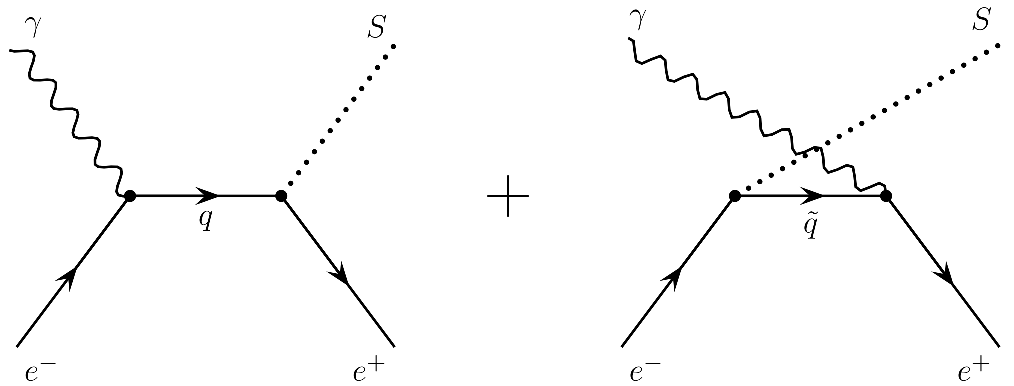
Problem **2**(a):

There are two tree diagrams for the $e^-e^+ \to S\gamma$ process, namely



(S.1)

These two diagrams are related by $t \leftrightarrow u$ crossing, and also by the charge conjugation (which exchanges the initial e^- and e^+). The net tree-level amplitude is

$$\begin{aligned}
\mathcal{M}_{\rm tree} &= \mathcal{E}^{*\mu}_{\mathbf{k},\lambda}(\gamma) \times \mathcal{M}_\mu\,, \\
\mathcal{M}^\mu &= \mathcal{M}^\mu_1 + \mathcal{M}^\mu_2\,, \\
\mathcal{M}^\mu_1 &= -i\,\bar v(e^+)(-ig)\frac{i}{\not q - m_e}\,(ie\gamma^\mu)\,u(e^-) \\
&= \frac{eg}{t-m^2} \times \bar v(\not q + m_e)\gamma^\mu u\,, \\
\mathcal{M}^\mu_2 &= -i\,\bar v(e^+)\,(ie\gamma^\mu)\,\frac{i}{\not{\tilde q} - m_e}(-ig)u(e^-) \\
&= \frac{eg}{u-m^2} \times \bar v\gamma^\mu(\not{\tilde q} + m_e)u\,,
\end{aligned} \tag{S.2}$$

where

$$q = p_- - k_\gamma = k_s - p_+ \quad \text{and} \quad \tilde q = p_- - k_s = k_\gamma - p_+\,. \tag{S.3}$$

Problem **2**(b):

The Ward identity for the one-photon amplitude (S.2) says $k_\gamma^\mu \times \mathcal{M}_\mu = 0$. To verify it, let's start with the first diagram:

$$\begin{aligned}
k_\gamma^\mu \times \bar{v}(\not{q} + m_e)\gamma_\mu u \;&=\; \bar{v}(\not{q} + m_e)\,\not{k}_\gamma u \\
&=\; \bar{v}(\not{p}_- - \not{k}_\gamma + m_e)\,\not{k}_\gamma u \\
&=\; \bar{v}(\not{p}_- + m_e)\,\not{k}_\gamma u \qquad \langle\!\langle \text{ because } \not{k}_\gamma\,\not{k}_\gamma = k_\gamma^2 = 0 \,\rangle\!\rangle \\
&=\; \bar{v}\Big(2(p_-k_\gamma) - \not{k}_\gamma(\not{p}_- - m_e)\Big)u \\
&=\; 2(p_-k_\gamma) \times \bar{v}u \;-\; 0 \qquad \langle\!\langle \text{ because } (\not{p}_- - m_e) \times u(e^-) = 0 \,\rangle\!\rangle \\
&=\; (m_e^2 - t) \times \bar{v}u\,,
\end{aligned} \tag{S.4}$$

and hence

$$k_\gamma^\mu \times \mathcal{M}_{1\mu} \;=\; -eg \times \bar{v}u\,. \tag{S.5}$$

We see that *by itself*, the first diagram does not satisfy the Ward entity. Instead, we need to add the second diagram's contribution

$$\begin{aligned}
k_\gamma^\mu \times \bar{v}\gamma_\mu(\not{\tilde{q}} + m_e)u \;&=\; \bar{v}\,\not{k}_\gamma(\not{\tilde{q}} + m_e)u \\
&=\; \bar{v}\,\not{k}_\gamma(\not{k}_\gamma - \not{p}_+ + m_e)u \\
&=\; \bar{v}\,\not{k}_\gamma(-\not{p}_+ + m_e)u \qquad \langle\!\langle \text{ because } \not{k}_\gamma\,\not{k}_\gamma = k_\gamma^2 = 0 \,\rangle\!\rangle \\
&=\; \bar{v}\Big(-2(p_+k_\gamma) + \not{k}_\gamma(\not{p}_+ + m_e)\Big)u \\
&=\; -2(p_+k_\gamma) \times \bar{v}u \;+\; 0 \qquad \langle\!\langle \text{ because } \bar{v})e^+) \times (\not{p}_+ + m_e) = 0 \,\rangle\!\rangle \\
&=\; (u - m_e^2) \times \bar{v}u\,,
\end{aligned} \tag{S.6}$$

and hence

$$k_\gamma^\mu \times \mathcal{M}_{2\mu} \;=\; +eg \times \bar{v}u\,. \tag{S.7}$$

Again, the second diagram does not satisfy the Ward identity *by itself*, but the net amplitude does:

$$k_\gamma^\mu \times (\mathcal{M}_\mu \;=\; \mathcal{M}_{1\mu} \,+\, \mathcal{M}_{2\mu}) \;=\; 0. \tag{S.8}$$

Problem **2**(c):

Thanks to the Ward identity, summing $|\mathcal{M}|^2$ over the photon's polarizations is easy:

$$\begin{aligned}\sum_\lambda |\mathcal{M}|^2 &= -\mathcal{M}^\mu \mathcal{M}^*_\mu \qquad \langle\!\langle \text{ see my notes on Ward identities } \rangle\!\rangle\\ &= -\mathcal{M}_1^\mu \mathcal{M}^*_{1\mu} - \mathcal{M}_2^\mu \mathcal{M}^*_{2\mu} - 2\,\mathrm{Re}\left(\mathcal{M}_1^\mu \mathcal{M}^*_{2\mu}\right)\\ &= -\frac{e^2 g^2}{(t-m_2^2)^2} \times \bar v(\not q + m_e)\gamma^\mu u \times \bar u \gamma_\mu (\not q + m_e) v\\ &\quad - \frac{e^2 g^2}{(u-m_2^2)^2} \times \bar v \gamma^\mu (\tilde{\not q} + m_e) u \times \bar u (\tilde{\not q} + m_e)\gamma_\mu v\\ &\quad - \frac{2e^2 g^2}{(t-m_2^2)(u-m_2^2)} \times \mathrm{Re}\Big(\bar v(\not q + m_e)\gamma^\mu u \times \bar u (\tilde{\not q} + m_e)\gamma_\mu v\Big).\end{aligned} \tag{S.9}$$

And given this formula, averaging over the electron's and positron's spins produces Dirac traces according to

$$\overline{|\mathcal{M}|^2} \equiv \frac{1}{4}\sum_{s_-,s_+}\sum_\lambda |\mathcal{M}|^2 = e^2 g^2 \left(\frac{A_{11}}{(t-m_2^2)^2} + \frac{A_{22}}{(u-m_e^2)^2} + \frac{2\,\mathrm{Re}\,A_{12}}{(t-m_e^2)(u-m_e^2)}\right) \tag{S.10}$$

where

$$\begin{aligned} A_{11} &= -\tfrac{1}{4}\,\mathrm{Tr}\Big((p_+ - m_e)(\not q + m_e)\gamma^\mu(\not p_- + m_e)\gamma_\mu(\not q + m_e)\Big),\\ A_{22} &= -\tfrac{1}{4}\,\mathrm{Tr}\Big((p_+ - m_e)\gamma^\mu(\tilde{\not q} + m_e)(\not p_- + m_e)(\tilde{\not q} + m_e)\gamma_\mu\Big),\\ A_{12} &= -\tfrac{1}{4}\,\mathrm{Tr}\Big((p_+ - m_e)(\not q + m_e)\gamma^\mu(\not p_- + m_e)(\tilde{\not q} + m_e)\gamma_\mu\Big).\end{aligned} \tag{S.11}$$

Evaluating these traces is straightforward but tedious. Fortunately, it becomes much simpler when we neglect the electron's mass. In that limit, the first trace becomes

$$\begin{aligned} A_{11} &\approx -\tfrac{1}{4}\,\mathrm{Tr}(\not p_+ \not q \gamma^\mu \not p_- \gamma_\mu \not q)\\ &= +\tfrac{1}{2}\,\mathrm{Tr}(\not p_+ \not q \not p_- \not q) \qquad \langle\!\langle \text{ using } \gamma^\mu \not p_- \gamma_\mu = -2\not p_- \rangle\!\rangle\\ &= 4(p_+q)(p_-q) - 2(p_+p_-)\,q^2\\ &\approx (M_s^2 - t)\times t - s\times t = (M_s^2 - t - s)\times t\\ &\approx u \times t\,,\end{aligned} \tag{S.12}$$

where the last two lines follow from

$$\begin{aligned}
q^2 \;&=\; t, \\
p_+p_- \;&=\; \tfrac{1}{2}(p_- + p_+)^2 - \cancel{m_e^2} \;\approx\; \frac{s}{2}\,, \\
p_-q \;&=\; p_-(p_- - k_\gamma) \;=\; \tfrac{1}{2}(p_- - k_\gamma)^2 + \cancel{\tfrac{1}{2}m_e^2} \;\approx\; \frac{t}{2}\,, \\
p_+q \;&=\; p_+(k_S - p_+) \;=\; -\tfrac{1}{2}(p_+ - k_S)^2 + \tfrac{1}{2}M_s^2 - \cancel{\tfrac{1}{2}m_e^2} \;\approx\; \frac{M_s^2 - t}{2}\,, \\
s + t + u \;&=\; M_s^2 + \cancel{2m_e^2} \;\approx\; M_s^2\,.
\end{aligned} \tag{S.13}$$

Likewise, the second trace becomes

$$\begin{aligned}
A_{22} \;&\approx\; -\tfrac{1}{4}\,\mathrm{Tr}(\not{p}_+\gamma^\mu\,\tilde{\not{q}}\,\not{p}_-\,\tilde{\not{q}}\gamma_\mu) \\
&=\; -\tfrac{1}{4}\,\mathrm{Tr}(\gamma_\mu\,\not{p}_+\gamma^\mu\,\tilde{\not{q}}\,\not{p}_-\,\tilde{\not{q}}) \\
&=\; +\tfrac{1}{2}\,\mathrm{Tr}(\not{p}_+\,\tilde{\not{q}}\,\not{p}_-\,\tilde{\not{q}}) \qquad \langle\!\langle\,\text{using } \gamma_\mu\,\not{p}_+\gamma^\mu = -2\,\not{p}_+\,\rangle\!\rangle \\
&=\; 4(p_+\tilde{q})(p_-\tilde{q}) \;-\; 2(p_+p_-)\,\tilde{q}^2 \\
&\approx\; (M_s^2 - u)\times u \;-\; s\times u \;=\; (M_s^2 - u - s)\times u \\
&\approx\; t\times u\,,
\end{aligned} \tag{S.14}$$

where the last two lines follow from (S.13) and

$$\begin{aligned}
\tilde{q}^2 \;&=\; u, \\
p_+\tilde{q} \;&=\; p_+(k_\gamma - p_+) \;=\; -\tfrac{1}{2}(k_\gamma - p_+)^2 - \cancel{\tfrac{1}{2}m_e^2} \;\approx\; -\frac{u}{2}\,, \\
p_-\tilde{q} \;&=\; p_-(p_- - k_s) \;=\; \tfrac{1}{2}(p_- - k_s)^2 - \tfrac{1}{2}M_s^2 + \cancel{\tfrac{1}{2}m_e^2} \;\approx\; \frac{u - M_s^2}{2}\,.
\end{aligned} \tag{S.15}$$

Finally, the third trace becomes

$$\begin{aligned}
A_{22} \;&\approx\; -\tfrac{1}{4}\,\mathrm{Tr}(\not{p}_+\,\not{q}\gamma^\mu\,\not{p}_-\,\tilde{\not{q}}\gamma_\mu) \\
&=\; -(p_-\tilde{q})\times\mathrm{Tr}(\not{p}_+\,\not{q})\langle\!\langle\,\text{using } \gamma^\mu\,\not{p}_-\,\tilde{\not{q}}\gamma^\mu = +4(p_-\tilde{q})\,\rangle\!\rangle \\
&=\; -4(p_-\tilde{q})(p_+q) \\
&\approx\; +(u - M_s^2)(t - M_s^2).
\end{aligned} \tag{S.16}$$

Now let's plug all these traces back into eq. (S.10). Neglecting m_e^2 in the denominators, we have

$$
\begin{aligned}
\overline{|\mathcal{M}|^2} \;&=\; e^2 g^2\left(\frac{tu}{t^2} \;+\; \frac{ut}{u^2} \;+\; \frac{2(t-M_s^2)(u-M_s^2)}{tu}\right)\\
&=\; \frac{e^2g^2}{tu}\times\Big(u^2 \;+\; t^2 \;+\; 2(t-M_s^2)(u-M_s^2)\Big)\\
&=\; \frac{e^2g^2}{tu}\times\Big((t+u-M_s^2)^2 \;+\; M_s^4\Big)\\
&=\; e^2g^2\times\frac{s^2+M_s^4}{tu}\,.
\end{aligned}
\tag{S.17}
$$

Problem **2**(d):

Given eq. (S.17) for the spin-averaged and polarization-summed $|\mathcal{M}|^2$, calculating the partial cross-section is just the matter of kinematics. In the center of mass frame, the initial electron and positron have $p_\mp^\mu = (E_e, \pm\mathbf{p})$ where $E_e \approx |\mathbf{p}|$. As to the final photon and scalar, they have equal and opposite 3-momenta but different energies: $k_\gamma^\mu = (\omega, +\mathbf{k})$ while $k_S^\mu = (E_s, -\mathbf{k})$, where $\omega = |\mathbf{k}| \neq E_s = \sqrt{\mathbf{k}^2 + M_s^2}$. By energy conservation

$$
\omega \;+\; E_s \;=\; 2E_e \;=\; \sqrt{s}\,. \tag{S.18}
$$

To solve this equation, we rewrite it as

$$
\omega^2 \;+\; M_s^2 \;=\; E_s^2 \;=\; (\sqrt{s}\;-\omega)^2 \;=\; s \;-\; 2\sqrt{s}\times\omega \;+\omega^2, \tag{S.19}
$$

which gives us

$$
\omega \;=\; \frac{s-M_s^2}{2\sqrt{s}} \quad\Longrightarrow\quad E_s \;=\; \frac{s+M_s^2}{2\sqrt{s}}\,. \tag{S.20}
$$

Given all these momenta,

$$
\begin{aligned}
t \;&=\; -2(p_-k_\gamma) \;=\; -2E_e\omega \;+\; 2\mathbf{p}\cdot\mathbf{k} \approx\; -2E_e\omega\times(1-\cos\theta) \;=\; -\tfrac{1}{2}(s-M_s^2)\times(1-\cos\theta),\\
u \;&=\; -2(p_+k_\gamma) \;=\; -2E_e\omega \;-\; 2\mathbf{p}\cdot\mathbf{k} \approx\; -2E_e\omega\times(1+\cos\theta) \;=\; -\tfrac{1}{2}(s-M_s^2)\times(1+\cos\theta),
\end{aligned}
\tag{S.21}
$$

and plugging these formulae into eq. (S.17) gives us

$$\overline{|\mathcal{M}|^2} \;=\; 4e^2g^2 \times \frac{s^2+M_S^4}{(s-M_S^2)^2} \times \frac{1}{\sin^2\theta}\,. \tag{S.22}$$

In the center-of-mass frame, the partial cross-section of 2 *particles* $\to$ 2 *particles* scattering is given by

$$\frac{d\sigma}{d\Omega_{\rm cm}} \;=\; \frac{|\mathcal{M}|^2}{64\pi^2 s} \times \frac{|\mathbf{p}'|}{|\mathbf{p}|}\,. \tag{S.23}$$

For the problem at hand, the inelasticity factor $|\mathbf{p}'|/|\mathbf{p}|$ is

$$\frac{|\mathbf{k}|}{|\mathbf{p}|} \;\approx\; \frac{\omega}{E_e} \;=\; \frac{s-M_s^2}{s}\,. \tag{S.24}$$

Combining this factor with eq. (S.22), we finally arrive at the following formula for the partial cross-section:

$$\frac{d\sigma(e^-e^+\to\gamma S)}{d\Omega_{\rm c.m.}} \;=\; \frac{\alpha g^2}{4\pi} \times \frac{s^2+M_s^4}{s^2(s-M_s^2)} \times \frac{1}{\sin^2\theta}\,. \tag{S.25}$$

Note the forward-backward symmetry $\theta \leftrightarrow \pi-\theta$ of this cross section. Physically, it is due to the charge-conjugation symmetry which exchanges the initial electron and positron.

As usual for annihilation processes in the ultra-relativistic limit, the cross-section (S.25) has divergent peaks in forward and backward directions, $\theta \to 0$ or $\theta\to\pi$. The divergence here is an artefact of the $m_e^2 = 0$ approximation, which becomes inaccurate at very small angles $\theta \lesssim (m_e/E)$ (or $\pi-\theta \lesssim (m_e/E)$). A more careful analysis shows that

$$\text{for } \theta\lesssim\gamma^{-1},\quad \frac{d\sigma(e^-e^+\to\gamma S)}{d\Omega_{\rm c.m.}} \;\approx\; \frac{\alpha g^2}{4\pi s} \times \left(\frac{s-M_s^2}{s} \times \frac{1}{\theta^2+\gamma^{-2}} \;+\; \frac{M_s^2}{s-M_s^2} \times \frac{2\theta^2}{(\theta^2+\gamma^{-2})^2}\right) \tag{S.26}$$

(where $\gamma^{-1} = m_e/E \ll 1$) instead of eq. (S.25). Consequently, the total cross-section turns out to be finite rather then divergent, namely

$$\sigma_{\rm tot}(e^-e^+\to\gamma S) \;=\; \alpha g^2 \times \frac{(s^2+M_s^4)}{s^2(s-M_s^2)} \left(\log\frac{2E_e}{m_e} \;-\; \frac{sM_s^2}{s^2+M_s^4} \;+\; O\left(\frac{m_e^2}{E_e^2}\right)\right). \tag{S.27}$$

Problem **3**(a):

The scalar potential part of the linear sigma model's Lagrangian (2) is

$$V(\phi) \;=\; \frac{\lambda}{8}\Bigl(\sum\nolimits_i \phi_i^2 \,-\, f^2\Bigr)^2 \;-\; \beta\lambda f^2\times\phi_{N+1}\,, \tag{S.28}$$

where the last term explicitly breaks the $O(N+1)$ symmetry of the first term down to $O(N)$. To find the minimum of this potential, let's first find the stationary points where all the first derivatives $\partial V/\partial\phi_i$ are zero:

$$\text{for } i=1,\ldots,N,\quad \frac{\partial V}{\partial\phi_i} \;=\; \frac{\lambda}{2}\Bigl(\sum\nolimits_j \phi_j^2 \,-\, f^2\Bigr)\times\phi_i \;=\; 0, \tag{S.29}$$

$$\text{and}\quad \frac{\partial V}{\partial\phi_{N+1}} \;=\; \frac{\lambda}{2}\Bigl(\sum\nolimits_j \phi_j^2 \,-\, f^2\Bigr)\times\phi_{N+1} \;-\; \beta\lambda f^2 \;=\; 0. \tag{S.30}$$

From eq. (S.30) we immediately see that at any stationary point $\bigl(\sum\phi^2 - f^2\bigr) \neq 0$, hence eqs. (S.29) tell us that $\phi_1 = \cdots = \phi_N = 0$. In other words, all the stationary points lie on the ϕ_{N+1} axis in the $(N+1)$ dimensional space of the scalar field values. And in this space, eq. (S.30) becomes a simple cubic equation

$$\phi_{N+1}^3 \;-\; f^2\times\Phi_{N+1} \;-\; 2\beta f^2 \;=\; 0. \tag{S.31}$$

For small $\beta \ll f$, this cubic equation has 3 real solutions, approximately

$$\langle\phi_{N+1}\rangle_1 \;\approx\; -2\beta,\qquad \langle\phi_{N+1}\rangle_2 \;\approx\; -f \;+\; \beta,\qquad \langle\phi_{N+1}\rangle_3 \;\approx\; +f \;+\; \beta. \tag{S.32}$$

Now let's find out which of the three stationary points is a minimum (or at least a local minimum) by looking at the second derivatives of the potential (S.28). Along the ϕ_{N+1} axis in the field space, the second derivatives amount to

$$\frac{\partial^2 V}{\partial\phi_i\partial\phi_j} \;=\; \frac{\lambda}{2}\times\begin{cases} \bigl(3\phi_{N+1}^2 - f^2\bigr) & \text{for } i=j=N+1,\\[1ex] 0 & \text{for } i\le N,\ j=N+1 \text{ or } j\le N,\ i=N+1,\\[1ex] \bigl(\phi_{N+1}^2 - f^2\bigr)\times\delta_{ij} & \text{for } i,j\le N. \end{cases} \tag{S.33}$$

Evaluating these derivatives for the 3 stationary points (S.32) — while assuming small $\beta > 0$

— gives us

$$\begin{aligned}
&@\ \langle\phi_{N+1}\rangle_1: \quad \frac{\partial^2 V}{(\partial\phi_{N+1})^2} < 0 \quad \text{while other} \quad \frac{\partial^2 V}{(\partial\phi_i})^2 < 0 \quad\Longrightarrow\quad \text{maximum},\\
&@\ \langle\phi_{N+1}\rangle_2: \quad \frac{\partial^2 V}{(\partial\phi_{N+1})^2} > 0 \quad \text{while other} \quad \frac{\partial^2 V}{(\partial\phi_i})^2 < 0 \quad\Longrightarrow\quad \text{saddle point},\\
&@\ \langle\phi_{N+1}\rangle_3: \quad \frac{\partial^2 V}{(\partial\phi_{N+1})^2} > 0 \quad \text{while other} \quad \frac{\partial^2 V}{(\partial\phi_i})^2 > 0 \quad\Longrightarrow\quad \text{minimum}.
\end{aligned} \tag{S.34}$$

Thus, the potential (S.28) has a unique minimum at

$$\langle\phi_1\rangle = \cdots = \langle\phi_N\rangle = 0, \quad \langle\phi_{N+1}\rangle = +f + \beta + O(\beta^2/f). \tag{3}$$

Quod erat demonstrandum.

Problem **3**(b):
Let's shift the fields as in eq. (4). In terms of the shifted fields,

$$T \stackrel{\text{def}}{=} \sum\nolimits_i \phi_i^2 - f^2 = \underset{\sim}{\pi}^2 + \bigl(\sigma + \langle\phi_{N+1}\rangle\bigr)^2 - f^2 = \underset{\sim}{\pi}^2 + \sigma^2 + 2\,\langle\phi_{N+1}\rangle\times\sigma + \bigl(\langle\phi_{N+1}\rangle^2 - f^2\bigr), \tag{S.35}$$

where $\underset{\sim}{\pi}$ is a short-hand for N-vector $(\pi^1,\ldots,\pi^N)$ of the pion fields, thus $\underset{\sim}{\pi}^2 = (\pi^1)^2+\cdots+(\pi^N)^2$. Therefore, expanding the scalar potential (S.28) into powers of the shifted fields, we obtain

$$\begin{aligned}
V &= \frac{\lambda}{8}\times T^2 - \beta\lambda f^2\times\bigl(\sigma + \langle\phi_{N+1}\rangle\bigr)\\
&= \frac{\lambda}{8}\times\bigl(\underset{\sim}{\pi}^2 + \sigma^2\bigr)^2 + \frac{\lambda\,\langle\phi_{N+1}\rangle}{2}\times\sigma\times\bigl(\underset{\sim}{\pi}^2 + \sigma^3\bigr)\\
&\quad + \frac{\lambda\,\langle\phi_{N+1}\rangle^2}{2}\times\sigma^2 + \frac{\lambda(\langle\phi_{N+1}\rangle^2 - f^2)}{4}\times\bigl(\underset{\sim}{\pi}^2 + \sigma^2\bigr)\\
&\quad + \left(\frac{\lambda\,\langle\phi_{N+1}\rangle}{2}\times\bigl(\langle\phi_{N+1}\rangle^2 - f^2\bigr) - \beta\lambda f^2\right)\times\sigma + \text{const.}
\end{aligned} \tag{S.36}$$

On the last line here, the coefficient of σ vanishes thanks to $\langle\phi_{N+1}\rangle$ obeying the cubic equation (S.31). For the same reason, the coefficient of $(\underset{\sim}{\pi}^2 + \sigma^2)$ on the line before the last may be simplified as

$$\frac{\lambda(\langle\phi_{N+1}\rangle^2 - f^2)}{4} \;=\; \frac{\beta\lambda f^2}{2\,\langle\phi_{N+1}\rangle}\,. \tag{S.37}$$

Altogether, we have

$$V(\sigma,\underset{\sim}{\pi}) \;=\; \frac{\lambda}{8}\times\left(\underset{\sim}{\pi}^2 \,+\, \sigma^2\right)^2 \;+\; \frac{\mu}{2}\times\left(\sigma^3 \,+\, \sigma\underset{\sim}{\pi}^2\right) \;+\; \frac{M_\sigma^2}{2}\times\sigma^2 \;+\; \frac{M_\pi^2}{2}\times\underset{\sim}{\pi}^2 \;+\; \text{const}, \tag{S.38}$$

where

$$\begin{aligned}
\text{quartic coupling } \lambda \;&=\; \lambda, \\
\text{cubic coupling } \mu \;&=\; \lambda\times\langle\phi_{N+1}\rangle \;\approx\; \lambda(f+\beta), \\
\text{pion mass}^2\ M_\pi^2 \;&=\; \frac{\beta\lambda f^2}{\langle\phi_{N+1}\rangle} \;\approx\; \beta\lambda f, \\
\text{sigma mass}^2\ M_\sigma^2 \;&=\; M_\pi^2 \;+\; \lambda\,\langle\phi_{N+1}\rangle^2 \;\approx\; \lambda f(f+3\beta).
\end{aligned} \tag{S.39}$$

Let's take a closer look at the pion's mass^2, $M_\pi^2 \approx \beta\times\lambda f$. In the $\beta = 0$ limit, the pions are massless in accordance with the Goldstone theorem. Indeed, for $\beta = 0$ the sigma model's Lagrangian has exact $SO(N+1)$ symmetry which is spontaneously broken down to $SO(N)$; there are N spontaneously broken generators, so there should be N massless Goldstone bosons. But for $\beta \neq 0$, the $SO(N+1)$ symmetry of the Lagrangian is only approximate, and its *explicit* breaking by the $\beta\lambda f^2\times\phi_{N+1}$ term spoils the Goldstone theorem. Thus, instead of exactly massless Goldstone bosons we should get light but not quite massless pseudo-Goldstone bosons; to the first order in β, their mass^2 should be proportional to β. And indeed, in the linear sigma model $M_\pi^2 \approx \beta\times\lambda f$.

Still, for $\beta \ll f$, the pions should be much lighter than the sigma particle. And indeed, according to eqs. (S.39),

$$\frac{M_\pi^2}{M_\sigma^2} \approx \frac{\beta\lambda f}{\lambda f^2} \;=\; \frac{\beta}{f} \;\ll\; 1. \tag{S.40}$$

Problem **3**(c): Back in homework#9 (problem 3), we had a very similar setup to the *shifted* fields of the linear sigma models: $N+1$ scalar fields $\sigma(x)$ and $\pi^i(x)$, with the Lagrangian

$$\begin{aligned} \mathcal{L} &= \tfrac{1}{2}(\partial_\mu\sigma)^2 \,+\, \tfrac{1}{2}(\partial_\mu\underset{\sim}{\pi})^2 \,-\, V, \\ V(\sigma,\underset{\sim}{\pi}) &= \frac{\lambda}{8}\Bigl(\underset{\sim}{\pi}^2 \,+\, \sigma^2 \,+\, 2f\times\sigma\Bigr)^2 \\ &= \frac{\lambda f^2}{2}\times\sigma^2 \,+\, \frac{\lambda f}{2}\times\bigl(\sigma^3 \,+\, \sigma\underset{\sim}{\pi}^2\bigr) \,+\, \frac{\lambda}{8}\times\bigl(\sigma^2 \,+\, \underset{\sim}{\pi}^2\bigr)^2. \end{aligned} \tag{9.3}$$

In particular, there is a mass term for the σ field but not for the pions, which are exactly massless — exactly as in the present sigma model with $\beta=0$. The cubic and quartic terms in the potential (9.3) also have exactly the same form as in eq. (S.38), and the couplings λ and $\mu=\lambda f$ are related to the σ field's mass as

$$\mu^2 \;=\; \lambda\times M_\sigma^2\,. \tag{S.41}$$

For the present sigma model, we have exactly similar relation for $\beta=0$. Indeed, according to eq. (S.39),

$$\text{for } \beta=0,\quad \mu \;=\; \lambda f,\quad M_\sigma^2 \;=\; \lambda f^2 \quad\Longrightarrow\quad \mu^2 \;=\; \lambda\times M_\sigma^2\,. \tag{S.42}$$

Therefore, the $\pi\pi\to\pi\pi$ scattering amplitudes in the linear sigma model for $\beta=0$ come out to be exactly as homework#9: At the tree level,

$$\begin{aligned} \mathcal{M}(\pi^j+\pi^k\to\pi^\ell+\pi^m) \;=\; &-\left(\lambda\,+\,\frac{\mu^2}{s-M_\sigma^2}\right)\times\delta^{jk}\delta^{\ell m} \;-\; \left(\lambda\,+\,\frac{\mu^2}{t-M_\sigma^2}\right)\times\delta^{j\ell}\delta^{km} \\ &-\; \left(\lambda\,+\,\frac{\mu^2}{u-M_\sigma^2}\right)\times\delta^{jm}\delta^{k\ell}, \end{aligned} \tag{S.43}$$

which in light of the relation (S.41) becomes

$$\begin{aligned} \mathcal{M}(\pi^j+\pi^k\to\pi^\ell+\pi^m) \;=\; &-\frac{\lambda s}{s-M_\sigma^2}\times\delta^{jk}\delta^{\ell m} \;-\; \frac{\lambda t}{t-M_\sigma^2}\times\delta^{j\ell}\delta^{km} \\ &-\; \frac{\lambda u}{u-M_\sigma^2}\times\delta^{jm}\delta^{k\ell}. \end{aligned} \tag{S.44}$$

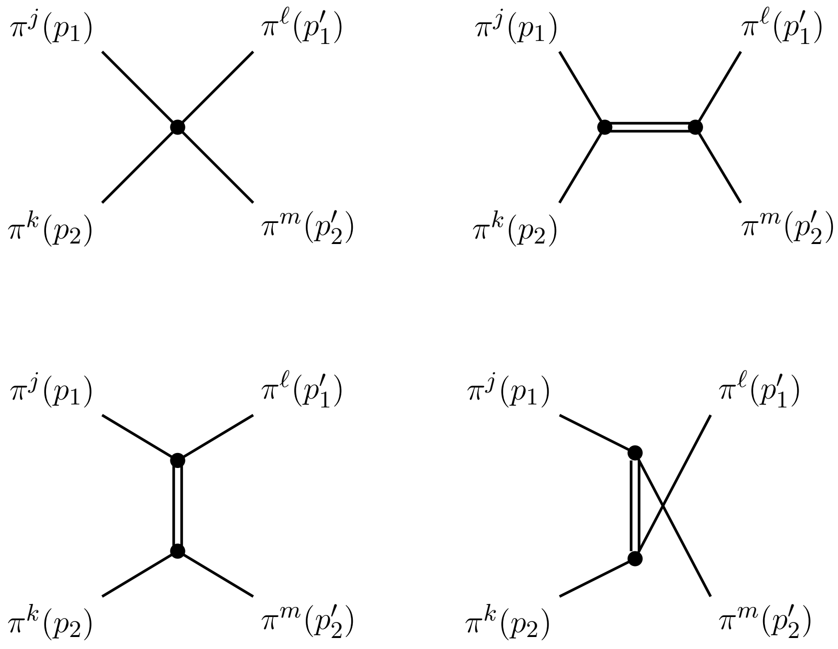
When any of the 4 pions' energy becomes small, we get $s,t,u \ll M_\sigma^2$, and the scattering amplitude becomes small as

$$\mathcal{M} \approx \frac{\lambda}{M_\sigma^2} \times \left(s \times \delta^{jk}\delta^{\ell m} + t \times \delta^{j\ell}\delta^{km} + u \times \delta^{jm}\delta^{k\ell} \right) = O\left(\frac{\lambda E_{\rm cm}^2}{M_\sigma^2} \right). \tag{S.45}$$

Problem **3**(d): For $\beta \neq 0$, the quartic and the cubic couplings of the σ and π^i to each other has similar overall form to what we had back in homework#9, but the overall coefficients λ and μ of those couplings are no longer related to the σ particle's mass by eq. (S.41). Instead, eq. (S.39) gives us

$$\mu = \lambda \langle \phi_{N+!} \rangle, \quad M_\sigma^2 = M_\pi^2 + \lambda \langle \phi_{N+!} \rangle^2 \quad \Longrightarrow \quad \mu^2 = \lambda \times \left(M_\sigma^2 - M_\pi^2 \right). \tag{S.46}$$

Now consider the $\pi\pi \to \pi\pi$ scattering. At the tree level, we have exactly the same 4 diagrams for such scattering as in the homework#9, namely



(S.47)

Altogether, these diagrams yield the scattering amplitude exactly as in eq. (S.43), but for the

β-modified couplings and masses. Consequently, in light of eq. (S.46) instead of (S.41), we have

$$\lambda\;+\;\frac{\mu^2}{s-M_\sigma^2}\;=\;\frac{\lambda s-\lambda M_\sigma^2+\mu^2}{s-M_\sigma^2}\;=\;\frac{\lambda s-\lambda M_\pi^2}{s-M_\sigma^2} \tag{S.48}$$

and likewise

$$\lambda\;+\;\frac{\mu^2}{t-M_\sigma^2}\;=\;\frac{\lambda(t-M_\pi^2)}{t-M_\sigma^2}\quad\text{and}\quad\lambda\;+\;\frac{\mu^2}{u-M_\sigma^2}\;=\;\frac{\lambda(u-M_\pi^2)}{u-M_\sigma^2}\,, \tag{S.49}$$

so the amplitude (S.43) becomes

$$\begin{aligned}\mathcal{M}(\pi^j+\pi^k\to\pi^\ell+\pi^m)\;=\;-\frac{\lambda(s-M_\pi^2)}{s-M_\sigma^2}\times\delta^{jk}\delta^{\ell m}\;&-\;\frac{\lambda(t-M_\pi^2)}{t-M_\sigma^2}\times\delta^{j\ell}\delta^{km}\\ &-\;\frac{\lambda(u-M_\pi^2)}{u-M_\sigma^2}\times\delta^{jm}\delta^{k\ell}.\end{aligned} \tag{S.50}$$

When the pions' energies become low compared to M_σ — or in Lorentz-invariant terms, when $s,t,u\ll M_\sigma^2$ — we may simplify this amplitude by approximating all the denominators as $-M_\sigma^2$, thus

$$\mathcal{M}(\pi^j+\pi^k\to\pi^\ell+\pi^m)\;\approx\;\left(\frac{\lambda}{M_\sigma^2}\;\approx\;\frac{1}{f^2}\right)\times\begin{pmatrix}(s-M_\pi^2)\times\delta^{jk}\delta^{\ell m}\;+\;(t-M_\pi^2)\times\delta^{j\ell}\delta^{km}\\ \;+\;(u-M_\pi^2)\times\delta^{jm}\delta^{k\ell}\end{pmatrix}. \tag{5}$$

What happens to this amplitude when one of the pions' momentum becomes very small? Alas, for $\beta\neq 0$ the pions are massive, so we cannot take all 4 components of a pion's p^μ to zero. The best we can do is to take $\mathbf{p}\to 0$ while $p^0\to m$, which is the *non-relativistic limit*. However, if only one pion is non-relativistic while the other 3 pions have $E\gg M_\pi$ (but $E\ll M_\sigma$), we generally have $s,t,u=O(E\times M_\pi)\gg M_\pi^2$ (albeit $s,t,u\ll M_\sigma^2$), and the scattering amplitude becomes

$$\mathcal{M}\;=\;O\left(\frac{E\times M_\pi}{f^2}\right)\;\not\to\;0. \tag{S.51}$$

The strongest low-energy limit we an take for massive pions is to make all four pions non-relativistic. In this limit, $s=E_{\rm cm}^2\approx 4M_\pi^2$ while $u,t=O(\mathbf{p}^2)\ll M_\pi^2$, so the scattering amplitude (5) becomes

$$\mathcal{M}(\pi^j+\pi^k\to\pi^\ell+\pi^m)\ \approx\ \left(\frac{\lambda M_\pi^2}{M_\sigma^2}\approx\frac{\beta\lambda}{f}\right)\times\left(3\delta^{jk}\delta^{\ell m}\,-\,\delta^{j\ell}\delta^{km}\,-\,\delta^{jm}\delta^{k\ell}\right). \tag{6}$$

This amplitude is suppressed by the factor β/f, but it does not vanish! And even if all 4 pions belong to the same species, the scattering amplitude does not vanish in the non-relativistic limit,

$$\mathcal{M}(\pi^1+\pi^1\to\pi^1+\pi^1)\ \approx\ \frac{\lambda\beta}{f}\ \neq\ 0, \tag{S.52}$$

unlike what we had in homework#9 in the low-energy limit for the massless pions.

Problem **4**(a):

$$\begin{aligned} \text{given}\quad \Phi\ &\to\ e^{+i\theta}U_L\Phi U_R^\dagger\,, && (9)\\ \text{we have}\quad \Phi^\dagger\ &\to\ e^{-i\theta}U_R\Phi^\dagger U_L^\dagger\,, && (\text{S.53})\\ \text{hence}\quad \Phi^\dagger\Phi\ &\to\ U_R(\Phi^\dagger\Phi)U_R^\dagger\,, && (\text{S.54})\\ \left(\Phi^\dagger\Phi\right)^2\ &\to\ U_R\Phi^\dagger\Phi U_R^\dagger\,U_R\Phi^\dagger\Phi U_R^\dagger\ =\ U_R\left(\Phi^\dagger\Phi\right)^2U_R^\dagger\,, && (\text{S.55})\\ \text{likewise}\quad \left(\Phi^\dagger\Phi\right)^n\ &\to\ U_R\left(\Phi^\dagger\Phi\right)^nU_R^\dagger\quad \forall n=1,2,3,\ldots, && (\text{S.56}) \end{aligned}$$

and therefore

$$\text{all traces}\quad \operatorname{tr}\left(\left(\Phi^\dagger\Phi\right)^n\right)\quad \text{are invariant under symmetries (9),} \tag{S.57}$$

thanks to the cyclic invariance rule for traces, $\operatorname{tr}\left(U_RXU_R^\dagger\right)=\operatorname{tr}\left(XU_R^\dagger U_R\right)=\operatorname{tr}\left(X\right)$ for any $X=\left(\Phi^\dagger\Phi\right)^n$. Consequently, the scalar potential (8) is invariant under symmetries (9).

For the global symmetries where $e^{i\theta}$, U_L, and U_R do not depend on x, the kinetic term in (7) is also invariant. Indeed,

$$\begin{aligned}
&\text{for constant } e^{i\theta},\ U_L,\ U_R, \\
\partial_\mu\Phi &\to e^{+i\theta} U_L (\partial_\mu\Phi) U_R^\dagger\,, \\
\partial_\mu\Phi^\dagger &\to e^{-i\theta} U_R (\partial_\mu\Phi^\dagger) U_L^\dagger\,, \\
\partial^\mu\Phi^\dagger\,\partial_\mu\Phi &\to U_R (\partial^\mu\Phi^\dagger\,\partial_\mu\Phi) U_R^\dagger\,, \\
\text{and}\quad \mathrm{tr}\big(\partial^\mu\Phi^\dagger\,\partial_\mu\Phi\big) &\quad \text{is invariant.}
\end{aligned} \tag{S.58}$$

Altogether, the whole Lagrangian (7) is invariant, $\mathcal{Q.E.D.}$

Problem **4**($\star$):

The kinetic term in (7) and the last two terms in the potential (8) have a much bigger symmetry than $G = SU(N)\times SU(N)\times U(1)$, namely $SO(2N^2)$ which does not care for the matrix structure of the $\Phi(x)$ and treats it as $2N^2$ real component fields. Indeed,

$$\mathrm{tr}\big(\Phi^\dagger\Phi\big) = \sum_{i,j} \left|\Phi_i^{\ j}\right|^2 = \sum_{i,j}\left(\left(\mathrm{Re}\,\Phi_i^{\ j}\right)^2 + \left(\mathrm{Im}\,\Phi_i^{\ j}\right)^2\right) \tag{S.59}$$

is invariant under all $SO(2N^2)$ “rotations” of the components, and so is the kinetic term.

On the other hand, the $\mathrm{tr}\left(\Phi^\dagger\Phi\,\Phi^\dagger\Phi\right)$ in the potential does depend on the packing of $2N^2$ real components into a complex $N\times N$ matrix, and it is this term which reduces the internal symmetry group of the theory to $G = SU(N)\times SU(N)\times U(1)$.

Proving that all the $SO(2N^2)/G$ symmetries are broken by the quartic trace term is a non-trivial exercise in group theory rather than field theory. You do not have to do it as part of this homework set, and I am not writing down the proof here.

Problem **4**(b):

Given the eigenvalues $(\kappa_1,\ldots,\kappa_N)$ of the $\Phi^\dagger\Phi$ matrix, the invariant traces (S.57) obtain as

$$\mathrm{tr}\left(\left(\Phi^\dagger\Phi\right)^n\right) = \sum_{i=1}^N \kappa_i^n\,. \tag{S.60}$$

Consequently, the scalar potential is

$$V \;=\; \frac{\alpha}{2}\sum_i \kappa_i^2 \;+\; \frac{\beta}{2}\left(\sum_i \kappa_i\right)^2 \;+\; m^2\sum_i \kappa_i\,. \tag{S.61}$$

Now let's minimize this potential. Since the matrix $\Phi^\dagger\Phi$ cannot have any negative eigenvalues, we are looking for a minimum of $V(\kappa_1,\ldots,\kappa_N)$ *under constraints* $\kappa_i \geq 0$. This requires

$$\forall i=1,\ldots,N,\quad \textbf{either } \kappa_i \geq 0 \textbf{ and } \frac{\partial V}{\partial \kappa_i} \;=\; 0,\quad \textbf{or else } \kappa_i = 0 \textbf{ and } \frac{\partial V}{\partial \kappa_i} \;>\; 0, \tag{S.62}$$

where

$$\frac{\partial V}{\partial \kappa_i} \;=\; \alpha\kappa_i \;+\; m^2 \;+\; \beta\sum_j \kappa_j\,. \tag{S.63}$$

These derivatives are linear functions of the eigenvalues κ_i, so all the non-zero eigenvalues must obey the same linear equation

$$\alpha\times\kappa_i \;=\; -m^2 \;-\; \times\sum_j \kappa_j\,,\quad \text{same for all } \kappa_i \neq 0,$$

which means that all non-zero κ_i have the same value. Thus, up to a permutation of eigenvalues,

$$\kappa_1 \;=\; \cdots \;=\; \kappa_k \;=\; C^2,\quad \kappa_{k+1} \;=\; \cdots \;=\; \kappa_N \;=\; 0, \tag{S.64}$$

for some $k-0,1,2,\ldots,N$, and C^2 obtains from

$$\alpha\times C^2 \;+\; m^2 \;+\; \beta\times kC^2 \;=\; 0 \quad\longrightarrow\quad C^2 \;=\; \frac{-m^2}{\alpha+k\beta}\,. \tag{S.65}$$

To make sure that the solution (S.64) is a minimum rather that a maximum or a saddle point, we need

$$\begin{aligned} C^2 \;&=\; \frac{-m^2}{\alpha+k\beta} \;>\; 0 \quad \text{unless } k=0,\\ m^2 \;+\; \beta kC^2 \;&=\; \frac{\alpha m^2}{\alpha+k\beta} \;>\; 0 \quad \text{unless } k=N. \end{aligned} \tag{S.66}$$

Depending on the signs of α, β and m^2 parameters, this limits the solutions to the following:

- For $\alpha > 0$, $\beta > 0$, and $m^2 > 0$, the only solution is $k = 0$, which means $\kappa_1 = \cdots \kappa_N = 0$ and hence $\langle\Phi\rangle = 0$.
- For $\alpha > 0$, $\beta > 0$, and $m^2 < 0$, the only solutions is $k = N$, which means

$$\kappa_1 \;=\; \cdots \;=\; \kappa_N \;=\; C^2 \;=\; \frac{-m^2}{\alpha + N\beta} \;>\; 0, \tag{10}$$

and hence $\langle\Phi\rangle = C \times$ a unitary matrix. We shall focus on this regime through the rest of this problem.

- For $\alpha < 0$ or $\beta < 0$, the situation is more complicated:
  - — When $\alpha + \beta < 0$ or $\alpha + N\beta < 0$, the scalar potential (8) is unbounded from below and the theory is sick.
  - — When $\alpha > 0$ and $\beta < 0$ but $\alpha + N\beta > 0$, the solutions are similar to the $\beta > 0$ case: For $m^2 > 0$ all $\kappa_i = 0$, while for $m^2 < 0$ the κ_i are as in eq. (10).
  - — When $\beta > 0$ and $\alpha < 0$ but $\alpha + \beta > 0$: for $m^2 > 0$ the only solution is $k = 0$, meaning $\langle\Phi\rangle = 0$, but for $m^2 <$ all the solutions (S.64) with $k = 1, 2, \ldots, N$ are good local minima.

    To find the global minimum, we compare the potentials at the local minima,

$$\begin{aligned} V(\text{minimum}\#k) \;&=\; \frac{\alpha}{2} \times kC^4 \;+\; \frac{\beta}{2} \times (kC^2)^2 \;+\; m^2 \times kC^2 \\ &=\; \frac{k\alpha + k^2\beta}{2} \times \frac{m^4}{(\alpha + k\beta)^2} \;+\; km^2 \times \frac{-m^2}{(\alpha + k\beta)} \\ &=\; -\frac{m^4}{2} \times \frac{k}{k\beta + \alpha}\,. \end{aligned} \tag{S.67}$$

    Since $\alpha < 0$ but $\alpha + \beta > 0$, the deepest minimum obtains for $k = 1$, thus

$$\kappa_1 \;=\; \frac{-m^2}{\alpha + \beta}\,, \quad \kappa_2 \;=\; \cdots \;=\; \kappa_N \;=\; 0. \tag{S.68}$$

Problem **4**(c):

Let's act with some $SU(N)_L \times SU(N)_R \times U(1)$ symmetry (9) on the vacuum expectation values (11):

$$\langle\Phi\rangle \;=\; C \times \mathbf{1}_{N\times N} \;\to\; e^{i\theta} U_L \,\langle\Phi\rangle\, U_R^\dagger \;=\; C \times e^{i\theta} U_L U_R^\dagger\,. \tag{S.69}$$

Clearly, to keep the VEVs $\langle\Phi\rangle$ invariant, we need

$$e^{i\theta} U_L U_R^\dagger \;=\; \mathbf{1}_{N\times N} \tag{S.70}$$

and hence

$$U_R \;=\; e^{i\theta} \times U_L\,. \tag{S.71}$$

Moreover, since the U_L and U_R matrices have unit determinants, this requires

$$\det\left(e^{i\theta} \times \mathbf{1}_{N\times N}\right) \;=\; 1 \quad\Longrightarrow\quad N \times \theta \;=\; 0 \pmod{2\pi}. \tag{S.72}$$

Such a phase can be absorbed into the $U_L \in SU(N)$, so without loss of generality we need

$$e^{i\theta} \;=\; 1 \quad\text{and}\quad U_L \;=\; U_R \;\in\; SU(N). \tag{S.73}$$

In other words, the *unbroken* symmetry group is $SU(N)$ which acts on the scalar fields as

$$\Phi(x) \;\to\; U\Phi(x)\,U^\dagger\,, \quad U \in SU(N). \tag{S.74}$$

Problem **4**(d):

In terms of the shifted fields (12),

$$\partial_\mu \Phi \;=\; \frac{1}{\sqrt{2}}\left(\partial_\mu\varphi_1 \,+\, i\partial_\mu\varphi_2\right), \qquad \partial_\mu \Phi^\dagger \;=\; \frac{1}{\sqrt{2}}\left(\partial_\mu\varphi_1 \,-\, i\partial_\mu\varphi_2\right), \tag{S.75}$$

hence the kinetic term in the Lagrangian becomes

$$\mathrm{tr}\bigl(\partial_\mu\Phi^\dagger\,\partial^\mu\Phi\bigr) \;=\; \tfrac{1}{2}\,\mathrm{tr}(\partial_\mu\varphi_1\,\partial^\mu\varphi_1) \;+\; \tfrac{1}{2}\,\mathrm{tr}(\partial_\mu\varphi_2\,\partial^\mu\varphi_2). \tag{S.76}$$

As to the potential terms, we have

$$\begin{aligned}\Phi^\dagger\Phi \;&=\; C^2\times\mathbf{1}_{N\times N} \;+\; C\times\bigl(\delta\Phi^\dagger+\delta\Phi\bigr) \;+\; \delta\Phi^\dagger\delta\Phi\\ &=\; C^2\times\mathbf{1}_{N\times N} \;+\; \sqrt{2}C\times\varphi_1 \;+\; \tfrac{1}{2}\varphi_1^2 \;+\; \tfrac{1}{2}\varphi_2^2 \;+\; \tfrac{i}{2}[\varphi_1,\varphi_2]\end{aligned} \tag{S.77}$$

and consequently

$$\mathrm{tr}\bigl(\Phi^\dagger\Phi\bigr) \;=\; NC^2 \;+\; \sqrt{2}C\,\mathrm{tr}(\varphi_1) \;+\; \tfrac{1}{2}\,\mathrm{tr}(\varphi_1^2) \;+\; \tfrac{1}{2}\,\mathrm{tr}(\varphi_2^2), \tag{S.78}$$

$$\begin{aligned}\mathrm{tr}^2\bigl(\Phi^\dagger\Phi\bigr) \;&=\; N^2C^4 \;+\; 2\sqrt{2}NC^3\,\mathrm{tr}(\varphi_1) \;+\; 2C^2\,\mathrm{tr}^2(\varphi_1) \;+\; NC^2\,\bigl(\mathrm{tr}(\varphi_1^2)\;+\;\mathrm{tr}(\varphi_2^2)\bigr)\\ &\quad+\; \sqrt{2}C\,\mathrm{tr}(\varphi_1)\times\bigl(\mathrm{tr}(\varphi_1^2)\;+\;\mathrm{tr}(\varphi_2^2)\bigr) \;+\; \tfrac{1}{4}\,\bigl(\mathrm{tr}(\varphi_1^2)\;+\;\mathrm{tr}(\varphi_2^2)\bigr)^2\end{aligned} \tag{S.79}$$

$$\begin{aligned}\mathrm{tr}\left(\bigl(\Phi^\dagger\Phi\bigr)^2\right) \;&=\; NC^4 \;+\; 2\sqrt{2}C^3\,\mathrm{tr}(\varphi_1) \;+\; 3C^2\,\mathrm{tr}(\varphi_1^2) \;+\; C^2\,\mathrm{tr}(\varphi_2^2)\\ &\quad+\; \sqrt{2}C\,\mathrm{tr}(\varphi_1^3) \;+\; \sqrt{2}C\,\mathrm{tr}(\varphi_1\varphi_2^2)\\ &\quad+\; \tfrac{1}{4}\,\mathrm{tr}(\varphi_1^4) \;+\; \tfrac{1}{4}\,\mathrm{tr}(\varphi_2^4) \;+\; \tfrac{3}{2}\,\mathrm{tr}(\varphi_1^2\varphi_2^2) \;-\; \tfrac{1}{2}\,\mathrm{tr}(\varphi_1\varphi_2\varphi_1\varphi_2).\end{aligned} \tag{S.80}$$

Plugging all these formulae into the potential (8) and expanding in powers of φ_1 and φ_2, we obtain

$$V \;=\; \mathrm{const} \;+\; V_1 \;+\; V_2 \;+\; V_3 \;+\; V_4 \tag{S.81}$$

where

$$\begin{aligned}V_1 \;&=\; \sqrt{2}C\times(m^2\;+\;\beta NC^2\;+\;\alpha C^2)\times\mathrm{tr}(\varphi_1) \;=\; 0\\ &\qquad\langle\!\langle\,\text{because } m^2\;+\;(\alpha+N\beta)C^2\;=\;0\,\rangle\!\rangle\end{aligned} \tag{S.82}$$

$$V_2 \;=\; \beta C^2\times\mathrm{tr}^2(\varphi_1)\;+\;\tfrac{1}{2}(m^2\,+\,\beta NC^2\,+\,3\alpha C^2)\times\mathrm{tr}(\varphi_1^2)$$
$$+\;\tfrac{1}{2}(m^2\,+\,\beta NC^2\,+\,\alpha C^2)\times\mathrm{tr}(\varphi_2^2)$$
$$=\;\beta C^2\times\mathrm{tr}^2(\varphi_1)\;+\;\alpha C^2\,\mathrm{tr}(\varphi_1^2)\;+\;0, \tag{S.83}$$

$$V_3 \;=\; \frac{\beta C}{\sqrt{2}}\times\mathrm{tr}(\varphi_1)\times\Big(\mathrm{tr}(\varphi_1^2)\,+\,\mathrm{tr}(\varphi_2^2)\Big)\;+\;\frac{\alpha C}{\sqrt{2}}\times\Big(\mathrm{tr}(\varphi_1^3)\,+\,\mathrm{tr}(\varphi_1\varphi_2^2)\Big), \tag{S.84}$$

$$V_4 \;=\; \frac{\beta}{8}\Big(\mathrm{tr}(\varphi_1^2)\,+\,\mathrm{tr}(\varphi_2^2)\Big)^2$$
$$+\,\frac{\alpha}{8}\Big(\mathrm{tr}(\varphi_1^4)\,+\,\mathrm{tr}(\varphi_2^4)\,+\,6\,\mathrm{tr}(\varphi_1^2\varphi_2^2)\,-\,2\,\mathrm{tr}(\varphi_1\varphi_2\varphi_1\varphi_2)\Big). \tag{S.85}$$

Combining the quadratic part (S.83) of this potential with the kinetic terms (S.76), we arrive at

$$\mathcal{L}_2 \;=\; \tfrac{1}{2}\,\mathrm{tr}(\partial_\mu\varphi_1\,\partial^\mu\varphi_1)\;-\;\beta C^2\times\mathrm{tr}^2(\varphi_1)\;-\;\alpha C^2\,\mathrm{tr}(\varphi_1^2)+\;\tfrac{1}{2}\,\mathrm{tr}(\partial_\mu\varphi_2\,\partial^\mu\varphi_2) \tag{S.86}$$

which gives us the mass spectrum of the theory: the $\varphi_1(x)$ matrix of fields is massive and the $\varphi_2(x)$ matrix is massless. Each matrix is $N\times N$ and hermitian, so it contains N^2 independent real scalar fields, which give rise to N^2 particles. Altogether, the spectrum comprises:

- N^2 massless particles from the $\varphi_2(x)$ matrix.
- N^2-1 massive particles with $M^2=2\alpha C^2$ from the traceless part of the $\varphi_1(x)$ matrix.
- One more massive particle with $M^2=2(\alpha+N\beta)C^2=-2m^2$ from the trace of $\varphi_1(x)$.

To see where the values of the masses come from, let's decompose the φ_1 matrix into the pure trace plus the traceless part,

$$\xi(x)\;\stackrel{\mathrm{def}}{=}\;\frac{\mathrm{tr}(\varphi_1(x))}{\sqrt{N}}\quad\text{and}\quad\tilde{\varphi}_1(x)\;\stackrel{\mathrm{def}}{=}\;\varphi_1(x)\;-\;\frac{\xi(x)}{\sqrt{N}}\times\mathbf{1}_{N\times N}\quad\Longrightarrow\quad\mathrm{tr}(\tilde{\varphi}_1)\;\equiv\;0. \tag{S.87}$$

Consequently,

$$\mathrm{tr}^2(\varphi_1)\;=\;N\times\xi^2,\qquad\mathrm{tr}(\varphi_1^2)\;=\;\xi^2\,+\,\mathrm{tr}(\tilde{\varphi}_1^2), \tag{S.88}$$

likewise

$$\mathrm{tr}\big(\partial_\mu\varphi_1\,\partial^\mu\varphi_1\big) \;=\; \partial_\mu\xi\,\partial^\mu\xi \;+\; \mathrm{tr}\big(\partial_\mu\tilde\varphi_1\,\partial^\mu\tilde\varphi_1\big),$$

so the free Lagrangian (S.86) becomes

$$\begin{aligned}\mathcal{L}_2 \;=\; & \tfrac12(\partial_\mu\xi)^2 \;-\; 2(\beta C^2N \,+\, \alpha C^2)\times\tfrac12\xi^2\\ &+\; \tfrac12\,\mathrm{tr}\big((\partial_\mu\tilde\varphi_1)^2\big) \;-\; 2\alpha C^2\times\tfrac12\,\mathrm{tr}(\tilde\varphi_1^2)\\ &+\; \tfrac12\,\mathrm{tr}\big((\partial_\mu\varphi_2)^2\big) \;-\; 0\times\tfrac12\,\mathrm{tr}(\varphi_2^2)\end{aligned}\tag{S.89}$$

where all the masses are manifest.

Problem **4**(e):

The unbroken $SU(N)$ symmetry acts on the scalar fields according to

$$\Phi(x) \;\to\; U\times\Phi(x)\times U^\dagger.\tag{S.74}$$

and since the VEV (11) is invariant, the shifted fields $\delta\Phi(x)=\Phi(x)\,\langle\Phi\rangle$ also transform according to

$$\delta\Phi(x) \;\to\; U\times\delta\Phi(x)\times U^\dagger.\tag{S.90}$$

Moreover, unitary transforms like these preserve hermiticity, so when we decompose $\delta\Phi(x)$ into a hermitian matrix $\varphi_1(x)$ and an antihermitian matrix $i\varphi_2(x)$, the transforms (S.90) do not mix the φ_1 and φ_2 with each other. Instead, they transform like

$$\varphi_1(x) \;\to\; U\varphi_1(x)\,U^\dagger,\qquad \varphi_2(x) \;\to\; U\varphi_2(x)\,U^\dagger,\tag{S.91}$$

which means that φ_1 and φ_2 comprise separate $SU(N)$ multiplets. Furthermore, the transforms (S.91) preserve trances $\mathrm{tr}(\varphi_1)$ and $\mathrm{tr}(\varphi_2)$, so to make the $SU(N)$ multiplet structure manifest, let's decompose both φ_1 and φ_2 into their traceless parts and pure traces along the lines of eq. (S.87),

$$\varphi_1(x) \;=\; \frac{\xi_1(x)}{\sqrt{N}}\times\mathbf{1}_{N\times N} \;+\; \tilde\varphi_1(x),\quad \varphi_2(x) \;=\; \frac{\xi_2(x)}{\sqrt{N}}\times\mathbf{1}_{N\times N} \;+\; \tilde\varphi_2(x),\quad \mathrm{tr}(\tilde\varphi_1)\;\equiv\;\mathrm{tr}(\tilde\varphi_2)\;\equiv\;0. \tag{S.92}$$

With this decomposition, the ξ_1 and the ξ_2 are both invariant under the $SU(N)$ — which puts each of them into its own singlet multiplet — while the each of the traceless parts $\tilde\varphi_1$ and $\tilde\varphi_2$ makes each own *adjoint multiplet.*

This multiplet structure agrees with the masses we obtained in part (d). Indeed, all N^2-1 members of the adjoint multiplet $\tilde\varphi_1$ have the same mass $2\alpha C^2$, while the singlet ξ_1 has a different mass $2(\alpha+N\beta)C^2$.

On the other hand, both the adjoint multiplet $\tilde\varphi_2$ and the singlet ξ_2 are massless. The reason for this degeneracy goes beyond the un-broken $SU(N)$ symmetry; instead, both the $\tilde\varphi_2$ and the ξ_2 are Goldstone bosons of the spontaneously broken symmetries in

$$G/H \;=\; \Big(SU(N)_L\times SU(N)_R\times U(1)\Big)\Big/\,SU(N)\,. \tag{S.93}$$

Specifically, the singlet ξ_2 is the Goldstone boson of the broken $U(1)$ symmetry. Indeed, the $U(1)$'s generator commutes with all the other generators, so it belongs in its own singlet of the symmetry, and the corresponding Goldstone particle should also be a singlet.

Now consider the non-abelian generators. Generators T^a_L of the $SU(N)_L$ form an adjoint multiplet of the $SU(N)_L$, but are invariant under the $SU(N)_R$. Likewise, generators T^a_R of the $SU(N)_R$ form an adjoint multiplet of the $SU(N)_R$, but are invariant under the $SU(N)_L$. In other words, under an $(U_L,U_R)\in SU(N)_L\times SU(N)_R$ they transform as

$$T^a_L \;\to\; U_LT^a_LU^\dagger_L\,,\qquad T^a_R\;\to\; U_RT^a_RU^\dagger_R\,. \tag{S.94}$$

When the $SU(N)_L\times SU(N)_R$ is broken down to a single $SU(N)$ spanning $U_L=U_R=U$, both T^a_L and T^a_R transform as

$$T^a_L\;\to\;UT^a_LU^\dagger,\qquad T^a_R\;\to\;UT^a_RU^\dagger, \tag{S.95}$$

which puts them into *two adjoint multiplets* of the unbroken $SU(N)$. Equivalently, we may form two adjoint multiplets out of

$$T_V^a \;=\; T_L^a + T_R^a \qquad \text{and} \qquad T_A^a \;=\; T_L^a - T_R^a\,, \tag{S.96}$$

which act on the scalar fields according to

$$T_V^a\Phi \;=\; \tfrac{i}{2}[\lambda^a,\Phi], \qquad T_A^a\Phi \;=\; \tfrac{i}{2}\{\lambda^a,\Phi\}. \tag{S.97}$$

The T_V^a generate the unbroken $SU(N)$ symmetry, *cf.* eq. (S.74). The T_A^a generators are spontaneously broken, hence there should be an adjoint multiplet of massless Goldstone bosons. And indeed there is — the $\tilde{\varphi}_2$.