; \frac{G_F^2 M_\mu}{2\pi^3} \int\limits_0^{M_\mu/2} dE_e \int\limits_0^{M_\mu/2} dE_{\bar\nu} E_{\bar\nu}(M_\mu - 2E_{\bar\nu}) \times \int\limits_0^{M_\mu/2} dE_\nu\, \delta(E_e + E_\nu + E_{\bar\nu} - M_\mu) \\
&=\; \frac{G_F^2 M_\mu}{2\pi^3} \int\limits_0^{M_\mu/2} dE_e \int\limits_0^{M_\mu/2} dE_{\bar\nu} E_{\bar\nu}(M_\mu - 2E_{\bar\nu}) \times \text{restrict to}(E_\nu = M - E_e - E_{\bar\nu} \le \tfrac12 M) \\
&=\; \frac{G_F^2 M_\mu}{2\pi^3} \int\limits_0^{\frac12 M_\mu} dE_e \int\limits_{\frac12 M_\mu - E_e}^{\frac12 M_\mu} dE_{\bar\nu}\, E_{\bar\nu}(M_\mu - 2E_{\bar\nu}) \\
&\qquad \langle\!\langle \text{ where the lower limit of the } \textstyle\int dE_{\bar\nu} \text{ comes from } E_\nu \le \tfrac12 M_\mu \Longrightarrow E_e + E_{\bar\nu} \ge \tfrac12 M_\mu \,\rangle\!\rangle \\
&=\; \frac{G_F^2 M_\mu}{2\pi^3} \int\limits_0^{\frac12 M_\mu} dE_e\, E_e^2(\tfrac12 M_\mu - \tfrac23 E_e).
\end{aligned} \tag{S.76}$$

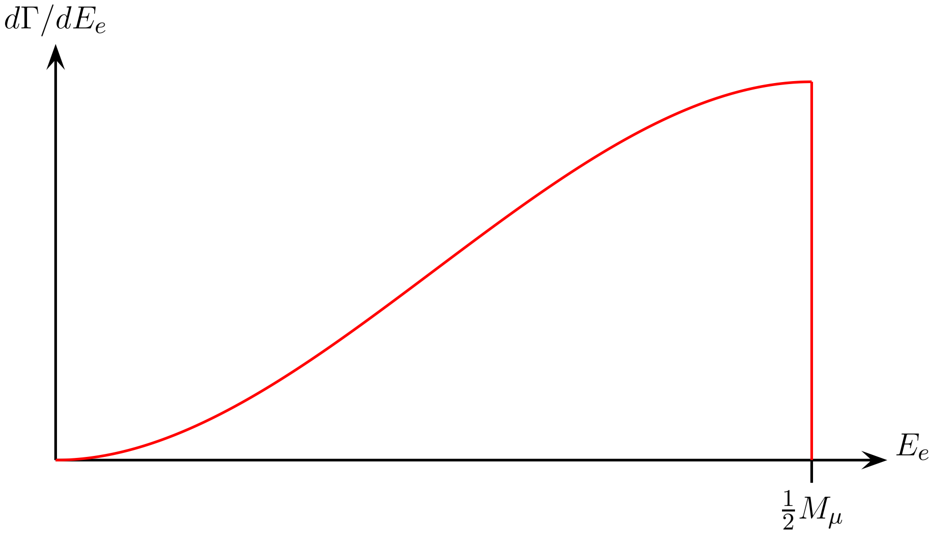
In other words, the partial muon decay rate with respect to the final electron's energy is given by

$$\frac{d\Gamma}{dE_e} \;=\; \frac{G_F^2 M_\mu}{12\pi^3} \times E_e^2(3M_\mu - 4E_e) \tag{S.77}$$

or rather

$$\frac{d\Gamma}{dE_e} \;\approx \begin{cases} \frac{G_F^2}{12\pi^3}\, M_\mu E_e^2(3M_\mu - 4E_e) & \text{for } E_e < \frac12 M_\mu, \\ 0 & \text{for } E_e > \frac12 M_\mu. \end{cases} \tag{S.78}$$

Graphically,



Note how this curve smoothly reaches its maximum at $E_e = \frac{1}{2}M_\mu$ and then abruptly falls down to zero.

It remains to calculate the total decay rate of the muon by integrating the partial rate (S.78) over the electron's energy. The result is

$$\Gamma_{\text{tot}}(\mu \to e\nu\bar{\nu}) \;=\; \frac{G_F^2 M_\mu}{12\pi^3} \times \int\limits_0^{\frac{1}{2}M_\mu} dE_e\, E_e^2(3M_\mu - 4E_e) \;=\; \frac{G_F^2 M_\mu^5}{192\pi^3}\,. \tag{S.79}$$